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DIFFERENTIAL OPERATORS AND FLAT CONNECTIONS ON A RIEMANN SURFACE

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We consider filtered holomorphic vector bundles on a compact Riemann surface X equipped with a holomorphic connection satisfying a certain transversality condition with respect to the filtration. If Q is a stable vector bundle of rank r and degree $(1-\text{genus}(X))nr$, then any holomorphic connection on the jet bundle $J^n(Q)$ satisfies this transversality condition for the natural filtration of $J^n(Q)$ defined by projections to lower-order jets. The vector bundle $J^n(Q)$ admits holomorphic connection. The main result is the construction of a bijective correspondence between the space of all equivalence classes of holomorphic vector bundles on X with a filtration of length n together with a holomorphic connection satisfying the transversality condition and the space of all isomorphism classes of holomorphic differential operators of order n whose symbol is the identity map.

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1. Introduction. Let E be a holomorphic vector bundle of rank $(n+1)r$ over a compact connected Riemann surface X equipped with a flat connection ∇. Let F be a holomorphic subbundle of E of rank r. The second fundamental form of ∇ gives a filtration of V

$$F_0 := 0 \subset F_1 := F \subset F_2 \subset F_3 \subset \cdots \subset F_m \tag{1.1}$$

by subbundles. More precisely, F_{i+1}/F_i is the subbundle of the vector bundle E/F_i generated by the image of the second fundamental form

$$S(F_i) : TX \otimes F_i \longrightarrow E/F_i \tag{1.2}$$

of the subbundle F_i for the connection ∇. This condition defines the filtration inductively. Let

$$S_i : F_i/F_{i-1} \longrightarrow K_X \otimes (F_{i+1}/F_i) \tag{1.3}$$

be the homomorphism induced by $S(F_i)$. We call the pair (∇, F) a coupled connection if $m = n+1$ with $F_{n+1} = E$, and S_i is an isomorphism for all $i \in [1,n]$.

Let $Q := E/F_n$ be the final quotient in the above filtration for the coupled connection (∇, F). We show that there is a natural isomorphism of E with the jet bundle $J^n(Q)$ (Lemma 3.2). Using Lemma 3.2, we construct a differential operator

$$D \in H^0(X, \mathrm{Diff}^{n+1}_X(Q, K_X^{n+1} \otimes Q)) \tag{1.4}$$

of order $n+1$, where K_X is the holomorphic cotangent bundle of X such that the symbol of D is the identity automorphism of Q.

Conversely, given such a differential operator whose symbol is the identity automorphism of Q, we construct a coupled connection on $J^n(Q)$ (Lemma 4.1).

Two coupled connections (∇, F) on E and (∇', F') on E' are called equivalent if there is an isomorphism of E with E' that takes F to F' and ∇ to ∇'. Similarly, two differential operators D and D', on Q and Q', respectively, of the above type with identity maps as symbol will be called equivalent if there is an isomorphism of Q with Q' that transports D to D'.

We prove that the space of all equivalence classes of coupled connections is in bijective correspondence with the space of all equivalence classes of differential operators. More precisely, both of the above-mentioned constructions, namely, from operators to coupled connections and from coupled connections to operators, give a bijective correspondence and they are inverses of each other (Theorem 4.5).

Let Q be a stable vector bundle over X of rank r and degree $(1-g)nr$, where g is the genus of X. We prove that $J^n(Q)$ admits a flat connection (Lemma 3.6). It is also shown that any flat connection on $J^n(Q)$, together with the subbundle $K_X^{\otimes n} \otimes Q$, defines a coupled connection (Lemma 3.5).

The special case of $n = 1$ was investigated in [2]. More on coupled connections can be found in [3].

2. Preliminaries. We recall the definition of jet bundles and its basic properties.

Let E be a holomorphic vector bundle over a Riemann surface X and let k be a nonnegative integer. The kth-order *jet bundle* of E, denoted by $J^k(E)$, is defined to be the following direct image on X:

$$J^k(E) := p_{1*}\left(\frac{p_2^*E}{p_2^*E \otimes \mathbb{O}_{X\times X}(-(k+1)\Delta)}\right), \tag{2.1}$$

where $p_i : X \times X \to X$, $i = 1,2$, is the projection onto the ith factor and Δ is the diagonal divisor on $X \times X$ consisting of all points of the form (x,x).

Let K_X denote the holomorphic cotangent bundle of X. There is a natural exact sequence

$$0 \longrightarrow K_X^{\otimes k} \otimes E \longrightarrow J^k(E) \longrightarrow J^{k-1}(E) \longrightarrow 0 \tag{2.2}$$

which is constructed using the obvious inclusion of $\mathbb{O}_{X\times X}(-(k+1)\Delta)$ in $\mathbb{O}_{X\times X}(-k\Delta)$. The inclusion map $K_X^{\otimes k}\otimes E \to J^k(E)$ is constructed by using the homomorphism

$$K_X^{\otimes k} \longrightarrow J^k(\mathbb{O}_X) \tag{2.3}$$

which is defined at any $x \in X$ by sending $(df)^{\otimes k}$, where f is any holomorphic function with $f(x) = 0$, to the jet of the function $f^k/k!$ at x. Any homomorphism $E \to F$ induces a homomorphism

$$J^k(E) \longrightarrow J^k(F) \tag{2.4}$$

for any $k \geq 0$.

The sheaf of *differential operators* $\mathrm{Diff}_X^k(E,F)$ is defined to be $\mathrm{Hom}(J^k(E),F)$. The homomorphism

$$\sigma : \mathrm{Diff}_X^k(E,F) \longrightarrow \mathrm{Hom}\left(K_X^{\otimes k}\otimes E, F\right) \tag{2.5}$$

obtained by restricting a homomorphism from $J^k(E)$ to F to the subsheaf $K_X^{\otimes k}\otimes E$ in (2.2), is known as the *symbol map*.

For all $k, l \geq 0$, there is a natural injective homomorphism

$$\tau : J^{k+l}(E) \longrightarrow J^k(J^l(E)). \tag{2.6}$$

A holomorphic section of E over an open subset U of X gives a section of $J^i(E)$ over U for each $i \geq 0$. This defines a map from a subspace of the space of sections $\Gamma(U, J^{k+l}(E))$ to a subspace of $\Gamma(U, J^k(J^l(E)))$. This map extends to the $\mathbb{O}_X$-linear homomorphism τ. We describe the image of τ for the special case $k = 1$. This will be done using induction on l.

So, first set $l = 1$. Using (2.4), the homomorphism $J^1(E) \to E$ in (2.2) gives a homomorphism $\gamma : J^1(J^1(E)) \to J^1(E)$. On the other hand, (2.2) gives a homomorphism $\gamma' : J^1(J^1(E)) \to J^1(E)$. The image $\tau(J^2(E))$ is the kernel of the difference $\gamma - \gamma'$. In other words, we have an exact sequence

$$0 \longrightarrow J^2(E) \xrightarrow{\tau} J^1(J^1(E)) \xrightarrow{\gamma-\gamma'} K_X\otimes E \longrightarrow 0. \tag{2.7}$$

Note that the image of $\gamma - \gamma'$ is contained in the subbundle $K_X \otimes E \subset J^1(E)$ since the two projections of $J^1(J^1(E))$ to E, obtained from γ and γ', respectively, coincide.

Now, suppose that

$$\tau_{l-1} : J^1(E) \longrightarrow J^1(J^{l-1}(E)) \tag{2.8}$$

is the homomorphism in (2.6). As before, the projection $J^l(E) \to J^{l-1}(E)$ in (2.2) induces, using (2.4), a homomorphism

$$\gamma : J^1(J^l(E)) \longrightarrow J^1(J^{l-1}(E)). \tag{2.9}$$

On the other hand, consider the homomorphism $J^1(J^l(E)) \to J^l(E)$ defined as in (2.2) and let

$$\gamma' : J^1(J^l(E)) \longrightarrow J^1(J^{l-1}(E)) \tag{2.10}$$

denote its composition with the homomorphism τ_{l-1} in (2.8). Just as before, the image of the homomorphism

$$\tau_l : J^{1+l}(E) \longrightarrow J^1(J^l(E)) \tag{2.11}$$

in (2.6) is precisely the kernel of $\gamma - \gamma'$.

A holomorphic connection on a holomorphic vector bundle E over X is a first-order operator

$$\nabla \in H^0(X, \mathrm{Diff}^1_X(E, K_X \otimes E)) \tag{2.12}$$

with the identity automorphism of E as its symbol. It is easy to see that this condition on E is equivalent to the *Leibniz identity*

$$\nabla(fs) = df \otimes s + f\nabla s, \tag{2.13}$$

where s is a local (holomorphic) section of E and f is a local holomorphic function. The (complex) dimension of X being one, the curvature of ∇ vanishes identically. Therefore, if $\overline{\partial}$ is the Dolbeault operator defining the holomorphic structure of E, then $\nabla + \overline{\partial}$ is a flat connection on E compatible with its holomorphic structure. Conversely, the $(1,0)$-part of any flat connection on E compatible with its holomorphic structure defines a holomorphic connection on E. By a flat connection on a holomorphic vector bundle we always mean one which is compatible with the holomorphic structure.

For a subbundle F of E, the *second fundamental form* of a holomorphic connection ∇ on E is the composition

$$F \hookrightarrow E \xrightarrow{\nabla} K_X \otimes E \xrightarrow{\mathrm{Id} \times q} K_X \otimes (E/F), \tag{2.14}$$

where the final homomorphism is defined using the natural projection $q : E \to E/F$. Note that the Leibniz identity ensures that the second fundamental form is a homomorphism of $\mathcal{O}_X$-modules. Therefore, it defines a vector bundle homomorphism

$$S \in H^0(X, K_X \otimes \mathrm{Hom}(F, E/F)). \tag{2.15}$$

Now, S generates a subbundle of E containing F. Indeed, denoting by T the torsion part of the cokernel $(E/F)/S(K_X^* \otimes F)$, for the homomorphism S, the inverse image of T for the quotient map $E/F \to (E/F)/S(K_X^* \otimes F)$ is a subbundle of E/F containing the image, under S, of $K_X^* \otimes F$. Denoting this subbundle by

F', the inverse image $F_2 := q^{-1}(F')$ is a subbundle of E containing F. Clearly, F is preserved by ∇ if and only if $F = F_2$.

We have a filtration

$$F_0 := 0 \subset F_1 := F \subset F_2 \subset F_3 \subset \cdots \subset F_n \subset F_{n+1} \tag{2.16}$$

of E by subbundles, where F_{i+1} is the subbundle obtained by substituting F_i for F in the above construction of F_2. If F_{n+1}, namely where the iterated construction of filtration stabilizes, is a proper subbundle of E, then we will define F_{n+2} to be E itself. Note that F_{n+1} is preserved by ∇.

Since the operator ∇ sends F_i to $K_X \otimes F_{i+1}$, the second fundamental forms for the subbundles in the filtration (2.16) of E give a homomorphism

$$S_i : F_i/F_{i-1} \longrightarrow K_X \otimes (F_{i+1}/F_i) \tag{2.17}$$

of vector bundles for each $i \in [1, n+1]$. So, S_1 coincides with S in (2.15). Also, if $F_{n+1} \neq E$, then $S_{n+1} = 0$ (so, S_{n+1} always vanishes) and $S_i \neq 0$ for all $i \leq n$.

3. Coupled connection. Let X be a compact connected Riemann surface of genus at least two. Let E be a holomorphic vector bundle, of rank $(n+1)r$ over X, equipped with a holomorphic connection ∇. Suppose F is a holomorphic subbundle of E of rank r.

Definition 3.1. The pair (∇, F) will be called a *coupled connection* if F_{n+1}, namely where the filtration (2.16) stabilizes, coincides with E and the vector bundle homomorphism S_i in (2.17) is an isomorphism for each $i \in [1, n]$. The rank of the subbundle F, namely r, will be called the *rank* of the coupled connection.

Take a coupled connection (∇, F) on a vector bundle E over X.

The final quotient E/F_n for the filtration (2.16) of E will be denoted by Q. Since each S_i in (2.17) is an isomorphism, we have

$$S_n \circ S_{n-1} \circ \cdots \circ S_1 : F \longrightarrow K_X^n \otimes Q \tag{3.1}$$

as an isomorphism. Here, S_i denotes the homomorphism

$$K_X^{i-1} \otimes (F_i/F_{i-1}) \longrightarrow K_X^{i-1} \otimes K_X \otimes (F_{i+1}/F_i) = K_X^i \otimes (F_i/F_{i-1}) \tag{3.2}$$

obtained by taking the tensor product of the identity map of K_X^{i-1} with the homomorphism in (2.17). Furthermore, $F_{i+1}/F_i = K_X^{\otimes i} \otimes F$.

Let $q : E \to Q$ denote the natural quotient homomorphism. For any $x \in X$ and any vector $v \in E_x$ in the fiber of E over x, let s_v denote the (unique) local flat section of E, defined around x, with $s_v(x) = v$. Let $\overline{v}$ denote the vector in $J^j(Q)$ representing the (local) section $q(s_v)$ of Q. Consequently, we have a

vector bundle homomorphism

$$\phi_j : E \longrightarrow J^j(Q) \tag{3.3}$$

that sends any v to $\overline{v}$ constructed above.

Lemma 3.2. *The homomorphism ϕ_n is an isomorphism. Furthermore, there is a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\phi_n} & J^n(Q) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle h} \\ E/F_i & \longrightarrow & J^{n-i}(Q) \end{array} \tag{3.4}$$

for every $i \in [1, n]$, where q is the natural quotient map and h is obtained from (2.2).

Proof. The lemma follows immediately by unraveling the definitions. Towards this, we first describe the second fundamental form and the filtration in (2.16).

As in (2.15), let F be a subbundle of a flat vector bundle E. We give an interpretation of the homomorphism S in (2.15). Take a point $x \in X$ and a vector $v \in F_x$ over x. Let v_1 denote the (unique) flat section of E over the first-order infinitesimal neighborhood of x, with $v_1(x) = v$. Let $\overline{v}$ be a section of F over the first-order infinitesimal neighborhood of x, with $\overline{v}(x) = v$. The difference $\overline{v} - v_1$ is clearly an element of $(K_X \otimes E)_x$. Its projection to $(K_X \otimes (E/F))_x$ coincides with $S(x)(v)$.

More generally, for $i \geq 1$, let v_i denote the (unique) flat section of E over the ith-order infinitesimal neighborhood of x, with $v_i(x) = v$. Let $\overline{v}$ be a section of F_i over the ith-order infinitesimal neighborhood of x, with $\overline{v}(x) = v$. Consider the difference $\overline{v}_i := \overline{v} - v_i$. Using induction, it is easy to see that v_i is a section of F_{i+1}. Consequently, by projecting $\overline{v}_i$, we get an element in $(K_X^{\otimes i} \otimes (F_{i+1}/F_i))_x$. This element in $(K_X^{\otimes i} \otimes (F_{i+1}/F_i))_x$ coincides with $(S_i \circ S_{i-1} \circ \cdots \circ S_1)(v)$ with S_j as in (3.1). From this observation the lemma follows immediately. □

From Lemma 3.2 it follows that we have the homomorphism

$$\phi_{n+1} \circ \phi_n^{-1} : J^n(Q) \longrightarrow J^{n+1}(Q) \tag{3.5}$$

which is a splitting of the surjective homomorphism $J^{n+1}(Q) \to J^n(Q)$ given by (2.2). In other words, the homomorphism in (3.5) gives a splitting of the exact sequence in (2.2). The resulting homomorphism

$$\psi : J^{n+1}(Q) \longrightarrow K_X^{n+1} \otimes Q \tag{3.6}$$

for this splitting is a differential operator

$$D \in H^0(X, \mathrm{Diff}_X^{n+1}(Q, K_X^{n+1} \otimes Q)) \tag{3.7}$$

of order $n+1$ on X. Since D is defined by a splitting of the jet sequence, its symbol, which is a holomorphic section of $K_X^{-n-1} \otimes Q^* \otimes K_X^{n+1} \otimes Q = \mathrm{End}(Q)$, coincides with the identity automorphism of Q.

In Section 4, we see that the above construction of differential operator from a coupled connection can be reversed.

Transport the connection ∇ on E to a connection on $J^n(Q)$ using the isomorphism ϕ_n in (3.3). Consider the filtration (2.16) of $J^n(Q)$ for the subbundle $K_X^n \otimes Q \subset J^n(Q)$ in (2.2) corresponding to this connection. Lemma 3.2 says that this filtration of $J^n(Q)$ coincides with the filtration obtained from (2.2). Therefore, we have the following equivalent definition of a coupled connection.

Definition 3.3. A coupled connection is a flat connection on $J^n(Q)$, where Q is a holomorphic vector bundle on X, such that the filtration of $J^n(Q)$ obtained from (2.2) coincides with the filtration (2.16) for the subbundle $K_X^n \otimes Q \subset J^n(Q)$.

Note that for the filtration of the vector bundle $J^n(Q)$ obtained from (2.2), the graded object is $\oplus_{i=0}^n K_X^{\otimes i} \otimes Q$. Therefore, if this filtration coincides with the filtration obtained from (2.16), then each homomorphism S_i in (2.17) must be an isomorphism. Indeed, this is a consequence of the general fact that any homomorphism between holomorphic vector bundles of the same degree over X, which is an isomorphism outside finitely many points, must be an isomorphism.

A holomorphic vector bundle V over X is called *stable* if for every proper holomorphic subbundle $W \subset V$ the inequality

$$\mathrm{degree}(W)/\mathrm{rank}(W) < \mathrm{degree}(V)/\mathrm{rank}(V) \tag{3.8}$$

is valid [4, 5].

Definition 3.4. A coupled connection (∇, F) will be called *stable* if the vector bundle F is stable.

Note that since $K_X^{\otimes n} \otimes Q = F$, the vector bundle Q is stable if and only if F is stable.

Lemma 3.5. *Let Q be a stable vector bundle and ∇ a flat connection on $J^n(Q)$. Then ∇ is a coupled connection.*

Proof. Consider the filtration

$$0 := V_0 \subset V_1 := K_X^{\otimes n} \otimes Q \subset V_2 \subset V_3 \subset \cdots \subset V_n \subset V_{n+1} = J^n(Q) \tag{3.9}$$

obtained from the jet sequence (2.2). In other words, V_i is the kernel of the projection $J^n(Q) \to J^{n-i}(Q)$ in (2.2). The quotient V_{i+1}/V_i is $K_X^{n-i} \otimes Q$, which is a stable vector bundle of degree $(g-1)(n-2i)r$, where $g = \text{genus}(X)$ and $r = \text{rank}(Q)$. The degree is computed from the fact that $\text{degree}(J^n(Q)) = 0$ since it admits a flat connection. From this, it follows that there is no nonzero homomorphism from V_i/V_{i-1} to $K_X \otimes (V/V_{i+1})$. Indeed, it is easy to see that $H^0(X, \text{Hom}(W_1, W_2)) = 0$, where W_1 and W_2 are stable vector bundles with $\text{degree}(W_1)/\text{rank}(W_1) > \text{degree}(W_2)/\text{rank}(W_2)$. Consequently, if $i \le n-1$, then

$$H^0(X, \text{Hom}(V_i/V_{i-1}, K_X \otimes (V_{n+1}/V_n))) = 0 \tag{3.10}$$

as $\text{degree}(K_X) > 0$. Using the same argument, it also follows that if

$$H^0(X, \text{Hom}(V_i/V_{i-1}, K_X \otimes (V_{n+1}/V_{j+1}))) = 0, \tag{3.11}$$

for some $j \ge i+1$, then $H^0(X, \text{Hom}(V_i/V_{i-1}, K_X \otimes (V_{n+1}/V_j))) = 0$. Therefore, using induction, it follows that

$$H^0(X, \text{Hom}(V_i/V_{i-1}, K_X \otimes (J^n(Q)/V_{i+1}))) = 0. \tag{3.12}$$

For a subbundle W of $J^n(Q)$, let $S(W) : W \to K_X \otimes (J^n(Q)/W)$ denote the second fundamental form of W for the connection ∇. For a subbundle W_1 of W, clearly, there is commutative diagram

$$\begin{array}{ccc} W_1 & \hookrightarrow & W \\ \downarrow{\scriptstyle S(W_1)} & & \downarrow{\scriptstyle S(W)} \\ K_X \otimes (J^n(Q)/W_1) & \xrightarrow{f} & K_X \otimes (J^n(Q)/W) \end{array} \tag{3.13}$$

of vector bundles, where f is the tensor product of the identity automorphism of K_X with the obvious quotient map $J^n(Q)/W_1 \to J^n(Q)/W$. Using this and (3.12) it follows that

$$S(V_i) \subset K_X \otimes (V_{i+1}/V_i) \subset K_X \otimes (J^n(Q)/V_i). \tag{3.14}$$

Indeed, (3.14) is immediate for $i = 1$. It is straightforward to deduce (3.14) using induction on i.

Let

$$\overline{S}(V_i) : V_i/V_{i-1} \longrightarrow K_X \otimes (V_{i+1}/V_i) \tag{3.15}$$

be the homomorphism induced by $S(V_i)$ in (3.14). Since V_i/V_{i-1} and $K_X \otimes (V_{i+1}/V_i)$ are both stable vector bundles of the same degree/rank quotient, the homomorphism $\overline{S}(V_i)$ must be either an isomorphism or the zero homomorphism. In view of (3.14), it is obvious that if $\overline{S}(V_i) = 0$, then V_i must be a flat subbundle of $J^n(Q)$. But $\text{degree}(V_i) \ne 0$ if $i \in [1, n]$, and hence such a

V_i does not admit a flat connection. Therefore, $\overline{S}(V_i)$ is an isomorphism. This completes the proof of the lemma. □

The following lemma complements Lemma 3.5.

Lemma 3.6. *Let Q be a stable vector bundle over X of rank r and* degree$(1-g)nr$. *Then $J^n(Q)$ admits a flat connection.*

Proof. A theorem of Weil [6] and Atiyah [1] says that a holomorphic vector bundle V over a compact Riemann surface admits a flat connection if and only if every direct summand of it is of degree zero. A holomorphic subbundle W of V is called a direct summand if there is another subbundle W' of V such that $W \oplus W'$ is isomorphic to V. Therefore, every irreducible, that is nondecomposable, holomorphic vector bundle of degree zero admits a flat connection. Consequently, it suffices to show that $J^n(Q)$ is irreducible.

Consider the filtration of $J^n(Q)$ in (3.9) obtained from jet sequence. We noted that the quotient V_{i+1}/V_i is a stable vector bundle of degree $(g-1)(n-2i)r$. Therefore, the filtration (3.9) is the Harder-Narasimhan filtration of $J^n(Q)$. This implies that any automorphism of $J^n(Q)$ preserves the filtration (3.9). See [4] for properties of Harder-Narasimhan filtration.

Assume that $J^n(Q) = W \oplus W'$. Take two nonzero numbers $c, c' \in \mathbb{C}^*$ with $c \neq c'$. Let T denote the automorphism $c\,\mathrm{Id}_W \oplus c'\,\mathrm{Id}_{W'}$ of $J^n(Q)$ that acts on W as multiplication by c and acts on W' as multiplication by c'.

Since T preserves the filtration (3.9) and V_1 is simple, we conclude that V_1 is contained in either W or W'. Say that W contains V_1. So $J^n(Q)/V_1 = (W/V_1) \oplus W'$, and T gives an automorphism of $J^n(Q)/V_1$. This automorphism of $J^n(Q)/V_1$ will be denoted by T_1.

Since

$$V_2/V_1 \subset V_3/V_1 \subset \cdots \subset V_n/V_1 \subset V_{n+1}/V_1 = J^n(Q)/V_1 \tag{3.16}$$

is the Harder-Narasimhan filtration of $J^n(Q)/V_1$, and V_2/V_1 is simple, it follows that V_2/V_1 is contained in either W/V_1 or W'. If V_2/V_1 is contained in W', then the subbundle $V_2 \cap W'$ of V_2 gives a splitting of the exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_2/V_1 \longrightarrow 0. \tag{3.17}$$

But each exact sequence

$$0 \longrightarrow V_i/V_{i-1} \longrightarrow V_{i+1}/V_{i-1} \longrightarrow V_{i+1}/V_i \longrightarrow 0 \tag{3.18}$$

is nonsplit, where $i \in [1,n]$. Indeed, the corresponding extension class in

$$H^1(X, \mathrm{Hom}(V_{i+1}/V_i, V_i/V_{i-1})) \quad (\supset H^1(X, K_X)) \tag{3.19}$$

coincides with the element $1 \in H^1(X, K_X)$. Recall that $\mathrm{Hom}(V_{i+1}/V_i, V_i/V_{i-1}) = K_X \otimes \mathrm{End}(V_{i+1}/V_i)$.

Therefore, we have $V_2 \subset W$. Now, clearly an inductive argument is valid. More precisely, if $V_i \subset W$ and V_{i+1} is not contained in W, then $V_{i+1} \cap W'$ gives a splitting of the exact sequence (3.18). Using induction, we conclude that $W = V_{n+1}$. In other words, $J^n(Q)$ is indecomposable. This completes the proof of the lemma. □

Isomorphism classes stable vector bundles of rank r form an algebraic variety of dimension $r^2(g-1)+1$ [5]. Consequently, Lemma 3.6 combines with Lemma 3.5 to produce examples of coupled connections.

4. Coupled connections from differential operator. Let Q be a holomorphic vector bundle over X of rank r and let

$$D \in H^0(X, \mathrm{Diff}^{n+1}_X(Q, K_X^{n+1} \otimes Q)) \tag{4.1}$$

be a differential operator whose symbol is the identity automorphism of Q. The condition on the symbol of D ensures that the homomorphism $J^{n+1}(Q) \to K_X^{n+1} \otimes Q$ defining D gives a splitting of the jet sequence

$$0 \longrightarrow K_X^{n+1} \otimes Q \longrightarrow J^{n+1}(Q) \longrightarrow J^n(Q) \longrightarrow 0. \tag{4.2}$$

Let

$$\gamma : J^n(Q) \longrightarrow J^{n+1}(Q) \tag{4.3}$$

be the homomorphism corresponding to this splitting.

Consider the commutative diagram of vector bundles

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_X^{n+1} \otimes Q & \longrightarrow & J^{n+1}(Q) & \longrightarrow & J^n(Q) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \tau & & \| & & \\ 0 & \longrightarrow & K_X \otimes J^n(Q) & \longrightarrow & J^1(J^n(Q)) & \longrightarrow & J^n(Q) & \longrightarrow & 0, \end{array} \tag{4.4}$$

where τ is defined in (2.6) and the horizontal exact sequences are as in (2.2). The homomorphism

$$\tau \circ \gamma : J^n(Q) \longrightarrow J^1(J^n(Q)) \tag{4.5}$$

is clearly a splitting of the bottom exact sequence in (4.4), where γ is defined in (4.3).

A splitting of the exact sequence

$$0 \longrightarrow K_X \otimes V \longrightarrow J^1(V) \longrightarrow V \longrightarrow 0 \tag{4.6}$$

in (2.2) defines a holomorphic connection on V [1]. More precisely, the space of flat connections on V is in bijective correspondence with the space of splittings of this exact sequence. As we noted earlier, any holomorphic connection on a Riemann surface is automatically flat since there are no nonzero holomorphic 2-forms on it. We recall that given a flat connection on V, the corresponding splitting homomorphism $V \to J^1(V)$ of the exact sequence sends any vector $v \in V_x$ in the fiber to the image, in $J^1(V)$, of the (unique) flat section s of V, defined around x, with $s(x) = v$.

Therefore, the homomorphism $\tau \circ \gamma$ in (4.5) defines a flat connection on $J^n(Q)$. This connection will be denoted by $\overline{\nabla}$.

Given a differential operator D_1 on X whose symbol homomorphism is an isomorphism, the sheaf of solutions of D_1 defines a local system on X. More precisely, if D_1 is of order m acting on a vector bundle of rank l, then the corresponding local system is of rank ml. It can be checked that the local system defined by the operator D in (4.1) coincides with the local system defined by the above flat connection $\overline{\nabla}$.

LEMMA 4.1. *The connection $\overline{\nabla}$ on $J^n(Q)$ is a coupled connection.*

PROOF. To prove the lemma, we first give a description of the second fundamental form.

Let V be a vector bundle over X and let W be its subbundle. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & W & & \\
 & & & & & & \downarrow{\scriptstyle \iota} & & \\
 & & & & J^1(V) & \xrightarrow{p} & V & & \\
 & & & & \downarrow{\scriptstyle j} & & \downarrow & & \\
0 & \longrightarrow & K_X \otimes (V/W) & \longrightarrow & J^1(V/W) & \xrightarrow{q} & V/W & \longrightarrow & 0 \\
 & & & & \downarrow & & \| & & \\
 & & & & V/W & = & V/W, & &
\end{array} \tag{4.7}$$

where ι is the inclusion map and all other maps are as in (2.2). Now let ∇ be a holomorphic connection on V. So, ∇ defines a homomorphism

$$h : V \longrightarrow J^1(V) \tag{4.8}$$

such that $p \circ h$ is the identity automorphism of V.

Consider the homomorphism $j\circ h\circ\iota : W \to J^1(V/W)$. The commutativity of the diagram implies that $q\circ j\circ h\circ\iota = 0$. Consequently, $j\circ h\circ\iota$ gives a homomorphism

$$S(W) : W \longrightarrow K_X\otimes (V/W). \tag{4.9}$$

It is easy to check that this homomorphism coincides with the second fundamental form of W for the connection ∇.

