IJMMS 2003:40, 2519–2539
PII. S0161171203212369
http://ijmms.hindawi.com

ON COMMON FIXED POINTS OF PAIRS OF A SINGLE AND A MULTIVALUED COINCIDENTALLY COMMUTING MAPPINGS IN D-METRIC SPACES

B. C. DHAGE, A. JENNIFER ASHA, and S. M. KANG

Received 10 December 2002 and in revised form 15 February 2003

The present paper studies some common fixed-point theorems for pairs of a single-valued and a multivalued coincidentally commuting mappings in D-metric spaces satisfying a certain generalized contraction condition. Our result generalizes more than a dozen known fixed-point theorems in D-metric spaces including those of Dhage (2000) and Rhoades (1996).

2000 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction. The concept of a D-metric space introduced by the first author in [1] is as follows. A nonempty set, together with a function $\rho : X\times X\times X \to [0,\infty)$, is called a *$D$-metric space* and denoted by (X,ρ) if the function ρ, called a *D-metric* on X, satisfies the following properties:

(i) $\rho(x,y,z) = 0 \Leftrightarrow x = y = z$ (coincidence),

(ii) $\rho(x,y,z) = 0 = \rho(p\{x,y,z\})$ (symmetry), where p is a permutation,

(iii) $\rho(x,y,z) \le \rho(x,y,a)+\rho(x,a,z)+\rho(a,y,z)$ for all $x,y,z,a\in X$ (tetrahedral inequality).

It is known that the D-metric ρ in a continuous function on X^3 in the topology of D-metric convergence is Hausdorff. The details of a D-metric space and its topological properties appear in Dhage [8]. Some specific examples of a D-metric space are presented in Dhage [2].

A sequence $\{x_n\} \subset X$ is called *convergent* and *converges* to a point x if $\lim_{m,n} \rho(x_m,x_n,x) = 0$. Again a sequence $\{x_n\} \subset X$ is called *D-Cauchy* if $\lim_{m,n,p} \rho(x_m,x_n,x_p) = 0$. A complete D-metric space X is one in which every D-Cauchy sequence converges to a point in X. A subset S of a D-metric space X is called *bounded* if there exists a constant $k > 0$ such that $\rho(x,y,z) \le k$ for all $x,y,z \in X$ and the constant k is called a *D-bound* of S. The smallest among all such D-bounds k of S is called the *diameter* of X and it is denoted by $\delta(S)$.

Let 2^X and $\mathrm{CB}(X)$ denote the classes of nonempty closed and nonempty, closed, bounded subsets of X, respectively. A correspondence $F : X \to 2^X$ is called a *multivalued mapping* on a D-metric space X, and a point $u \in X$ is called a *fixed point* of F if $u \in Fu$.

In [3], the first author has defined a notion of the generalized or Kasusai D-metric on X. Let $\kappa:(\mathrm{CB}(X))^3\to[0,\infty)$ be a function defined by

$$\kappa(A,B,C)=\inf\{\epsilon>0\mid A\cup B\subset N(c,\epsilon),\ B\cup C\subset N(A,\epsilon),\ C\cup A\subset N(B,\epsilon)\},\tag{1.1}$$

where $N(A,\epsilon)=\cup_{a\in A}N(a,\epsilon)$, $N(a,\epsilon)=\{x\in N^*(a,\epsilon)\mid \rho(a,x,y)<\epsilon$ for all $y\in N^*(a,\epsilon)\}$, and $N^*(a,\epsilon)=\{x\in X\mid \rho(a,x,x)<\epsilon\}$.

The definition (1.1) is equivalent to

$$\kappa(A,B,C)=\max\Big\{\sup_{a\in A,\,b\in B}D(a,b,c),\ \sup_{b\in B,\,c\in C}D(b,c,A),\ \sup_{c\in C,\,a\in A}D(c,a,B)\Big\},\tag{1.2}$$

where $D(a,b,c)=\inf\{\rho(a,b,c)\mid c\in C\}$.

