

## KKM THEOREM WITH APPLICATIONS TO LOWER AND UPPER BOUNDS EQUILIBRIUM PROBLEM IN $G$ -CONVEX SPACES

M. FAKHAR and J. ZAFARANI

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We give some new versions of KKM theorem for generalized convex spaces. As an application, we answer a question posed by Isac et al. (1999) for the lower and upper bounds equilibrium problem.

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**1. Introduction.** In [5], Isac et al. raised the following open problem which is closely related to the equilibrium problem. Given a closed nonempty subset  $K$  in a locally convex semireflexive topological space, a mapping  $f : K \times K \rightarrow \mathbb{R}$ , and two real numbers  $\alpha, \beta$ , where  $\alpha \leq \beta$ , it is interesting to know under what conditions there exists an  $\bar{x} \in K$  such that

$$\alpha \leq f(\bar{x}, y) \leq \beta, \quad \forall y \in K. \quad (1.1)$$

First, Li [8] gave some answers to the open problem (1.1) by introducing and using the concept of extremal subsets. Then Chadli et al. [1] gave some answers to this open problem by a method different from that Li used. Our goal in this paper is to derive some more results in answering this problem in  $G$ -convex spaces. In fact, we will derive some results of problem (1.1) for bifunctions that are defined on  $X \times X$ , for which  $X$  is a  $G$ -convex space.

Let  $X$  be nonempty set. We denote by  $2^X$  the family of all subsets of  $X$ , by  $\mathcal{F}(X)$  the family of all nonempty finite subsets of  $X$ , and by  $|A|$  the cardinality of  $A \in \mathcal{F}(X)$ .

Let  $Y$  be a nonempty set and let  $X$  be a topological space. If  $F : Y \rightarrow 2^X$  is a multivalued map, then we say that  $F$  is transfer closed-valued if, for any  $(y, x) \in Y \times X$  with  $x \notin F(y)$ , there exists  $y' \in Y$  such that  $x \notin \text{cl}F(y')$ ; see Tian [14]. It is clear that this definition is equivalent to saying that  $\bigcap_{y \in Y} F(y) = \bigcap_{y \in Y} \text{cl}F(y)$ . If  $B \subseteq Y$  and  $A \subseteq X$ , then we say that  $F : B \rightarrow 2^A$  is transfer closed-valued if the multivalued map  $y \rightarrow F(y) \cap A$  is transfer closed-valued. In the case when  $X = Y$  and  $A = B$ , we say that  $F$  is transfer closed-valued on  $A$ .

Let  $f$  be a bifunction on  $X \times Y$ , then  $f$  is called  $\lambda$ -transfer lower semicontinuous (l.s.c.) on the first variable on  $X$  if, for each  $(x, y) \in X \times Y$  with  $f(x, y) > \lambda$ , there exist  $y' \in Y$  and a neighborhood  $U(x)$  of  $x$  in  $X$  such that  $f(z, y') > \lambda$  for

all  $z \in U(x)$ . The bifunction  $f$  is said to be  $\lambda$ -transfer upper semicontinuous (u.s.c.) on the first variable on  $X$  if  $-f$  is  $\lambda$ -transfer l.s.c. on the first variable. If  $f$  is defined on  $Y \times X$ , then  $\lambda$ -transfer l.s.c. (u.s.c.) bifunction on second variable on  $X$  is defined by a similar method. It is easily seen that an l.s.c. (u.s.c.) bifunction is  $\lambda$ -transfer l.s.c. (u.s.c.) bifunction for each  $\lambda$ .

A generalized convex space or  $G$ -convex space was first introduced by Park and Kim [12], and more recently, it has been generalized by Park [10]. A  $G$ -convex space  $(X, D; \Gamma)$  consists of a topological space  $X$ , a nonempty set  $D$ , and a multivalued map  $\Gamma : \mathcal{F}(D) \rightarrow 2^X \setminus \{\emptyset\}$  such that, for each  $A \in \mathcal{F}(D)$  with the cardinality  $|A| = n + 1$ , there exists a continuous function  $\Phi_A : \Delta_n \rightarrow \Gamma(A)$  such that each  $J \in \mathcal{F}(A)$  implies  $\Phi_A(\Delta_J) \subset \Gamma(J)$ , for which if  $A = \{a_0, a_1, \dots, a_n\}$  and  $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_j}\}$ , then  $\Delta_J = \text{co}\{e_{i_0}, \dots, e_{i_j}\}$ . When  $D = X$ , we will write  $(X; \Gamma)$  in place of  $(X, X; \Gamma)$ . If  $(X, D; \Gamma)$  is a  $G$ -convex space,  $D \subseteq X$ , and  $K \subset X$ , then  $K$  is  $G$ -convex if for each  $A \in \mathcal{F}(D)$ ,  $A \subset K$  implies  $\Gamma(A) \subset K$ . The  $G$ -convex hull of  $K$  denoted by  $G\text{-co}K$  is the set  $\bigcap \{B \subset X : B \text{ is a } G\text{-convex subset of } X \text{ containing } K\}$ .