For any $i\in[0,n-1]$, let W_i denote the kernel of the homomorphism $J^n(Q)\to J^i(Q)$ in (2.2). We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & W_i & & \\
 & & & & & & \downarrow{\scriptstyle \iota} & & \\
 & & & & J^{n+1}(Q) & \xrightarrow{p} & J^n(Q) & & \\
 & & & & \downarrow{\scriptstyle j} & & \downarrow & & \\
0 & \longrightarrow & K_X^{i+1}\otimes Q & \longrightarrow & J^{i+1}(Q) & \xrightarrow{q} & J^i(Q) & \longrightarrow & 0 \\
 & & & & \downarrow & & \| & & \\
 & & & & J^i(Q) & = & J^i(Q), & &
\end{array} \tag{4.10}$$

where ι is the inclusion map and all other maps are as in (2.2). Let $h : J^n(Q)\to J^{n+1}(Q)$ be a splitting of p, that is, $p\circ h$ is the identity automorphism of $J^n(Q)$. Just as before, the homomorphism $j\circ h\circ\iota$ gives a homomorphism

$$S_i : W_i \longrightarrow K_X^{i+1}\otimes Q \tag{4.11}$$

since $q\circ j\circ h\circ\iota = 0$. On the other hand, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_i & \longrightarrow & J^n(Q) & \longrightarrow & J^i(Q) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\
0 & \longrightarrow & K_X^{i+1}\otimes Q & \longrightarrow & J^{i+1}(Q) & \longrightarrow & J^i(Q) & \longrightarrow & 0.
\end{array} \tag{4.12}$$

It is straightforward to check that the above homomorphism f coincides with S_i constructed in (4.11).

Since the connection $\overline{\nabla}$ is defined by $\tau\circ\gamma$, where γ is the splitting of the homomorphism p in (4.10), from the earlier description of the second fundamental form it follows that the second fundamental form of the subbundle $W_i\subset J^n(Q)$, for the connection $\overline{\nabla}$, coincides with the homomorphism S_i in

(4.11) constructed by substituting h with γ. But S_i coincides with f. Therefore, the filtration (2.16) constructed from the subbundle $W_{n-1} = K_X^n \otimes Q \subset J^n(Q)$ for the connection $\overline{\nabla}$ coincides with the filtration (3.9) constructed from (2.2). This completes the proof of the lemma. □

In Section 3, given a coupled connection, we constructed a differential operator (see (3.7)). Let

$$\overline{D} \in H^0(X, \mathrm{Diff}_X^{n+1}(Q, K_X^{n+1} \otimes Q)) \tag{4.13}$$

be the differential operator for the coupled connection $(J^n(Q), \overline{\nabla})$ constructed in Lemma 4.1 from the operator D in (4.1).

Proposition 4.2. *The differential operator $\overline{D}$ coincides with the operator D in (4.1).*

Proof. It is straightforward to check that the homomorphism γ in (4.3) coincides with the homomorphism $\phi_{n+1} \circ \phi_n^{-1}$ in (3.5) for the coupled connection $(J^n(Q), \overline{\nabla})$. The proposition follows immediately from this observation. □

Let

$$D' \in H^0(X, \mathrm{Diff}_X^{n+1}(Q', K_X^{n+1} \otimes Q')) \tag{4.14}$$

be a differential operator with the identity automorphism of Q' as its symbol (the same condition for D in (4.1)). The two operators D and D' will be called *equivalent* if there is an isomorphism

$$\rho : Q \longrightarrow Q' \tag{4.15}$$

which intertwines D and D'. In other words, $(\mathrm{Id} \otimes \rho) \circ D = D' \circ \rho$, where $\mathrm{Id} \otimes \rho$ is the automorphism of $K_X^{n+1} \otimes Q$ defined by ρ and the identity automorphism of K_X^{n+1}.

Let $\mathcal{B}(r, n)$ denote the space of all equivalence classes of differential operators

$$D \in H^0(X, \mathrm{Diff}_X^{n+1}(Q, K_X^{n+1} \otimes Q)), \tag{4.16}$$

where Q is some holomorphic vector bundle of rank r and the symbol of D is the identity automorphism of Q.

If $D_1 \in H^0(X, \mathrm{Diff}_X^{n+1}(Q, Q_1))$ is a differential operator whose symbol is an isomorphism, then Q_1 is isomorphic to $K_X^{n+1} \otimes Q$. Therefore, $\mathcal{B}(r, n)$ coincides with the space of equivalence classes of differential operators of order $n+1$, between two vector bundles of rank r, such that the symbol is an isomorphism.

If D and D' are two equivalent differential operators from Q to $K_X^{n+1} \otimes Q$, then the automorphism ρ of Q that intertwines D and D' induces an automorphism of $J^n(Q)$. Let ρ' denote the automorphism of $J^n(Q)$. Let $\overline{\nabla}$ (resp., $\overline{\nabla}'$)

be the coupled connection constructed from D (resp., D') in Lemma 4.1. It is easy to see that the automorphism ρ' commutes with the connections $\overline{\nabla}$ and $\overline{\nabla}'$.