Define

$$\begin{aligned}D(A,B,C)&=\inf\{\rho(a,b,c)\mid a\in A,\ b\in B,\ c\in C\},\\ \delta(A,B,C)&=\sup\{\rho(a,b,c)\mid a\in A,\ b\in B,\ c\in C\}.\end{aligned}\tag{1.3}$$

Notice that D and δ are continuous functions on $(\mathrm{CB}(X))^3$ and satisfy

$$D(A,B,C)\le\kappa(A,B,C)\le\delta(A,B,C).\tag{1.4}$$

A multivalued mapping $F:X\to\mathrm{CB}(X)$ is called *continuous* if

$$\lim_{m,n}\rho(x_m,x_n,x)=0\Longrightarrow\kappa(Fx_m,Fx_n,Fx)=0.\tag{1.5}$$

In [3], the first author has proved some fixed-point theorem for multivalued contraction mappings in D-metric spaces, and in [5] he has proved some common fixed-point theorems for coincidentally commuting single-valued mappings in D-metric spaces satisfying a condition of generalized contraction.

In this paper, we prove some common fixed-point theorems for a pair of singlevalued and multivalued mappings in a D-metric space satisfying a contraction condition more general than that given in Dhage [1, 2, 3, 4, 5, 7] and Rhoades [12]. The results of this paper are new to the fixed-point theory in D-metric spaces and include nearly a dozen of known fixed-point theorems as special cases (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12]).

2. Preliminaries.

Before going to the main results of this paper, we give some preliminaries needed in the sequel.

Let $F:X\to 2^X$. Then by an orbit of F at a point $x\in X$ we mean a set $O_F(x)$ in X defined by

$$O_F(x)=\{x_0=x,\ x_{n+1}\in Fx_n,\ n\ge 0\}.\tag{2.1}$$

An orbit $O_F(x)$ is called *bounded* if $\delta(O_F(x)) < \infty$, and a D-metric space X is called F-*orbitally bounded* if $O_F(x)$ is bounded for each $x \in X$. Again an F-orbit $O_F(x)$ is called *complete* if every D-Cauchy sequence in $O_F(x)$ converges to a point in X. A D-metric space X is said to be F-*orbitally complete* if $O_F(x)$ is complete for each $x \in X$. Finally, F is called F-*orbitally continuous* if for any sequence $\{x_n\} \subseteq O_F(x)$, we have

$$\lim_{m,n} \rho(x_m, x_n, x^*) = 0 \Longrightarrow \lim_{m,n} \kappa(Fx_m, Fx_n, Fx^*) = 0 \tag{2.2}$$

for each $x \in X$.

Let Φ denote the class of all functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following properties:

(i) ϕ is continuous,

(ii) ϕ is nondecreasing,

(iii) $\phi(t) < t$, $t > 0$,

(iv) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t \in [0,\infty)$.

The function ϕ is called a *Lipschitz control function* or *Lipschitz growth function* and the usual growth function is $\phi(t) = \alpha t$, $0 \le t < 1$. The following lemma concerning the function ϕ appears in [7].

Lemma 2.1. *If $\phi \in \Phi$, then $\phi^n(t) = 0$ for each $n \in \mathbb{N}$ and $\lim_n \phi^n(t) = 0$ for each $t \in [0,\infty)$.*

We need the following D-Cauchy principle of Dhage [7] in the sequel.

Lemma 2.2 (D-Cauchy principle)**.** *Let $\{x_n\}$ be a bounded sequence in a D-metric space X with D-bound k satisfying, for some positive real number r,*

$$\rho(x_n, x_{n+1}, x_m) \le \left[\phi^n(k^r)\right]^{1/r} \tag{2.3}$$

for all $m > n \in \mathbb{N}$, where $\phi : [0,\infty) \to [0,\infty)$ satisfies $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t \in [0,\infty)$. Then $\{x_n\}$ is a D-Cauchy sequence in X.

Proof. The proof appears in [7], but for the sake of completeness we give the details. Let $p, t \in \mathbb{N}$ be arbitrary but fixed. Then from (2.3) it follows that

$$\begin{aligned} \rho(x_n, x_{n+1}, x_{n+p}) &\le \left[\phi^n(k^r)\right]^{1/r}, \\ \rho(x_n, x_{n+1}, x_{n+p+t}) &\le \left[\phi^n(k^r)\right]^{1/r}, \end{aligned} \tag{2.4}$$

for all $n \in \mathbb{N}$.