Notice that  $G$ -convex spaces contain most of the well-know spaces such as topological vector spaces, convex spaces, generalized  $H$ -spaces,  $L$ -spaces,  $C$ -spaces, and hyperconvex metric spaces (see [10, 11, 12, 13] and the references therein).

Let  $(X, D; \Gamma)$  be a  $G$ -convex space, then the multivalued mapping  $F : D \rightarrow 2^X$  is called a KKM map if, for each finite subset  $A$  of  $D$ , we have  $\Gamma(A) \subseteq F(A)$ ; see Park and Lee [13]. If  $x \mapsto \text{cl}F(x)$  is a KKM map, then we say that  $\text{cl}F$  is a KKM map.

**2. Main results.** The KKM theorem is a very important tool in the study of the equilibrium problem. To solve problem (1.1) on  $G$ -convex spaces, we first give some refined versions of the KKM theorem. The following KKM theorem, due to Park and Lee [13, Theorem 1], is essential for obtaining our main results.

**THEOREM 2.1.** *Let  $(X, D; \Gamma)$  be a  $G$ -convex space and let  $F : D \rightarrow 2^X$  be a multimap such that*

- (1)  *$F$  has closed (resp., open) values,*
- (2)  *$F$  is a KKM map.*

*Then  $\{F(z) : z \in D\}$  has the finite intersection property. More precisely, for each  $N \in \mathcal{F}(D)$ ,  $\Gamma(N) \cap (\bigcap_{z \in N} F(z) \neq \emptyset)$ . Further, if*

- (3)  *$\bigcap_{z \in M} \text{cl}F(z)$  is compact for some  $M \in \mathcal{F}(D)$ , then  $\bigcap_{z \in D} \text{cl}F(z) \neq \emptyset$ .*

As a consequence of the above theorem, we obtain the following result which is a refinement of [3, Theorem 1.1] and [7, Theorem 3.3].

**THEOREM 2.2.** *Let  $(X, D; \Gamma)$  be a  $G$ -convex space such that, for each  $A, B \in \mathcal{F}(D)$  with  $A \subseteq B$ ,  $\Gamma(A) \subseteq \Gamma(B)$ . Suppose that  $F : D \rightarrow 2^X \setminus \{\emptyset\}$  and  $G : D \rightarrow 2^X \setminus \{\emptyset\}$  are two multivalued maps such that*

- (1)  *$F(x) \subseteq G(x)$  for all  $x \in D$ ,*
- (2)  *$F$  is a KKM map,*

- (3) for some  $M \in \mathcal{F}(D)$ ,  $\bigcap_{x \in M} \text{cl} F(x)$  is compact,  
 (4) for each  $A \in \mathcal{F}(D)$  with  $M \subseteq A$ ,  $G : A \rightarrow 2^{\Gamma(A)}$  is transfer closed-valued,  
 (5) for each  $A \in \mathcal{F}(D)$  with  $M \subseteq A$ ,

$$\text{cl} \left( \bigcap_{x \in A} G(x) \right) = \bigcap_{x \in A} G(x). \quad (2.1)$$

Then  $\bigcap_{x \in D} G(x) \neq \emptyset$ .

**PROOF.** Let  $A \in \mathcal{F}(D)$  with  $M \subseteq A$ . Consider a multivalued map  $F_A : A \rightarrow 2^{\Gamma(A) \setminus \{\emptyset\}}$  defined by  $F_A(x) := \text{cl}_{\Gamma(A)}(F(x) \cap \Gamma(A))$  for all  $x \in A$ . Then  $F_A(x)$  is closed in  $\Gamma(A)$ . Also  $F_A$  is a KKM map. In fact, if  $B \in \mathcal{F}(A)$ , then  $\Gamma(B) \subseteq \Gamma(A)$  and  $\Gamma(B) \subseteq \bigcup_{x \in B} F(x)$ , thus  $\Gamma(B) \subseteq (\bigcup_{x \in B} F(x)) \cap \Gamma(A) \subseteq \bigcup_{x \in B} F_A(x)$ . So, by [Theorem 2.1](#), we have

$$\bigcap_{x \in A} F_A(x) \neq \emptyset. \quad (2.2)$$

Let  $\{A_i : i \in I\}$  be the family of all finite subsets of  $D$  containing the set  $M$ , partially ordered by  $\subseteq$ . Now, for each  $i \in I$ , let  $X_i = \Gamma(A_i)$ . By (2.2),

$$\bigcap_{x \in A_i} \text{cl}_{X_i}(F(x) \cap X_i) \neq \emptyset, \quad \text{for each } i \in I. \quad (2.3)$$

