SHORT PROOFS OF THEOREMS OF LEKKERKERKER
AND BALLIEU

MAX RIEDERLE

Eberhardstr. 14
79 Ulm/Donau
West Germany

ABSTRACT.   For any irrational number ξ let $A(\xi)$ denote the set of all accumulation

points of $\{z:\ z=q(q\xi - p),\ p \in \mathbb{Z},\ q \in \mathbb{Z} - \{0\}$, p and q relatively prime$\}$.   In this

paper the following theorem of Lekkerkerker is proved in a short and elementary way:

The set $A(\xi)$ is discrete and does not contain zero if and only if ξ is a quadratic

irrational.   The procedure used for this proof simultaneously takes care of a

theorem of Ballieu.

1.   INTRODUCTION.

This paper is easily readable by anyone familiar with the elements of continued

fractions, as far as Lagrange's theorem on the periodic representation of quadratic

irrationals.   Throughout this paper, ξ denotes an irrational number which is repre-

sented by the regular continued fraction $\xi = [b_0,b_1,b_2,\ldots] = [b_0,b_1,\ldots,b_{n-1},\xi_n]$

and $A(\xi)$ stands for the set of all the real accumulation points of $\{z:\ z = q(q\xi-p)$,

$p \in \mathbb{Z},\ q \in \mathbb{Z} - \{0\}$, p and q relatively prime$\}$.   Obviously, $A(\xi)$ describes those

Dirichlet approximation qualities which occur infinitely often.   Furthermore, for

any sequence (a_n) let $H(a_n)$ denote the set of all its limit points and for $x \in \mathbb{R}$

and $\varepsilon > 0$ set $B(x,\varepsilon) = (x - \varepsilon,\ x + \varepsilon)$.   The main purpose of this paper is to give a

simple proof of the following theorem of Lekkerkerker [1] (cf. also [2])·

The set $A(\xi)$ is discrete and does not contain zero if and only if ξ is a quadratic irrational.

The proof of the sufficient part of the theorem mainly depends on the irreducible polynomial of ξ, whereas the necessary part is a consequence of the relation between $A(\xi)$ and the sequence (ξ_n) and simultaneously establishes the following theorem of Ballieu [3]:

The set $H(\xi_n)$ is finite and (ξ_n) is bounded if and only if ξ is a quadratic irrational.

Finally for any quadratic irrational ξ we shall show how to evaluate $A(\xi)$ in an easy way.

2. BASIC FORMULAS.

In this section we state the formulas used in the sequel. Let A_n/B_n denote the n-th convergent of $[b_0, b_1, b_2, \ldots]$ where A_n and B_n are relatively prime. Set $\rho_n = B_n/B_{n-1}$ and put $\delta_n = B_n(B_n\xi - A_n)$. Then the following formulas hold for all $n \in \mathbb{N}_0$:

$$\delta_n = \frac{(-1)^n}{\xi_{n+1} + 1/\rho_n} \tag{2.1}$$

$$\delta_{n-1} = \frac{(-1)^{n-1}}{\rho_n + 1/\xi_{n+1}} \tag{2.2}$$

$$\xi_{n+1} = \frac{1 + \sqrt{1 + 4\delta_n\delta_{n-1}}}{2\delta_n}(-1)^n. \tag{2.3}$$

PROOF. Equation (2.1) is an easy consequence of the well known identity $\xi - A_n/B_n = (-1)^n/(B_n(B_n\xi_{n+1} + B_{n-1}))$, formula (2.2) can be derived from (2.1) when using the identities $\xi_n = b_n + 1/\xi_{n+1}$ and $\rho_n = b_n + 1/\rho_{n-1}$ and, finally, (2.3) can be obtained when combining (2.1) and (2.2).

3. PROOF OF LEKKERKERKER'S THEOREM.

(i) Suppose ξ is a quadratic irrational. There exists an indefinite quadratic form $f(x,y) = ax^2 + bxy + cy^2$ with $a,b,c \in \mathbb{Z}$ and $f(\xi,1) = 0$. If ζ denotes the algebraic conjugate of ξ then if follows by Vieta's theorem that

$$f(p,q) = a(p - q\xi)(p - q\zeta) = aq(q\xi - p)(\zeta - p/q) \tag{3.1}$$

for all $(p,q) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$ which implies that

$$q(q\xi - p) = \frac{f(p,q)}{a(\zeta - p/q)} \ . \tag{3.2}$$

When using the notation $\tilde{Z} = \{(x,y): \ x,y \in \mathbb{Z} \ , \ y \neq 0, \ x \text{ and } y \text{ relatively prime}\}$ equation (3.2) implies that

$$A(\xi) \subset \frac{f(\tilde{Z})}{a(\zeta - \xi)} \ , \tag{3.3}$$

since if $(p_n,q_n) \in \tilde{Z}$ with $\lim_{n \to \infty} q_n(q_n\xi - p) \in A(\xi)$ then $\lim_{n \to \infty} p_n/q_n = \xi$. Clearly, $f(Z) \subset \mathbb{Z}$ and we conclude that $A(\xi)$ is discrete. Now from (3.1) we can see that $f(p,q) \neq 0$ for all $(p,q) \in \tilde{Z}$ and hence it follows by (3.3) that $0 \notin A(\xi)$.