Now by repeated application of the tetrahedral inequality, we obtain

$$\begin{aligned} &\rho(x_n, x_{n+p}, x_{n+p+t}) \\ &\quad \le \rho(x_n, x_{n+1}, x_{n+p}) + \rho(x_n, x_{n+1}, x_{n+p+t}) + \rho(x_{n+1}, x_{n+p}, x_{n+p+t}) \\ &\quad \le \rho(x_n, x_{n+1}, x_{n+p}) + \rho(x_n, x_{n+1}, x_{n+p+t}) + \rho(x_{n+1}, x_{n+2}, x_{n+p}) \\ &\qquad + \rho(x_{n+1}, x_{n+2}, x_{n+p+t}) + \rho(x_{n+2}, x_{n+p}, x_{n+p+t}) \end{aligned}$$

$$\leq 2[\phi^n(k^r)]^{1/r} + 2[\phi^{n+1}(k^r)]^{1/r} + \rho(x_{n+2}, x_{n+p}, x_{n+p+t})$$
$$\leq 2\Big\{[\phi^n(k^r)]^{1/r} + \cdots + [\phi^{n+p-2}(k^r)]^{1/r}\Big\} + \rho(x_{n+p-1}, x_{n+p}, x_{n+p+t})$$
$$\leq 2\sum_{j=n}^{n+p-1}[\phi^j(k^r)]^{1/r}. \tag{2.5}$$

Since $\sum_{n=1}^{\infty}\phi^n(t) < \infty$ for each $t \in [0,\infty)$, we have $\sum_{j=1}^{\infty}[\phi^j(k^r)]^{1/r} < \infty$ and so $\lim_n \sum_{j=n}^{n+p-1}[\phi^j(k^r)]^{1/r} = 0$. Now from (2.5) it follows that

$$\lim_{n\to\infty}\rho(x_n, x_{n+p}, x_{n+p+t}) = 0. \tag{2.6}$$

This proves that $\{x_n\}$ is a D-Cauchy sequence in X and the proof of the lemma is complete. □

As a direct application of Lemma 2.2, we obtain the following result proved in [5].

Lemma 2.3. *Let $\{x_n\}$ be a bounded sequence in a D-metric space X with D-bound k satisfying*

$$\rho(x_n, x_{n+1}, x_m) \leq \lambda^n k \tag{2.7}$$

for all $m > n \in \mathbb{N}$, where $0 \leq \lambda < 1$. Then $\{x_n\}$ is D-Cauchy.

We use contractive conditions of the form

$$a^r \leq \phi(b^r) \tag{2.8}$$

for some positive real number r, where a and b are nonnegative real numbers and $\phi \in \Phi$, because sometimes inequality (2.8) holds, but for the same real numbers a and b, the inequality

$$a \leq \phi(b) \tag{2.9}$$

does not hold. To see this, let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function defined by

$$\phi(t) = \frac{\alpha t}{1+t}, \quad 0 \leq \alpha < 1. \tag{2.10}$$

Obviously the function ϕ is continuous, nondecreasing and satisfies $\phi(t) = \alpha t/(1+t) < t$ for $t > 0$. Again since

$$\sum_{n=1}^{\infty}\phi^n(t) = \sum_{n=1}^{\infty}\frac{\alpha^n t}{1+t+\cdots+\alpha^{n-1}t} < \sum_{n=1}^{\infty}\alpha^n < \infty, \tag{2.11}$$

we have that $\phi \in \Phi$.