Take any  $x_i \in \bigcap_{x \in A_i} \text{cl}_{X_i}(F(x) \cap X_i)$ . For each  $i \in I$ , let  $Y_i = \{x_j : j \geq i, j \in I\}$ . Clearly, we have that  $\{Y_i : i \in I\}$  has finite intersection property, and  $Y_i \subseteq \bigcap_{x \in M} \text{cl} F(x)$ , for all  $i \in I$ . Hence, by condition (3),  $\text{cl} Y_i$  is compact. Therefore  $\bigcap_{i \in I} \text{cl} Y_i \neq \emptyset$ . Choose any  $\tilde{x} \in \bigcap_{i \in I} \text{cl} Y_i$ . Also, for any  $i, j \in I$  with  $j \geq i$ , we have

$$\begin{aligned} x_j \in \bigcap_{x \in A_j} \text{cl}_{X_j}(F(x) \cap X_j) &\subseteq \bigcap_{x \in A_j} \text{cl}_{X_j}(G(x) \cap X_j) \\ &= \bigcap_{x \in A_j} (G(x) \cap X_j) \subseteq \bigcap_{x \in A_i} (G(x) \cap X_j) \\ &\subseteq \bigcap_{x \in A_i} G(x). \end{aligned} \quad (2.4)$$

Therefore,  $Y_i \subseteq \bigcap_{x \in A_i} G(x)$ . Now, for any  $x \in D$ , there exists  $i_0 \in I$  such that  $x \in A_{i_0}$ . It follows that

$$\tilde{x} \in \text{cl} Y_{i_0} \subseteq \text{cl} \left( \bigcap_{z \in A_{i_0}} G(z) \right) = \bigcap_{z \in A_{i_0}} G(z) \subseteq G(x). \quad (2.5)$$

Then  $\tilde{x} \in G(x)$  for all  $x \in X$ , and the proof is completed.  $\square$

By [Theorem 2.1](#) and the fact that  $\bigcap_{x \in D} G(x) = \bigcap_{x \in D} \text{cl} G(x)$ , when  $G$  is transfer closed-valued, we can obtain the following result.

**THEOREM 2.3.** Let  $(X, D; \Gamma)$  be a  $G$ -convex space. Suppose that  $F : D \rightarrow 2^X \setminus \{\emptyset\}$  and  $G : D \rightarrow 2^X \setminus \{\emptyset\}$  are two multivalued maps such that

- (1)  $F(x) \subseteq G(x)$  for all  $x \in D$ ,
- (2)  $\text{cl}F$  is a KKM map,
- (3) for some  $M \in \mathcal{F}(D)$ ,  $\bigcap_{x \in M} \text{cl}F(x)$  is compact,
- (4)  $G$  is transfer closed-valued.

Then  $\bigcap_{x \in D} G(x) \neq \emptyset$ .

The following examples show that Theorems 2.2 and 2.3 are different.

**EXAMPLE 2.4.** Assume that  $X = \mathbb{R}$  and  $D = \mathbb{N}$ . If we define  $\Gamma(A) = \text{co}(A + 1)$  for every  $A \in \mathcal{F}(D)$ , then  $(X, D; \Gamma)$  is a  $G$ -convex space and  $\Gamma(A) \neq G\text{-co}A$ . Suppose that  $F : D \rightarrow 2^X$  is defined as

$$F(x) = \begin{cases} \{1, 2\} \cup ((-\infty, 0) \cap \mathbb{Q}) & \text{if } x = 1, \\ (1, +\infty) & \text{if } x = 2, \\ \mathbb{R} & \text{if } x \neq 1, 2. \end{cases} \quad (2.6)$$

By taking  $M = \{1, 2\}$  and  $F = G$ , all the conditions of Theorem 2.2 are satisfied and  $\bigcap_{x \in D} F(x) = \{2\}$ , but  $\bigcap_{x \in D} \text{cl}F(x) = \{1, 2\}$ . Therefore,  $F$  is not transfer closed-valued and so we cannot apply Theorem 2.3.

The following example is a modified form of [14, Example 1].

**EXAMPLE 2.5.** If  $X = [0, 1]$ ,  $D = \mathbb{Q} \cap X$ , and  $\Gamma(A) = [\min A, 1]$ , for every  $A \in \mathcal{F}(D)$ , then  $(X, D; \Gamma)$  is a  $G$ -convex space. Suppose that  $F : D \rightarrow 2^X$  is defined by  $F(x) = [x, 1] \cap \mathbb{Q}$ . If  $F = G$ , then all the conditions of Theorem 2.3 are satisfied. But  $F$  is not KKM map and moreover for  $A = \{0, 0.5\}$ , conditions (4) and (5) are not satisfied.

By a method similar to that of the proof of Theorem 2.2, we can obtain the following result which is an improvement of [2, Lemma 2] and [6, Lemma 3.1] on  $G$ -convex spaces.

**THEOREM 2.6.** Let  $(X; \Gamma)$  be a  $G$ -convex space and let  $G\text{-co}A$  be closed for each  $A \in \mathcal{F}(X)$ . Suppose that  $F : X \rightarrow 2^X \setminus \{\emptyset\}$  and  $G : X \rightarrow 2^X \setminus \{\emptyset\}$  are two multivalued maps such that

- (1)  $F(x) \subseteq G(x)$  for all  $x \in X$ ,
- (2)  $F$  is a KKM map,
- (3) for some  $M \in \mathcal{F}(X)$ ,  $\bigcap_{x \in M} \text{cl}F(x)$  is compact,
- (4) for each  $A \in \mathcal{F}(X)$  with  $M \subseteq A$ ,  $G$  is transfer closed-valued on  $G\text{-co}A$ ,
- (5) for each  $A \in \mathcal{F}(X)$  with  $M \subseteq A$ ,

$$\text{cl} \left( \bigcap_{x \in G\text{-co}A} G(x) \right) \cap G\text{-co}A = \left( \bigcap_{x \in G\text{-co}A} G(x) \right) \cap G\text{-co}A. \quad (2.7)$$

Then  $\bigcap_{x \in X} G(x) \neq \emptyset$ .

**REMARK 2.7.** (a) If, in [Theorem 2.3](#),  $X$  is Hausdorff and  $X = D$ , then condition (3) can be replaced by the following condition:

(3') there exists a compact subset  $K$  of  $X$  such that, for each  $N \in \mathcal{F}(X)$ , there exists a nonempty compact  $G$ -convex subset  $L_N$  of  $X$  such that

$$\bigcap_{x \in L_N} \text{cl} F(x) \subseteq K.$$

(b) If, in [Theorem 2.6](#), for each  $A \in \mathcal{F}(X)$ ,  $G\text{-co} A$  is compact, then, instead of conditions (3) and (4) we can assume that

(3') there exists  $M \in \mathcal{F}(X)$  such that  $\text{cl}(\bigcap_{x \in M} F(x))$  is compact,

(4') for each  $A \in \mathcal{F}(X)$  with  $M \subseteq A$ ,  $F$  is transfer closed-valued on  $G\text{-co} A$ .

Then the conclusion of [Theorem 2.6](#) holds. In this case, we obtain a refinement of Lemma 2.3 of Ding and Tarafdar [4]. Also condition (3) of [Theorem 2.6](#) can be replaced by the following condition:

(3'') there exists  $M \in \mathcal{F}(X)$  such that  $\text{cl}(\bigcap_{x \in M} G(x))$  is compact.

(c) [Example 2.4](#) shows that, in general,  $\Gamma(A) \neq G\text{-co} A$ . Therefore, [Theorem 2.6](#) has its own applications.

Now, by [Theorem 2.2](#), we obtain the following result, which gives an answer to problem (1.1).

**THEOREM 2.8.** *Let  $(X, D; \Gamma)$  be a  $G$ -convex space such that for each  $A, B \in \mathcal{F}(D)$  with  $A \subseteq B$ ,  $\Gamma(A) \subseteq \Gamma(B)$ . Suppose that  $f$  and  $g$  are two real bifunctions defined on  $X \times D$  such that*

(1) *for each  $(x, y) \in X \times D$ , if  $\alpha \leq f(x, y) \leq \beta$ , then  $\alpha \leq g(x, y) \leq \beta$ ;*

(2) *for each  $A \in \mathcal{F}(D)$  and  $B \subseteq A$  with  $\emptyset \neq B \neq A$ , either*

(i)  *$\alpha \leq \inf_{x \in \Gamma(A)} \max_{y \in B} f(x, y)$  or*

(ii)  *$\sup_{x \in \Gamma(A)} \min_{y \in A \setminus B} f(x, y) \leq \beta$ .*

*For  $B = A$ , condition (i) holds, and for  $B = \emptyset$ , condition (ii) is satisfied;*

(3) *there exist a compact subset  $K$  of  $X$  and  $M \in \mathcal{F}(D)$  such that, for every  $x \in X \setminus K$ , there are a point  $y \in M$  and a neighborhood  $U(x)$  of  $x$  such that for any  $z \in U(x)$ ,  $f(z, y) < \alpha$  or  $f(z, y) > \beta$ ;*

(4) *for each  $A \in \mathcal{F}(D)$  with  $M \subseteq A$ ,  $g : \Gamma(A) \times A \rightarrow \mathbb{R}$  is  $\alpha$ -transfer u.s.c. and  $\beta$ -transfer l.s.c. on the first variable on  $\Gamma(A)$ ;*

(5) *for each  $A \in \mathcal{F}(D)$  with  $M \subseteq A$ ,  $x \in X$  and for each net  $(x_\lambda)$  in  $X$  converging to  $x$ , if  $\alpha \leq g(x_\lambda, y) \leq \beta$  for all  $y \in A$ , then  $\alpha \leq g(x, y) \leq \beta$ .*

*Then there exists  $\tilde{x} \in X$  such that  $\alpha \leq g(\tilde{x}, y) \leq \beta$  for all  $y \in D$ .*

**PROOF.** Assume that  $F, G : D \rightarrow 2^X$  are defined by

$$\begin{aligned} F(y) &= \{x \in X : \alpha \leq f(x, y) \leq \beta\}, \\ G(y) &= \{x \in X : \alpha \leq g(x, y) \leq \beta\}. \end{aligned} \tag{2.8}$$

By condition (1),  $F(y) \subseteq G(y)$  for all  $y \in D$ . Condition (2) implies that  $F$  is a KKM map, because if there exists  $A \in \mathcal{F}(D)$  such that  $\Gamma(A) \not\subseteq \bigcup_{y \in A} F(y)$ , then there is a point  $\hat{x} \in \Gamma(A)$  such that  $f(\hat{x}, y) < \alpha$  or  $f(\hat{x}, y) > \beta$ , for all  $y \in A$ . Let  $B = \{y \in A : f(\hat{x}, y) < \alpha\}$ , then  $B = A$  or  $\emptyset$ , or  $\emptyset \neq B \neq A$ . In the case when

$B = A$  or  $B = \emptyset$ , we have  $\max_{y \in A} f(\hat{x}, y) < \alpha$  or  $\min_{y \in A} f(\hat{x}, y) > \beta$ . If  $\emptyset \neq B \neq A$ , then  $\max_{y \in B} f(\hat{x}, y) < \alpha$  and  $\min_{y \in A \setminus B} f(\hat{x}, y) > \beta$  which contradicts condition (2). Also, by condition (3) we have  $\bigcap_{y \in M} \text{cl} F(y) \subseteq K$ . Now, we show that condition (4) implies that  $G : A \rightarrow 2^{\Gamma(A)}$  is transfer closed-valued for each  $A \in \mathcal{F}(D)$  with  $M \subseteq A$ . Let  $(x, y)$  be a point in  $\Gamma(A) \times A$  and  $x \notin \Gamma(A) \cap G(y)$ . Then  $g(x, y) < \alpha$  or  $g(x, y) > \beta$ . If  $g(x, y) < \alpha$ , then there exist  $y' \in A$  and a neighborhood  $U(x)$  of  $x$  in  $\Gamma(A)$  such that  $g(z, y') < \alpha$  for all  $z \in U(x)$ . Thus,  $x \notin \text{cl}_{\Gamma(A)}(\Gamma(A) \cap G(y'))$ . Similarly, we can prove the case when  $g(x, y) > \beta$ . Moreover if  $x \in \text{cl}(\bigcap_{y \in A} G(y))$ , then there exists a net  $(x_\lambda)$  in  $\bigcap_{y \in A} G(y)$  such that  $x_\lambda \rightarrow x$ . Therefore,  $\alpha \leq g(x_\lambda, y) \leq \beta$  for all  $y \in A$ , and by condition (5), we have  $\alpha \leq g(x, y) \leq \beta$ . Hence  $x \in \bigcap_{y \in A} G(y)$  and so, by [Theorem 2.2](#), we have  $\bigcap_{y \in D} G(y) \neq \emptyset$ .  $\square$

**REMARK 2.9.** (a) If in [Theorem 2.8](#) instead of condition (4) we assume the following condition:

(4')  $g$  is  $\alpha$ -transfer u.s.c. and  $\beta$ -transfer l.s.c. on the first variable on  $X$ , then, by [Theorem 2.3](#) and without condition (5), we can obtain another answer for problem (1.1). In the above case, if  $X = D$  and  $X$  is Hausdorff, then by [Remark 2.7\(a\)](#), condition (3) can be replaced by the following condition:

(3') there exists a compact subset  $K$  of  $X$  such that, for every  $N \in \mathcal{F}(X)$  there is a nonempty compact  $G$ -convex subset  $L_N$  of  $X$  such that for every  $x \in X \setminus K$ , there are a point  $y \in L_N$  and a neighborhood  $U(x)$  of  $x$  such that for any  $z \in U(x)$  we have  $f(z, y) < \alpha$  or  $f(z, y) > \beta$ .

(b) If in [Theorem 2.8](#)  $X = D$  and  $G\text{-co}A$  is compact for any  $A \in \mathcal{F}(X)$ , then we can conclude [Theorem 2.8](#) by replacing conditions (3), (4), and (5) by the following conditions:

- (3') there exist a compact subset  $K$  of  $X$  and  $M \in \mathcal{F}(X)$  such that, for every  $x \in X \setminus K$ , there is a point  $y \in M$  such that  $f(x, y) < \alpha$  or  $f(x, y) > \beta$ ;
- (4') for each  $A \in \mathcal{F}(X)$  with  $M \subseteq A$ ,  $f : G\text{-co}A \times G\text{-co}A \rightarrow \mathbb{R}$  is  $\alpha$ -transfer u.s.c. and  $\beta$ -transfer l.s.c. on the first variable on  $G\text{-co}A$ ;
- (5') for each  $A \in \mathcal{F}(X)$  with  $M \subseteq A$ ,  $x, y \in G\text{-co}A$ , and for each net  $(x_\lambda)$  in  $X$  converging to  $x$ , if  $\alpha \leq g(x_\lambda, z) \leq \beta$  for all  $z \in \Gamma(\{x, y\})$ , then  $\alpha \leq g(x, y) \leq \beta$ .

(c) In part (a), if  $X$  is a nonempty convex subset of a Hausdorff topological vector space, then we can obtain a refinement of [1, Theorem 2.3] and [8, Theorem 3.1].

**THEOREM 2.10.** Let  $(X; \Gamma)$  be a Hausdorff  $G$ -convex space, for any finite subset  $A$  of  $X$ , and let  $G\text{-co}A$  be compact. Suppose that  $f$ ,  $g_1$ , and  $g_2$  are real bifunctions on  $X \times X$  satisfying the following conditions:

- (1)  $g_1(x, x) \geq \alpha$  and  $g_2(x, x) \leq \beta$ , for all  $x \in X$ ;
- (2) for every  $x \in X$  and for every  $A \in \mathcal{F}(X)$  if  $A \subseteq \{y \in X : f(x, y) < \alpha \text{ or } f(x, y) > \beta\}$ ,  $\Gamma(A) \subseteq \{y \in X : g_1(x, y) < \alpha \text{ or } g_2(x, y) > \beta\}$ ;
- (3) there exist compact subset  $K$  of  $X$  and  $M \in \mathcal{F}(X)$  such that the set  $\{y \in M : f(x, y) < \alpha \text{ or } f(x, y) > \beta\}$  is nonempty for each  $x \in X \setminus K$ ;

- (4) for each  $A \in \mathcal{F}(X)$  with  $M \subseteq A$ ,  $f : G\text{-co}A \times G\text{-co}A \rightarrow \mathbb{R}$  is  $\alpha$ -transfer u.s.c. and  $\beta$ -transfer l.s.c. on the first variable on  $G\text{-co}A$ ;
- (5) for each  $A \in \mathcal{F}(X)$  with  $M \subseteq A$ ,  $x, y \in G\text{-co}A$ , and for each net  $(x_\lambda)$  in  $X$  converging to  $x$ , if  $\alpha \leq f(x_\lambda, z) \leq \beta$  for all  $z \in \Gamma(\{x, y\})$ , then  $\alpha \leq f(x, y) \leq \beta$ .

Then there exists  $\bar{x} \in X$  such that  $\alpha \leq f(\bar{x}, y) \leq \beta$  for each  $y \in X$ .

**PROOF.** Let  $F : X \rightarrow 2^X$  be defined by

$$F(y) = \{x \in X : \alpha \leq f(x, y) \leq \beta\}. \quad (2.9)$$

First, we show that  $F$  is a KKM map. Assume that there exists  $A \in \mathcal{F}(X)$  such that  $\Gamma(A) \not\subseteq \bigcup_{y \in A} F(y)$ . Therefore,  $\Gamma(A)$  contains a point  $x_0$  which is not in  $\bigcup_{y \in A} F(y)$ . Hence, by condition (2), we have  $g_1(x_0, x_0) < \alpha$  or  $g_2(x_0, x_0) > \beta$ . This contradicts condition (1). Condition (3) implies that  $\bigcap_{y \in M} F(y) \subseteq K$ . As in the proof of [Theorem 2.8](#), condition (4) implies condition (4') of [Remark 2.7](#), and condition (5) implies condition (5) of [Theorem 2.6](#). Therefore, by [Theorem 2.6](#) and part (b) of [Remark 2.7](#), we have  $\bigcap_{y \in X} F(y) \neq \emptyset$ .  $\square$

**REMARK 2.11.** If, in [Theorem 2.10](#), instead of conditions (3) and (4), we have the following conditions:

- (3') there exists a compact subset  $K$  of  $X$  such that for every  $N \in \mathcal{F}(X)$  there is a nonempty compact  $G$ -convex subset  $L_N$  of  $X$  such that for every  $x \in X \setminus K$  there are a point  $y \in L_N$  and a neighborhood  $U(x)$  of  $x$  such that for any  $z \in U(x)$ , we have  $f(z, y) < \alpha$  or  $f(z, y) > \beta$ ;
- (4')  $f$  is  $\alpha$ -transfer u.s.c. and  $\beta$ -transfer l.s.c. on the first variable on  $X$ .

