

PARAMETERS AND SOLUTIONS OF LINEAR AND NONLINEAR OSCILLATORS

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ABSTRACT. Relationship between existence of solutions for certain classes of nonlinear boundary value problems and the smallest or the largest eigenvalue of the corresponding linear problem is obtained. Behavior of the solutions, as the parameter increases, is also studied.

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1. INTRODUCTION.

Equations of the form

$$y''(x) + p(x) y(x) + \lambda q(x) y^n(x) = 0 , \quad (1.1)$$

where λ is a parameter and n is a positive integer, arise in many physical problems, for examples, in linear ($n = 1$) and nonlinear ($n \neq 1$) oscillation problems

[1, 2, 3, 4], and in nuclear energy distribution [5, 6]. In these problems, the parameter has physical significance, such as the energy level or the stiffness factor of the system under consideration.

In this work, relationship between existence of solutions for classes of nonlinear boundary value problems with equations of the form (1.1) and the smallest or the largest eigenvalue of the corresponding linear problem is obtained. The case of the coefficient $q(x)$ being a negative constant has been investigated in [7]. Conditions on the coefficients of the equation, under which the solution remains bounded as the parameter increases, are obtained.

2. EXISTENCE OF SOLUTIONS FOR NONLINEAR BOUNDARY VALUE PROBLEMS AND EIGENVALUE OF CORRESPONDING LINEAR PROBLEMS.

In this section, relationship between existence of solutions for equations of the form (1.1) with zero boundary conditions and the smallest or largest eigenvalue of the corresponding linear problem is obtained. The analysis used here is similar to that in [7]. It would be assumed that the functions $p(x)$ and $q(x)$ are in the class $C[0,1]$.

In the first two theorems, the nonlinear boundary value problem

$$y''(x) + p(x) y(x) - q(x) y^n(x) = 0 , \quad (2.1)$$

$$y(0) = 0 , \quad y(1) = 0 , \quad (2.2)$$

and the corresponding linear eigenvalue problem

$$z''(x) + p(x) z(x) - \lambda q(x) z(x) = 0 , \quad (2.3)$$

$$z(0) = 0 , \quad z(1) = 0 , \quad (2.4)$$

are considered.

THEOREM 2.1. If (1) $p(x) > 0$ and (2) $q(x) > 0$, then (2.1) and (2.2) has a positive solution if and only if the largest eigenvalue of (2.3) and (2.4) is positive.

PROOF. Suppose (2.1) and (2.2) has a positive solution. To show that the largest eigenvalue λ_1 of (2.3) and (2.4) is positive, let z_1 be a corresponding eigenfunction of (2.3) and (2.4) satisfying $z_1 \neq 0$ for $0 < x < 1$ [8]. Multiplying (2.1) by z_1 , (2.3) by y and subtracting the equations, we get

$$y''z_1 - yz_1'' - q(x)y^n z_1 + \lambda_1 q(x)yz_1 = 0. \quad (2.5)$$

Integration of (2.5) from 0 to 1 and the boundary conditions (2.2) and (2.4) lead to

$$-\int_0^1 q(x)y^n z_1 \, dx + \lambda_1 \int_0^1 q(x)yz_1 \, dx = 0,$$

therefore,

$$\lambda_1 = \frac{\int_0^1 q(x)y^n z_1 \, dx}{\int_0^1 q(x)yz_1 \, dx}$$

and λ_1 is positive.

Suppose now that the largest eigenvalue of (2.3) and (2.4) is positive. Note first that if y is positive and M denotes its maximum, then

$$y \leq M < R^{\frac{1}{n-1}},$$

where

$$R = \max_{x \in [0, 1]} \frac{p(x)}{q(x)}.$$

To apply an existence theorem for nonlinear eigenvalue problems in [9], equation (2.1) is written in the form

$$Ly = F(x, y),$$

where

$$Ly = -y'' + a(x)y, \quad a(x) > 0,$$

$$F(x, y) = [p(x) + a(x)]y - q(x)y^n.$$

To show that a positive solution of (2.1) and (2.2) exists, we must find curves

$u(x)$, $v(x)$ such that

$$0 < u(x) \leq v(x), \text{ for all } x \in (0, 1),$$

$$v(0) \geq 0, v(1) \geq 0, Lv \geq F(x, v),$$

$$u(0) \leq 0, u(1) \leq 0, Lu \leq F(x, u),$$

and $a(x)$ must be chosen so that $F(x, y)$ is a monotonic increasing function of y for all (x, y) in the set

$$S = \{(x, y) \mid 0 \leq x \leq 1, u(x) \leq y \leq v(x)\}.$$

Let

$$v(x) = R^{\frac{1}{n-1}},$$

then

$$\begin{aligned} Lv - F(x, v) &= a(x)v - [p(x) + a(x)]v + q(x)v^n \\ &= v[q(x)v^{n-1} - p(x)] \\ &= R^{\frac{1}{n-1}}[q(x)R - p(x)] \\ &\geq 0 \end{aligned}$$

and v satisfies all the requirements.

Let

$$u(x) = z_1(x),$$

where $z_1(x)$ is normalized such that

$$0 < z_1(x) \leq R^{\frac{1}{n-1}} \text{ and } z_1(x) \leq R^{\frac{1}{n-1}}, \text{ for } x \in (0, 1),$$

then

$$\begin{aligned} Lu &= -z_1'' + a(x)z_1 \\ &= p(x)z_1 - \lambda_1 q(x)z_1 + a(x)z_1 \\ &= [p(x) + a(x)]z_1 - \lambda_1 q(x)z_1 \\ &\leq [p(x) + a(x)]z_1 - q(x)z_1^n \\ &= F(x, u). \end{aligned}$$

From the fact that

$$\frac{\partial F}{\partial y} = p(x) + a(x) - q(x)ny^{n-1},$$

$F(x, y)$ is increasing in y in S if

$$a(x) \geq q(x)ny^{n-1} - p(x),$$

so choose

$$a(x) \geq Q n R - p_0,$$

where

$$Q = \max_{x \in [0, 1]} q(x), \quad p_0 = \min_{x \in [0, 1]} p(x).$$

By [9], the nonlinear problem (2.1) and (2.2) has at least one solution in S .

THEOREM 2.2. (1) If $p(x) > 0$, (2) $q(x) > 0$ and (3) n is odd, then (2.1) and (2.2) has a negative solution if and only if the largest eigenvalue of (2.3) and (2.4) is positive.

PROOF. Suppose (2.1) and (2.2) has a negative solution. Then as in the proof of Theorem 2.1, it can be shown that if λ_1 is the largest eigenvalue of (2.3) and (2.4) and z_1 is a corresponding eigenfunction, then

$$\frac{\int_0^1 q(x) y^n z_1 dx}{\int_0^1 q(x) y z_1 dx}$$

and since n is odd, λ_1 is positive.

Conversely, suppose that the largest eigenvalue of (2.3) and (2.4) is positive. Note first that if y is negative and m denotes its minimum at say x_0 , then

$$y'' = -p(x_0) m + q(x_0)m^n > 0,$$

$$m^n > \frac{p(x_0)}{q(x_0)} m \quad ,$$

$$m^{n-1} < \frac{p(x_0)}{q(x_0)} \quad .$$

Since $(n - 1)$ is even,

$$-\left[\frac{p(x_0)}{q(x_0)}\right]^{\frac{1}{n-1}} < m \leq y$$

and so

$$-\frac{1}{R^{\frac{1}{n-1}}} \leq y \quad .$$

To apply the existence theorem in [9], equation (2.1) is written in a form as in the proof of Theorem 2.1. To show that a negative solution of (2.1) and (2.2) exists, this time we must find curves $u(x)$, $v(x)$ such that

$$\begin{aligned} u(x) &\leq v(x) < 0, \quad \text{for } x \in (0, 1), \\ v(0) &= 0, \quad v(1) = 0, \quad Lv \geq F(x, v), \\ u(0) &\leq 0, \quad u(1) \leq 0, \quad Lu \leq F(x, u) \end{aligned}$$

and $a(x)$ must be chosen so that $F(x, y)$ is a monotonic increasing function of y for all (x, y) in the set

$$S = \{(x, y) \mid 0 \leq x \leq 1, \quad u(x) \leq y \leq v(x)\} \quad .$$

Let

$$u(x) = -R^{\frac{1}{n-1}} \quad ,$$

then

$$\begin{aligned} Lu - F(x, u) &= a(x)u - [p(x) + a(x)]u + q(x)u^n \\ &= u[q(x)u^{\frac{n-1}{n-1}} - p(x)] \\ &= -R^{\frac{1}{n-1}}[q(x)R - p(x)] \\ &\leq 0 \end{aligned}$$

and u satisfies all the requirements.

Let

$$v(x) = z_1(x) ,$$

where $z_1(x)$ is normalized such that

$$-\lambda_1^{\frac{1}{n-1}} \leq z_1(x) < 0 \quad \text{and} \quad -R^{\frac{1}{n-1}} \leq z_1(x), \quad \text{for } x \in (0, 1) ,$$

then

$$\lambda_1 \geq z_1^{n-1}$$

$$-\lambda_1 q(x) z_1 \geq -q(x) z_1^n$$

and so

$$\begin{aligned} Lv &= [p(x) + a(x)]z_1 - \lambda_1 q(x) z_1 \\ &\geq [p(x) + a(x)] z_1 - q(x) z_1^n \\ &= F(x, v) . \end{aligned}$$

From the fact that

$$\frac{\partial F}{\partial y} = p(x) + a(x) - q(x)n y^{n-1} ,$$

$F(x, y)$ is increasing in y in S if

$$a(x) \geq q(x)n y^{n-1} - p(x)$$

$$\leq Q n R - p_0 ,$$

so let

$$a(x) \geq Q n R - p_0 .$$

It follows from [9] that the nonlinear problem (2.1) and (2.2) has at least one solution in S .

In the next Theorem, the nonlinear problem

$$y''(x) + p(x)y + q(x) y^n = 0 , \quad (2.6)$$

$$y(0) = y(1) = 0 , \quad (2.7)$$

and the corresponding linear eigenvalue problem

$$z''(x) + p(x)z + \lambda q(x)z = 0 , \quad (2.8)$$

$$z(0) = z(1) = 0 , \quad (2.9)$$

are considered.

THEOREM 2.3. If (1) $p(x) > 0$, (2) $q(x) > 0$ and (3) n is even, then (2.6) and (2.7) has a negative solution if and only if the smallest eigenvalue of (2.8) and (2.9) is negative.

PROOF. Let

$$y(x) = -Y(x) , \text{ then}$$

$$-Y''(x) - p(x)Y + q(x) [-Y(x)]^n = 0$$

and so $Y(x)$ satisfies

$$Y''(x) + p(x)Y - q(x)Y^n = 0 , \quad (2.10)$$

$$Y(0) = 1 , \quad Y(1) = 0 . \quad (2.11)$$

By Theorem 2.1, (2.10) and (2.11) has a positive solution Y if and only if the largest eigenvalue of (2.3) and (2.4) is positive, and hence if and only if the smallest eigenvalue of (2.8) and (2.9) is negative. The conclusion of the theorem now follows.

3. BOUNDEDNESS OF THE SOLUTION AS THE PARAMETER INCREASES.

In this section, boundedness of the solution of

$$y''(x) + p(x)y + \lambda q(x)y^n = 0 , \quad (3.1)$$

$$y(0) = 0 , \quad (3.2)$$

as the parameter λ increases, is studied. It would be assumed that the functions $p(x)$ and $q(x)$ are in the class $C^1[0, 1]$.