Now for $a = 1/2$ and $b = 2/3$, we have, by (2.9),

$$\frac{1}{2} \leq \phi\left(\frac{2}{3}\right) = \frac{(2/3)\alpha}{1+2/3} = \frac{2}{5}\alpha, \tag{2.12}$$

which is not true since $0 \leq \alpha < 1$. But for the same values of a and b, we have a positive real number $r = 2$ such that

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \leq \frac{4\alpha}{13} = \phi\left(\left(\frac{2}{3}\right)^2\right) \tag{2.13}$$

for $13/16 \leq \alpha < 1$. Hence inequality (2.8) holds. Thus inequality (2.9) does not imply inequality (2.8). Actually, inequalities (2.8) and (2.9) are independent. To show that inequality (2.8) does not imply inequality (2.9), let $a = 1/4$, $b = 4/9$, and $r = 1/2$. Clearly, inequality (2.8) does not hold, but for the same values of a, b, and r, one has

$$\frac{1}{4} \leq 4\frac{\alpha}{13} = \frac{\alpha(4/9)}{1+4/9} = \phi\left(\frac{4}{9}\right) \tag{2.14}$$

for $\alpha \geq 13/16$, and so inequality (2.9) holds. Thus inequalities (2.8) and (2.9) are independent.

In the following sections, we will prove the main results of this paper.

3. Weak commuting mappings in D-metric spaces. Let $F : X \to 2^X$ and $g : X \to X$. Then the pair $\{F,g\}$ of maps is called *limit coincident* if $\lim_n Fx_n = \{\lim_n gx_n\}$ for some sequence $\{x_n\}$ in X, and *coincident* if there exists a point $u \in X$ such that $Fu = \{gu\}$. Again two maps F and g are called *limit commuting* if $\lim_n Fgx_n = \{\lim_n gFx_n\}$, where $\{x_n\}$ is a sequence in X, and *commuting* if $Fgx = \{gFx\}$ for all $x \in X$. Two maps F and g are called *limit coincidentally commuting* if their limit coincidence implies the limit commutativity on X. Similarly, they are called *coincidentally commuting* if they are commuting at the coincidence points. Again two maps F and g are said to be *limit pseudocommuting* if $\lim_n Fgx_n \cap \lim_n gFx_n \neq \phi$, that is, $\lim_n D(Fgx_n, gFx_n, gFx_n) = 0$, where $\{x_n\}$ is a sequence in X, and *pseudocommuting* if $Fgx \cap gFx \neq \emptyset$ for each $x \in X$. Finally, the pair $\{F,g\}$ is called *limit coincidentally pseudocommuting* if its limit coincidence implies the limit pseudocommutativity on X, and *coincidentally pseudocommuting* if it is pseudocommuting at the coincidence points. It is known that a coincidentally commuting pair is limit coincidentally commuting and a coincidentally pseudocommuting pair is limit coincidentally pseudocommuting, but the converse implications need not hold. A pair of maps $\{F,g\}$ is *weak commuting* if it is either limit commuting, coincidentally commuting, limit pseudocommuting, or coincidentally pseudocommuting on X. Below, we will prove some common fixed-point theorems for each of these weak commuting mappings on D-metric spaces.

3.1. Limit coincidentally commuting maps in D-metric spaces. Let $F : X \to 2^X$ and $g : X \to X$. By an (F/g)-orbit of the pair $\{F,g\}$ of maps at a point $x \in X$, we mean a set $O_F(gx)$ in X defined by

$$O_F(gx) = \{y_n \mid y_0 = gx_0,\ y_n = gx_n \in Fx_{n-1},\ n \in \mathbb{N},\ \text{where } x_0 = x\} \tag{3.1}$$

for some sequence $\{x_n\}$ in X. The orbit $O_F(gx)$ is well defined for each $x \in X$ if $F(X) \subseteq g(X)$. By $\overline{O_F(gx)}$ we denote the closure of the set $O_F(gx)$ in X.

A D-metric space X is called (F/g)-*orbitally bounded* if $\delta(O_F(gx)) < \infty$ for each $x \in X$. Further X is called (F/g)-*orbitally complete* if every D-Cauchy sequence $\{x_n\} \subset O_F(gx)$ converges to a point in X for each $x \in X$. Finally, a mapping $T : X \to \mathrm{CB}(X)$ is called (F/g)-*orbitally continuous* if for any $\{x_n\} \subset O_F(gx)$, $x_n \to x^*$ implies that $Tx_n \to Tx^*$ for each $x \in X$.