(ii) Suppose that $A(\xi)$ is discrete and $0 \notin A(\xi)$. From equation (2.1) we can see that $|\delta_n| \leq 1$ for all $n \in \mathbb{N}_0$. Therefore and since all the numbers δ_n are distinct, $H(\delta_n)$ is a compact subset of $A(\xi)$ and hence $H(\delta_n)$ is finite and $0 \notin H(\delta_n)$. Now by (2.3) it is easy to see that $H(\xi_n)$ is finite and (ξ_n) is bounded. Therefore, in order to complete the proof, it suffices to prove Ballieu's theorem.

4. PROOF OF BALLIEU'S THEOREM.

(i) Suppose that (ξ_n) is bounded and $H(\xi_n)$ is finite, say $H(\xi_n) = \{z_1,\ldots,z_m\}$. It follows from the identity $\xi_n = [b_n,b_{n+1},\ldots]$ that there exists a $k \in \mathbb{N}$ such that $b_n \leq k$ for all $n \in \mathbb{N}$, and hence $b_n + 1/k \leq \xi_n \leq b_n + 1 - 1/(k + 1)$. Therefore, the set $H(\xi_n) \cap \mathbb{Z}$ is empty and we can find a number $\varepsilon > 0$ such that the sets $B(z_\nu,\varepsilon)$ are pairwise disjoint and contained in $\mathbb{R} - \mathbb{Z}$.

Let $I(z)$ denote the greatest integer not exceeding z and for $z \notin \mathbb{Z}$ put $\tau(z) = (z - I(z))^{-1}$. Clearly, $\xi_{n+1} = \tau(\xi_n)$ for all $n \in \mathbb{N}_0$. Also the function τ is continuous on $H(\xi_n)$, therefore $\tau(H(\xi_n)) \subset H(\xi_n)$ and we can find a δ, $0 < \delta < \varepsilon$, such that $\tau(B(z_\nu,\delta)) \subset B(\tau(z_\nu),\varepsilon)$ for all $\nu \in \{1,\ldots,m\}$. There exists a number n_0 such that $\xi_{n_0} \in B(z_1,\delta)$ and $\xi_n \in \bigcup_{\nu=1}^{m} B(z_\nu,\delta)$ for all $n \geq n_0$. Therefore, when writing $\tau^{(p)}$ for the p-th composition map of τ, we obtain by induction that $\xi_{n_0 + p} \in B(\tau^{(p)}(z_1),\delta)$ for all $p \in \mathbb{N}_0$. Since $\tau(H(\xi_n)) \subset H(\xi_n)$ and $H(\xi_n)$ is finite, the sequence $(\tau^{(p)}(z_1))$, $p \in \mathbb{N}_0$, is periodic. From the identities $b_{n_0 + p} = I(\xi_{n_0 + p}) = I(\tau^{(p)}(z_1))$ we conclude that the sequence $(b_{n_0 + p})$, $p \in \mathbb{N}_0$, is periodic and thus, by Lagrange's theorem, ξ is a quadratic irrational.

(ii) The other direction of Ballieu's theorem is an easy consequence of

Lagrange's theorem.

5. <u>CONCLUDING REMARKS</u>.

The inclusion in (3.3) is actually an equality. In order to prove this, we need the following well known theorem (cf. [4], p. 22-23):

Let $f(x,y)$ be an indefinite quadratic form with integer coefficients and let ξ be one of its roots. Then for any pair $(p,q) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$ there are infinitely many relatively prime integers p_n, q_n such that $f(p_n,q_n) = f(p,q)$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} (q_n \xi - p) = 0$.

In fact, this result combined with (5) and (6), leads to the following:

THEOREM. Suppose that ξ is a quadratic irrational, say $f(\xi,1) = 0$ for some indefinite quadratic form $f(x,y)$ with integer coefficients. Moreover, let ζ be the algebraic conjugate of ξ. Then

$$A(\xi) = \frac{f(\widetilde{Z})}{f(1,0)(\zeta - \xi)} .$$

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Celso Grebogi, Department of Physics, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk