

## A SINGULAR FUNCTIONAL-DIFFERENTIAL EQUATION

P.D. SIAFARIKAS

Department of Mathematics  
University of Patras  
Patras - GREECE

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**ABSTRACT.** The representation of the Hardy-Lebesgue space by means of the shift operator is used to prove an existence theorem for a singular functional-differential equation which yields, as a corollary, the well known theory of Frobenius for second order differential equations.

**KEY WORDS AND PHRASES.** Singular functional-differential equation, Hardy-Lebesgue space, Shift-operator.

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### 1. INTRODUCTION.

Consider the singular functional-differential equation

$$z^2 y''(z) + zp(z)y'(z) + q(z)y(z) + \sum_{i=1}^m a_i(z)y(q^i z) = 0, \quad |q| \leq 1 \quad (1.1)$$

where

$$p(z) = \sum_{n=0}^{\infty} a_n z^n, \quad q(z) = \sum_{n=0}^{\infty} b_n z^n \quad \text{and} \quad a_i(z) = \sum_{j=0}^{\infty} a_{ij} z^j, \quad i = 1, 2, \dots, m$$

are analytic functions in some neighborhood of the closed unit disk  $\bar{\Delta} = \{z \in \mathbb{C}: |z| \leq 1\}$ .

We consider the problem of finding conditions for Equation (1.1) to have solutions in the space  $H_2(\Delta)$ , i.e. the Hilbert space of functions  $f(z) = \sum_{n=1}^{\infty} a(n)z^{n-1}$  which are analytic in the open unit disk  $\Delta = \{z \in \mathbb{C}: |z| < 1\}$  and satisfy the condition  $\sum_{n=1}^{\infty} |a(n)|^2 < +\infty$ . We shall prove the following.

THEOREM. Let

$$k(k-1) + a_0k + b_0 = 0 \quad (1.2)$$

be the indicial equation of the unperturbed equation (1.1).

(i) If  $2k + a_0 - 1 = \delta = k_1 - k_2 \neq \pm n$ ,  $n = 1, 2, \dots$ , then Equation (1.1) has two linearly independent solutions of the form:

$$y_1(z) = z^{k_1}u(z) \quad \text{and} \quad y_2(z) = z^{k_2}u(z),$$

where  $k_1$  and  $k_2$  are the roots of Equation (1.2) and  $u(z)$  belongs to  $H_2(\Delta)$ .

(ii) If  $2k + a_0 - 1 = \delta = k_1 - k_2 = 0$ , i.e.  $k_1 = k_2$ , then Equation (1.1) has only one solution of the form:

$$y(z) = z^k u(z),$$

where  $k$  is the double root of Equation (1.2) and  $u(z)$  belongs to  $H_2(\Delta)$ .

(iii) If  $2k + a_0 - 1 = \delta = k_1 - k_2 = n$ ,  $n = 1, 2, \dots$ , then Equation (1.2) has always a solution of the form:

$$y(z) = z^{k_1} u(z),$$

where  $k_1$  is the greatest root of Equation (1.2) and  $u(z)$  belongs to  $H_2(\Delta)$ .

This theorem obviously generalizes the well known Frobenius theory [1] for the Fuchs differential equations:

$$z^2 y''(z) + zp(z)y'(z) + q(z)y(z) = 0,$$

which is a particular case of Equation (1.1).

Denote an abstract separable Hilbert space over the complex field by  $H$ , the Hardy-Lebesgue space by  $H_2(\Delta)$ , an ortho-normal basis in  $H$  by  $\{e_n\}_{n=1}^{\infty}$ , and the unilateral shift operator on  $H$  ( $\forall e_n = e_{n+1}$ ) by  $V$ . We can easily see that the following statements hold:

(i) Every value  $z$  in the unit disk ( $|z| < 1$ ) is an eigenvalue of  $V^*(V^*: V^*e_n = e_{n-1}, n \neq 1, V^*e_1 = 0)$ , the adjoint of  $V$ . The eigenelements  $f_z = \sum_{n=1}^{\infty} z^{n-1} e_n$  form a complete system in  $H$ , in the sense that if  $f$  is orthogonal to  $f_z$ , for every  $z: |z| < 1$  then  $f = 0$ .

(ii) The mapping  $f(z) = (f_z, f)$ ,  $f \in H$  is an isomorphism from  $H$  onto  $H_2(\Delta)$ .

(iii) The diagonal operator  $C_0: C_0 e_n = n e_n, n = 1, 2, \dots$ , has a self-adjointed extension in  $H$  with a compact inverse  $B: B e_n = \frac{1}{n} e_n, n = 1, 2, \dots$ . Moreover, if  $f(z) = (f_z, f)$  then

$$z^n f(z) = (f_z, V^n f) \quad (1.3)$$

$$f^{(n)}(z) = (f_z, (C_0 V^*)^n f) \quad (1.4)$$

$$z f'(z) = (f_z, (C_0 - I) f). \quad (1.5)$$

We shall use the proposition 1 of Reference [2].