Theorem 3.1. *Let $F : X \to \mathrm{CB}(X)$ and $g : X \to X$ be two mappings satisfying, for some positive real number r,*

$$\begin{aligned}\delta^r&(Fx,Fy,Fz)\\ &\le \phi(\max\{\rho^r(gx,gy,gz),\delta^r(Fx,Fy,gz),\delta^r(gx,Fx,gz),\\ &\qquad \delta^r(gy,Fy,gz),\delta^r(gx,Fy,gz),\delta^r(gy,Fx,gz)\})\end{aligned} \tag{3.2}$$

for all $x,y,z \in X$, where $\phi \in \Phi$. Suppose that

(a) *$F(X) \subseteq g(X)$ and $g(X)$ is bounded,*
(b) *$\{F,g\}$ is limit coincidentally commuting,*
(c) *F or g is (F/g)-orbitally continuous.*

Further if X is (F/g)-orbitally complete D-metric space, then F and g have a unique common fixed point $u \in X$ such that $Fu = \{u\} = gu$. Moreover, if g is continuous at u, then F is also continuous at u in the Kasubai D-metric on X.

Proof. Let $x \in X$ be arbitrary and define a sequence $\{y_n\}$ in X as follows. Take $x_0 = x$ and $y_0 = gx_0$. Choose a point $y_1 \in Fx_0 = X_1$. Since $F(X) \subseteq g(X)$, there is a point $x_1 \in X$ such that $y_1 = gx_1$. Again choose a point $y_2 \in Fx_1 = X_2$. By hypothesis (a), there is a point $x_2 \in X$ such that $y_2 = gx_2$. Proceeding in this way, by induction there is a sequence $\{x_n\}$ of points in X such that

$$y_0 = gx_0, \qquad y_{n+1} = gx_{n+1} \in X_{n+1} = Fx_n, \quad n = 0,1,2,\dots. \tag{3.3}$$

From hypothesis (a), it follows that

$$\delta(X_m,X_n,X_p) \le \delta(g(X)) = k < \infty \tag{3.4}$$

for all $m,n,p \in \mathbb{N}$.

Now there are two cases.

Case 1. Suppose that $y_r = y_{r+1}$ for some $r \in \mathbb{N}$. Then we have $gx_r = gx_{r+1} = u$ for some $u \in X$.

We will show that $Fx_r = \{u\}$. Suppose not. Then by (3.2),

$$\begin{aligned}
&\delta^r(Fx_r,Fx_r,u)\\
&\quad = \delta^r(Fx_r,Fx_r,gx_{r+1})\\
&\quad \le \delta^r(Fx_r,Fx_r,Fx_r)\\
&\quad \le \phi(\max\{\rho^r(gx_r,gx_r,gx_r),\delta^r(gx_r,Fx_r,gx_r),\delta^r(Fx_r,Fx_r,gx_r)\})\\
&\quad \le \phi(\max\{0,\delta^r(gx_r,Fx_r,gx_r),\delta^r(Fx_r,Fx_r,u)\})\\
&\quad = \phi(\max\{\delta^r(u,Fx_r,u),\delta^r(Fx_r,Fx_r,u)\})\\
&\quad = \phi(\delta^r(u,Fx_r,u))
\end{aligned} \tag{3.5}$$

because $\delta^r(Fx_r,Fx_r,u) \le \phi(\delta^r(Fx_r,Fx_r,u))$ is not possible in view of $\phi \in \Phi$.

Again by (3.2),

$$\begin{aligned}
\delta^r(Fx_r,u,u) &= \delta^r(Fx_r,gx_{r+1},gx_{r+1})\\
&\le \delta^r(Fx_r,Fx_r,Fx_r)\\
&\le \phi(\max\{\delta^r(u,Fx_r,u),\delta^r(Fx_r,Fx_r,u)\})\\
&= \phi(\delta^r(Fx_r,Fx_r,u)).
\end{aligned} \tag{3.6}$$

Substituting (3.6) in (3.5), we obtain

$$\delta^r(Fx_r,Fx_r,u) \le \phi^2(\delta^r(Fx_r,Fx_r,u)), \tag{3.7}$$

which is a contradiction since $\phi \in \Phi$. Hence $Fx_r = u$. Since F and g are limit coincidentally commuting, one has $Fgx_r = \{gFx_r\}$.

We will show that u is a common fixed point of F and g such that $Fu = \{u\} = gu$.

Now,

$$\begin{aligned}
\delta^r(Fu,gu,u) &= \delta^r(FFx_r,Fgx_r,Fx_r)\\
&\le \phi(\max\{\rho^r(gFx_r,ggx_r,gx_r),\delta^r(FFx_r,Fgx_r,gx_r),\\
&\qquad\qquad \delta^r(gFx_r,FFx_r,gx_r),\delta^r(ggx_r,Fgx_r,gx_r),\\
&\qquad\qquad \delta^r(gFx_r,Fgx_r,gx_r),\delta^r(ggx_r,FFx_r,gx_r)\})\\
&= \phi(\max\{\rho^r(gFx_r,ggx_r,gx_r),\delta^r(ggx_r,FFx_r,gx_r)\})\\
&= \phi(\delta^r(Fu,gu,u)),
\end{aligned} \tag{3.8}$$

which is possible only when $Fu = \{u\} = gu$ since $\phi \in \Phi$.

Case 2. Assume that $y_n \ne y_{n+1}$ for each $n \in \mathbb{N}$. We will show that $\{y_n\}$ is a D-Cauchy sequence in X. Let $x = x_0$, $y = x_1$, and $z = x_{m-1}$, $m \ge 1$. Then by (3.2),

$$\begin{aligned}
&\rho^r(y_1,y_2,y_m)\\
&\quad\le \delta^r(Fx_0,Fx_1,Fx_{m-1})\\
&\quad\le \phi(\max\{\rho^r(gx_0,gx_1,gx_{m-1}),\delta^r(Fx_0,Fx_1,gx_{m-1}),\delta^r(gx_0,Fx_0,gx_{m-1}),\\
&\qquad\qquad \delta^r(gx_1,Fx_1,gx_{m-1}),\delta^r(gx_0,Fx_1,gx_{m-1}),\delta^r(gx_1,Fx_0,gx_{m-1})\})\\
&\quad\le \phi(\max\{\delta^r(X_0,X_1,X_{m-1}),\delta^r(X_1,X_2,X_{m-1}),\delta^r(X_0,X_1,X_{m-1}),\\
&\qquad\qquad \delta^r(X_1,X_2,X_{m-1}),\delta^r(X_0,X_2,X_{m-1}),\delta^r(X_1,X_1,X_{m-1})\})\\
&\quad\le \phi\Big(\max_{0\le a\le 1,\,1\le b\le 2}\delta^r(X_a,X_b,X_{m-1})\Big)\\
&\quad\le \phi(k^r),
\end{aligned}\tag{3.9}$$

that is,

$$\rho(y_1,y_2,y_m)\le\left[\phi(k^r)\right]^{1/r}. \tag{3.10}$$

Similarly, letting $x=x_1$, $y=x_2$, and $z=z_{m-1}$, $m\ge 2$ in (3.2), we obtain

$$\begin{aligned}
&\rho^r(y_2,y_3,y_m)\\
&\quad\le \delta^r(Fx_1,Fx_2,Fx_{m-1})\\
&\quad\le \phi(\max\{\rho^r(gx_1,gx_2,gx_{m-1}),\delta^r(Fx_1,Fx_2,gx_{m-1}),\\
&\qquad\qquad \delta^r(gx_1,Fx_1,gx_{m-1}),\delta^r(gx_2,Fx_2,gx_{m-1}),\\
&\qquad\qquad \delta^r(gx_1,Fx_2,gx_{m-1}),\delta^r(gx_2,Fx_1,gx_{m-1})\})\\
&\quad\le \phi(\max\{\delta^r(Fx_0,Fx_1,Fx_{m-2}),\delta^r(Fx_1,Fx_2,Fx_{m-2}),\\
&\qquad\qquad \delta^r(Fx_0,Fx_1,Fx_{m-2}),\delta^r(Fx_1,Fx_2,Fx_{m-2}),\\
&\qquad\qquad \delta^r(Fx_0,Fx_2,Fx_{m-2}),\delta^r(Fx_1,Fx_1,Fx_{m-2})\})\\
&\quad\le \phi\Big(\phi\Big(\max_{0\le a\le 2,\,1\le b\le 3}\delta^r(X_a,X_b,X_{m-2})\Big)\Big)\\
&\quad\le \phi(\phi(k^r))\\
&\quad= \phi^2(k^r),
\end{aligned}\tag{3.11}$$