Then, by [Remark 2.7\(a\)](#) and without condition (5) we can obtain a refinement of [[1](#), Theorem 2.2]. Also if  $g_1$  and  $g_2$  are identical and equal to  $f$ , then we obtain an improvement of [[8](#), Theorem 3.1].

**3. Some applications.** In this section, we give some applications of [Theorem 2.8](#) and [Remark 2.9](#).

**THEOREM 3.1.** Let  $(X, D; \Gamma)$  be a  $G$ -convex space such that for each  $A, B \in \mathcal{F}(D)$  with  $A \subseteq B$ ,  $\Gamma(A) \subseteq \Gamma(B)$ . Suppose that  $f_1$  and  $g_1$  are two real bifunctions defined on  $D \times X$  such that

- (1) for each  $(y, x) \in D \times X$ , if  $f_1(y, x) \leq c$ , then  $g_1(y, x) \leq c$ ,
- (2) for each  $A \in \mathcal{F}(D)$ ,  $\sup_{x \in \Gamma(A)} \min_{y \in A} f_1(y, x) \leq c$ ,
- (3) there exist a compact subset  $K$  of  $X$  and  $M \in \mathcal{F}(D)$  such that, for every  $x \in X \setminus K$ , there exist a point  $y \in M$  and a neighborhood  $U(x)$  of  $x$  such that for any  $z \in U(x)$ ,  $f_1(y, z) > c$ ,
- (4) for each  $A \in \mathcal{F}(D)$  with  $M \subseteq A$ ,  $g_1 : A \times \Gamma(A) \rightarrow \mathbb{R}$  is  $c$ -transfer l.s.c. on the second variable on  $\Gamma(A)$ ,
- (5) for each  $A \in \mathcal{F}(D)$  with  $M \subseteq A$  and each net  $(x_\lambda)$  in  $X$  converging to  $x$ , if  $g_1(y, x_\lambda) \leq c$  for all  $y \in A$ , then  $g_1(y, x) \leq c$ .

Then there exists  $\bar{x} \in X$  such that  $g_1(y, \bar{x}) \leq c$  for all  $y \in D$ .

**PROOF.** Define  $f, g : X \times D \rightarrow \mathbb{R}$  by  $f(x, y) = e^{f_1(y, x)}$  and  $g(x, y) = e^{g_1(y, x)}$ . If  $\alpha = 0$  and  $\beta = e^c$ , then it is easy to see that all of the conditions of [Theorem 2.8](#) are satisfied. Therefore, there is a point  $\tilde{x} \in X$  such that  $0 \leq g(\tilde{x}, y) \leq e^c$  for all  $y \in D$ , that is,  $g_1(y, \tilde{x}) \leq c$  for all  $y \in D$ .  $\square$

**COROLLARY 3.2.** *Let  $(X, D; \Gamma)$  be a  $G$ -convex space such that for each  $A, B \in \mathcal{F}(D)$  with  $A \subseteq B$ ,  $\Gamma(A) \subseteq \Gamma(B)$ . Suppose that  $\varphi$  and  $\psi$  are two real bifunctions defined on  $X \times D$  such that*

- (1) *for each  $(x, y) \in X \times D$ , if  $\varphi(x, y) \geq 0$ , then  $\psi(x, y) \geq 0$ ,*
- (2) *for each  $A \in \mathcal{F}(D)$ ,  $\inf_{x \in \Gamma(A)} \max_{y \in A} \varphi(x, y) \geq 0$ ,*
- (3) *there exist a compact subset  $K$  of  $X$  and  $M \in \mathcal{F}(D)$  such that for every  $x \in X \setminus K$  there exist a point  $y \in M$  and a neighborhood  $U(x)$  of  $x$  such that for any  $z \in U(x)$ ,  $\varphi(z, y) < 0$ ,*
- (4) *for each  $A \in \mathcal{F}(D)$  with  $M \subseteq A$ ,  $\psi : \Gamma(A) \times A \rightarrow \mathbb{R}$  is 0-transfer u.s.c. on the first variable on  $\Gamma(A)$ ,*
- (5) *for each  $A \in \mathcal{F}(D)$  with  $M \subseteq A$  and each net  $(x_\lambda)$  in  $X$  converging to  $x$ , if  $\psi(x_\lambda, y) \geq 0$  for all  $y \in A$ , then  $\psi(x, y) \geq 0$ .*

*Then there exists  $\tilde{x} \in X$  such that  $\psi(\tilde{x}, y) \geq 0$  for all  $y \in D$ .*

**PROOF.** It is enough in [Theorem 3.1](#) to set  $c = 0$ ,  $f_1(y, x) = -\varphi(x, y)$ , and  $g_1(y, x) = -\psi(x, y)$ .  $\square$

If  $(X, \Gamma)$  is a  $G$ -convex space, then  $g : X \rightarrow \mathbb{R}$  is  $G$ -quasiconvex if  $\{x \in X : g(x) < \lambda\}$  is  $G$ -convex for each  $\lambda \in \mathbb{R}$ .