Let E and E' be two holomorphic vector bundles over X of rank $(n+1)r$. Let (∇, F) and (∇', F') be two coupled connections of rank r on E and E', respectively. The two coupled connections will be called *equivalent* if there is an automorphism

$$T : E \longrightarrow E' \tag{4.17}$$

such that $T(F) = F'$ and T takes ∇ to ∇', that is, $\nabla' \circ T = T \circ \nabla$.

Since T takes F to F' and ∇ to ∇', it follows immediately that T takes the filtration (2.16) for F to the corresponding filtration for F'. In particular, T gives an isomorphism of the final quotient $Q = E/F_n$ with the corresponding quotient of E'.

Let $\mathcal{A}(r,n)$ denote the space of all equivalence classes of coupled connections of rank r on some holomorphic vector bundle of rank $(n+1)r$ over X.

From the above comment on the construction in Lemma 4.1 it follows that we have a map

$$\Phi : \mathcal{B}(r,n) \longrightarrow \mathcal{A}(r,n) \tag{4.18}$$

that sends an operator D to the coupled connection $\overline{\nabla}$ constructed in Lemma 4.1 from D.

Let the two coupled connections (∇, F) on E and (∇', F') on E', as above, be equivalent. As before, Q and Q' denote the final quotient F_{n+1}/F_n for (∇, F) and (∇', F'), respectively. Consider the isomorphism

$$T' : Q \longrightarrow Q' \tag{4.19}$$

induced by T in (4.17). Let

$$T_j : J^j(Q) \longrightarrow J^j(Q') \tag{4.20}$$

denote the isomorphism induced by T'. The diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi_j} & J^j(Q) \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T_j} \\ E' & \xrightarrow{\phi'_j} & J^j(Q') \end{array} \tag{4.21}$$

is commutative, where ϕ_j and ϕ'_j are constructed in (3.3).

From the commutativity of (4.21) it is immediate that the differential operators D and D' constructed, as in (3.7), from (∇, F) and (∇', F'), respectively, are equivalent. More precisely, the isomorphism T' takes D to D'.

Therefore, we have a map

$$\Psi : \mathcal{A}(r,n) \longrightarrow \mathcal{B}(r,n) \tag{4.22}$$

that sends a coupled connection (∇, F) to the differential operator D constructed in (3.7).

We now have the following corollary of Proposition 4.2.

Corollary 4.3. *The composition $\Psi \circ \Phi$ is the identity map of $\mathcal{B}(r,n)$, where Φ and Ψ are constructed in (4.18) and (4.22), respectively. In particular, Ψ is surjective and Φ is injective.*

The following lemma complements Corollary 4.3.

Lemma 4.4. *The composition $\Phi \circ \Psi$ is the identity map of $\mathcal{A}(r,n)$.*

Proof. Let $\alpha := (\nabla, F)$ be a coupled connection on E and $\beta := (\nabla', F')$ a coupled connection on E' such that the differential operator $\Psi(\alpha)$ is equivalent to $\Psi(\beta)$.

Let Q (resp., Q') denote the final quotient F_{n+1}/F_n in (2.16) for α (resp., β). Since $\Psi(\alpha)$ and $\Psi(\beta)$ are equivalent, there is an isomorphism

$$A : Q \longrightarrow Q' \tag{4.23}$$

that takes $\Psi(\alpha)$ to $\Psi(\beta)$. In other words, $(\mathrm{Id} \otimes A) \circ \Psi(\alpha) = \Psi(\beta) \circ A$, where Id denotes the identity automorphism of K_X^{n+1}. Let

$$\overline{A} : J^n(Q) \longrightarrow J^n(Q') \tag{4.24}$$

be the isomorphism induced by A.

Let ϕ_n (resp., ϕ'_n) denote the isomorphism from E (resp., E') to $J^n(Q)$ (resp., $J^n(Q')$) constructed in Lemma 3.2. Finally, consider the isomorphism

$$T := (\phi'_n)^{-1} \circ \overline{A} \circ \phi_n : E \longrightarrow E'. \tag{4.25}$$

The isomorphism ϕ_n takes the filtration on E obtained from (2.16) to the filtration (3.9) of $J^n(Q)$ and the same for ϕ'_n. Furthermore, $\overline{A}$ evidently takes the filtration (3.9) of $J^n(Q)$ to the corresponding filtration of $J^n(Q')$. Therefore, the isomorphism T in (4.25) is compatible with the filtrations. It is easy to check that T takes the connection ∇ to the connection ∇'. This completes the proof of the lemma. □

Corollary 4.3 and Lemma 4.4 combine together to give the following theorem.

Theorem 4.5. *The two maps Φ and Ψ are bijective. Moreover, they are inverses of each other.*

Let $\mathcal{A}'(r,n) \subset \mathcal{A}(r,n)$ be the subset defined by stable coupled connections (see Definition 3.4). Similarly, let $\mathcal{B}'(r,n) \subset \mathcal{B}(r,n)$ denote the subset consisting of all operators defined on a stable vector bundle.

Theorem 4.5 has the following corollary.

Corollary 4.6. *Each of the two maps Φ and Ψ defines a bijection between $\mathcal{A}'(r,n)$ and $\mathcal{B}'(r,n)$. Furthermore, they are inverses of each other.*

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