## 2. PROOF OF THE THEOREM.

The transformation  $y(z) = z^k u(z)$ , reduces Equation (1.1) in the following:

$$zu''(z) + (h_0 + h_1 z + h_2 z^2 + \dots) u'(z) + (\rho_0 + \rho_1 z + \rho_2 z^2 + \dots) u(z) + \sum_{i=1}^m q^{ik} a_i(z) u(q^i z) = 0, \quad (2.1)$$

where  $k(k-1) + ka_0 + b_0 = 0, 2k + a_0 = h_0, a_1 = h_1, a_2 = h_2, a_3 = h_3, \dots$  and  $ka_1 + b_1 = \rho_0, ka_2 + b_2 = \rho_1, ka_3 + b_3 = \rho_2, \dots$ . Following Reference [2], we define the operators  $R_1, R_2, \dots, R_m$  on  $H_2(\Delta)$  as

$$R_1 u(z) = u(qz), |q| \leq 1, R_2 u(z) = u(q^2 z) = R_1^2(u(z)) \dots R_m u(z) = u(q^m z) = R_1^m u(z).$$

Thus the operator  $R: Ru(z) = \sum_{i=1}^m q^{ik} a_i(z) u(q^i z), |q| \leq 1$ , on  $H_2(\Delta)$  is represented in the space  $H$  by the operator

$$\tilde{R}: \tilde{R}u = \sum_{i=1}^m q^{ik} a_i^*(V) (\tilde{R}_1^*)^i u$$

where  $\tilde{R}_1$  is defined on  $H$  as  $R_1 e_n = q^{n-1} e_n, n = 1, 2, \dots$ . The equation (2.1) has a solution in  $H_2(\Delta)$  if and only if the operator equation

$$[V(C_0 V^*)^2 + \phi_1(V) C_0 V^* + \phi_2(V) + \tilde{R}]u = 0 \quad (2.2)$$

has a solution  $u$  in the abstract separable Hilbert space  $H$ .

$$\text{Here } u = \sum_{n=1}^{\infty} (\overline{u, e_n}) e_n, \phi_1(V) = (2k + a_0)I + h_1 V + h_2 V^2 + \dots,$$

$$\phi_2(V) = \rho_0 I + \rho_1 V + \rho_2 V^2 + \dots,$$

where the bar denotes complex conjugation.

Taking into account the relations

$$V^2 C_0 V^* = V(C_0 - I) \quad \text{and} \quad V C_0 - C_0 = -V,$$

Equation (2.2) can be written as

$$\left[ [C_0 + (2k + a_0 - 1)I + B\phi(V) - B^2 V \phi'_1(V)] V^* + B\phi_2(V) + B\tilde{R} \right] u = 0, \quad (2.3)$$

where

$$\phi(V) = h_1 V + h_2 V^2 + h_3 V^3 + \dots \quad \text{and} \quad \phi'_1(V) = h_1 + 2h_2 V + 3h_3 V^2 + \dots$$

Also, if we put  $2k + a_0 - 1 = \delta$  in Equation (2.3), we have

$$V^*[I + VK] u = 0 \quad (2.4)$$

where the operator

$$K = \delta BV^* + B^2 \phi(V) C_0 V^* + B^2 \phi_2(V) + B^2 \tilde{R}$$

is compact. Relation (2.4) implies that

$$(I + VK) u = c e_1, \quad c = \text{const.} \quad (2.5)$$

Now it follows that the operator  $(I + VK)^{-1}$  exists. In fact,

$$(I + VK)u = 0 \Rightarrow u = -VKu \Rightarrow (u, e_1) = -(Ku, V^* e_1) = 0. \quad \text{Also,}$$

$$(u, e_2) = -(u, K^* e_1) \Rightarrow (u, e_2)(1 + \delta) = 0 \quad (2.6)$$

Relation (2.6) if  $\delta \neq -1 \Rightarrow (u, e_2) = 0$ . Similarly,

$$(u, e_3) = -(u, K^* e_2) \Rightarrow (u, e_3)(1 + \frac{\delta}{2}) = 0. \quad (2.7)$$

Relation (2.7) if  $\delta \neq -2 \Rightarrow (u, e_3) = 0$ . By the same way and if  $\delta \neq -n$ ,  $n = 1, 2, \dots$ ,

we find

$$u = \sum_{n=1}^{\infty} (\overline{u, e_n}) e_n = 0.$$

Since also the operator  $VK$  is compact Fredholm alternative implies that the operator

$(I + VK)^{-1}$  is defined every where. Thus from Equation (2.5), we have

$$u = c \cdot (I + VK)^{-1} e_1.$$

This means that

(i) If  $2k + a_0 - 1 = \delta = k_1 - k_2 \neq \pm n$  with  $n = 1, 2, \dots$ , then the operator  $(I + VK)^{-1}$  always exists. Therefore, Equation (1.1) has two linearly independent solutions of the form

$$y_1(z) = z^{k_1} u(z) \quad \text{and} \quad y_2(z) = z^{k_2} u(z),$$

where  $k_1$  and  $k_2$  are the roots of Equation (1.2) and  $u(z)$  belongs to  $H_2(\Delta)$  and is given by the relation

$$u(z) = (u_z, u), \quad u_z = \sum_{n=1}^{\infty} z^{n-1} e_n, \quad u = c \cdot (I + VK)^{-1} e_1.$$

(ii) If  $2k + a_0 - 1 = \delta = k_1 - k_2 = 0$ , i.e.  $k_1 = k_2$ , then the operator  $(I + VK)^{-1}$  always exists. Therefore, Equation (1.1) has only one solution of the form

$$y(z) = z^k u(z),$$

where  $k$  is the double root of Equation (2.1) and  $u(z)$  as in (i).

(iii) If  $2k + a_0 - 1 = \delta = k_1 - k_2 = n$ ,  $n = 1, 2, \dots$ , then

$$2k_1 + a_0 - 1 = n, \quad n = 1, 2, \dots,$$

$$2k_2 + a_0 - 1 = -n, \quad n = 1, 2, \dots,$$

From the above and the Relations (2.6) and (2.7), we see that Equation (1.1) has always a solution of the form

$$y(z) = z^{k_1} u(z),$$

where  $k_1$  is the greatest root of Equation (1.2) and  $u(z)$  as in (i). All the above complete the proof of the theorem.

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