**REMARK 3.3.** If in [Corollary 3.2](#)  $X = D$ , for each  $x \in X$ ,  $y \mapsto \varphi(x, y)$  is  $G$ -quasiconvex, and  $\varphi(x, x) \geq 0$ , then condition (2) of [Corollary 3.2](#) is satisfied. So [Corollary 3.2](#) improves [9, Corollary 2].

If  $X = D$ ,  $X$  is Hausdorff space and  $G\text{-co}A$  is compact for any  $A \in \mathcal{F}(X)$ , then instead of conditions (3), (4), and (5) of [Theorem 3.1](#) we can suppose that

- (3') *there exist a compact subset  $K$  of  $X$  and  $M \in \mathcal{F}(X)$  such that, for every  $x \in X \setminus K$ , there exists a point  $y \in M$  such that  $f_1(y, x) > c$ ;*
- (4') *for each  $A \in \mathcal{F}(X)$  with  $M \subseteq A$ ,  $f_1$  is  $c$ -transfer l.s.c. on the second variable on  $G\text{-co}A$ ,*
- (5') *for each  $A \in \mathcal{F}(X)$  with  $M \subseteq A$ ,  $x, y \in G\text{-co}A$ , and each net  $(x_\lambda)$  in  $X$  converging to  $x$ , if  $g_1(z, x_\lambda) \leq c$  for all  $z \in \Gamma(\{x, y\})$ , then  $g_1(y, x) \leq c$ .*

In the above case we obtain a refinement of [2, Theorem 2], [6, Theorem 3.2], and [15, Theorems 2.2 and 2.3].

The following corollary improves [9, Corollary 3].

**COROLLARY 3.4.** *Let  $(X; \Gamma)$  be a Hausdorff  $G$ -convex space and let  $G\text{-co}A$  be compact for all  $A \in \mathcal{F}(X)$ . Suppose that  $Y$  is a topological space,  $T : X \rightarrow 2^Y$  is a multivalued mapping having a continuous selection  $f$ , and  $\phi : X \times Y \times X \rightarrow \mathbb{R}$  is a function such that*

- (1)  $\phi(x, y, z)$  is  $G$ -quasiconvex in  $z$ ,
- (2)  $\phi(x, f(x), z) \geq 0$  for all  $x \in X$ ,
- (3) there exist a compact subset  $K$  of  $X$  and  $M \in \mathcal{F}(X)$  such that, for every  $x \in X \setminus K$  and  $y \in Y$  there exists a point  $z \in M$  such that  $\phi(x, y, z) < 0$ ,
- (4) for each  $A \in \mathcal{F}(X)$  with  $M \subseteq A$ ,  $\phi(x, y, z)$  is 0-transfer u.s.c. in  $(x, y)$  on  $G\text{-co } A$ ,
- (5) for each  $A \in \mathcal{F}(X)$  with  $M \subseteq A$ ,  $x, z \in G\text{-co } A$ , and for each net  $(x_\lambda)$  in  $X$  converging to  $x$ , if  $\phi(x_\lambda, f(x_\lambda), z') \geq 0$  for all  $z' \in \Gamma(\{x, z\})$ , then  $\phi(x, f(x), z) \geq 0$ .

Then there exist an  $\bar{x} \in X$  and  $\bar{y} \in T(\bar{x})$  such that  $\phi(\bar{x}, \bar{y}, z) \geq 0$  for all  $z \in X$ .

**PROOF.** Let  $\varphi(z, x) = \psi(z, x) = -\phi(x, f(x), z)$  for  $(x, z) \in X \times X$ . Then  $\psi$  satisfies all of the requirements of [Remark 3.3](#). Therefore, by [Theorem 3.1](#), we have the conclusion.  $\square$

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M. Fakhar: Department of Mathematics, University of Isfahan, Isfahan 81745-163, Iran

*E-mail address:* [fakhar@sci.ui.ac.ir](mailto:fakhar@sci.ui.ac.ir)

J. Zafarani: Department of Mathematics, University of Isfahan, Isfahan 81745-163, Iran

*E-mail address:* [jzaf@sci.ui.ac.ir](mailto:jzaf@sci.ui.ac.ir)

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