THEOREM 3.1. If (1) $p(x) > 0$, $p'(x) \leq 0$, (2) $q(x) > 0$, $q'(x) \leq 0$, (3) n is odd or $y \geq 0$ and (4) $\frac{y'(0)}{\sqrt{\lambda}}$ is bounded as $\lambda \rightarrow \infty$, then y is bounded as $\lambda \rightarrow \infty$.

PROOF. Multiplication of (3.1) by y' and integration of the resulting equation over $[0, x]$ lead to

$$\frac{y'^2}{2} \Big|_0^x + p(s) \frac{y^2}{2} \Big|_0^x - \int_0^x p'(s) \frac{y^2}{2} ds + \lambda q(s) \frac{y^{n+1}}{n+1} \Big|_0^x - \lambda \int_0^x q'(s) \frac{y^{n+1}}{n+1} ds = 0,$$

$$y'^2(x) + p(x)y^2(x) - \int_0^x p'(s)y^2 ds + \frac{2}{n+1} \lambda q(x)y^{n+1}(x) - \frac{2}{n+1} \lambda \int_0^x q'(s)y^{n+1} ds = y'^2(0).$$

Therefore,

$$\frac{2}{n+1} \lambda q(x) y^{n+1}(x) \leq y'^2(0),$$

$$y^{n+1}(x) \leq \frac{n+1}{2} \frac{y'^2(0)}{\lambda q(x)}.$$

and the conclusion follows.

THEOREM 3.2. If (1) $p(x) > 0$, $p'(x) \geq 0$, (2) $q(x) > 0$, $q'(x) \leq 0$, (3) n is odd or $y \geq 0$ and (4) $y'(0)$ is bounded as $\lambda \rightarrow \infty$, then y is bounded as $\lambda \rightarrow \infty$.

PROOF. As in Theorem 3.1, equation (3.1) is multiplied by y' and the resulting equation integrated over $[0, x]$, obtaining

$$p(x)y^2(x) \leq y'^2(0) + \int_0^x p'(s)y^2 ds,$$

$$p(x)y^2(x) \leq y'^2(0) + \int_0^x p(s)y^2 \frac{p'(s)}{p(s)} ds$$

and by Gronwall's inequality [10],

$$\begin{aligned} p(x)y^2(x) &\leq y'^2(0) \exp \int_0^x \frac{p'(s)}{p(s)} ds \\ &= y'^2(0) \frac{p(x)}{p(0)}, \end{aligned}$$

therefore,

$$y^2(x) \leq \frac{y'^2(0)}{p(0)}$$

and the result follows.

REFERENCES

1. Ames, W. F., *Nonlinear Ordinary Differential Equations in Transport Processes*, Academic Press, New York and London, 1968.
2. Keller, J. B., *Lower Bounds and Isoperimetric Inequalities for Eigenvalues of the Schrödinger Equation*, J. Math. Phys. 2 (1961) 262-266.
3. McLachlan, N. W., *Ordinary Non-linear Differential Equations in Engineering and Physical Sciences*, 2nd ed., Oxford University Press, London and New York, 1958.
4. Struble, R. A., *Nonlinear Differential Equations*, McGraw-Hill Co., New York and London, 1962.
5. Canosa, J. and J. Cole, *Asymptotic Behavior of Certain Nonlinear Boundary-value Problems*, J. Math. Phys. 9 (1968) 1915-1921.
6. Ergen, W. K., *Self-limiting Power Excursions in Large Reactors* Trans. Am. Nucle. Soc. 8 (1965) 221.
7. Shampine, L. F., *Existence of Solutions for Certain Nonlinear Boundary-value Problems*, J. Math. Phys. 10 (1969) 1177-1178.
8. Ince, E. L., *Ordinary Differential Equations*, Dover Publications, New York, 1956.
9. Shampine, L. F., *Some Nonlinear Eigenvalue Problems*, J. Math. Mech. 17 (1968) 1065-1072.
10. Hartman, P., *Ordinary Differential Equations*, John Wiley and Sons, New York and London, 1964.

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