that is,

$$\rho(y_2,y_3,y_m)\le\left[\phi^2(k^r)\right]^{1/r}. \tag{3.12}$$

In general, by induction,

$$\rho(y_n,y_{n+1},y_m)\le\left[\phi^n(k^r)\right]^{1/r} \tag{3.13}$$

for all $m>n\in\mathbb{N}$.

Hence, the application of Lemma 2.2 yields that $\{y_n\}$ is a D-Cauchy sequence in X. The D-metric space X being complete, there is a point $u\in X$ such that $\lim_n y_n=u$. The definition of $\{y_n\}$ implies that $\lim_n gx_n=u$. We will show that $\lim_n Fx_n=\{u\}$.

Now,

$$\begin{aligned}
&\lim_n \delta^r(Fx_n, Fx_n, u)\\
&\quad = \lim_n \delta^r(Fx_n, Fx_n, y_{n+1})\\
&\quad \le \lim_n \delta^r(Fx_n, Fx_n, Fx_n)\\
&\quad \le \lim_n \phi(\max\{\rho^r(gx_n, gx_n, gx_n), \delta^r(Fx_n, Fx_n, gx_n), \delta^r(gx_n, Fx_n, gx_n)\})\\
&\quad = \lim_n \phi(\max\{\delta^r(Fx_n, Fx_n, u), 0\})\\
&\quad = \phi\Big(\lim_n \delta^r(Fx_n, Fx_n, u)\Big),
\end{aligned} \tag{3.14}$$

which implies that $\lim_n Fx_n = u$. Thus we have

$$\lim_n Fx_n = \{u\} = \lim_n gx_n. \tag{3.15}$$

Since F and g are limit coincidentally commuting, one has

$$\lim_n Fgx_n = \Big\{\lim_n gFx_n\Big\}. \tag{3.16}$$

Suppose that g is (F/g)-orbitally continuous on X. Then we have

$$\lim_n Fgx_n = \lim_n gFx_n = \lim_n ggx_n = gu. \tag{3.17}$$

First, we will show that u is a common fixed point of F and g. Suppose not. Then we have

$$\begin{aligned}
\delta^r(u, u, gu) &= \lim_n \delta^r(Fx_n, Fx_n, gFx_n)\\
&= \lim_n \delta^r(Fx_n, Fx_n, Fgx_n)\\
&\le \lim_n \phi(\max\{\rho^r(gx_n, gx_n, ggx_n),\\
&\qquad\qquad \delta^r(Fx_n, Fx_n, ggx_n), \delta^r(gx_n, Fx_n, ggx_n)\})\\
&= \phi\Big(\max\Big\{\lim_n \delta^r(gx_n, gx_n, ggx_n), \lim_n \delta^r(Fx_n, Fx_n, ggx_n)\Big\}\Big)\\
&= \phi(\delta^r(u, u, gu)),
\end{aligned} \tag{3.18}$$

which is a contradiction and hence $gu = u$.

Again

$$\begin{aligned}
&\delta^r(Fu, gu, u)\\
&\quad = \lim_n \delta^r(Fu, Fx_n, Fgx_n)\\
&\quad \le \lim_n \phi(\max\{\rho^r(gu, gx_n, ggx_n), \delta^r(Fu, Fx_n, ggx_n), \delta^r(gu, Fu, ggx_n),\\
&\qquad\qquad \delta^r(gx_n, Fx_n, ggx_n), \delta^r(gu, Fx_n, ggx_n), \delta^r(gx_n, Fu, ggx_n)\})
\end{aligned}$$

$$\begin{aligned}&= \phi(\max\{\rho^r(gu,u,gu),\delta^r(Fu,u,gu),\delta^r(gu,Fu,gu),\\&\qquad \delta^r(u,u,gu),\delta^r(gu,u,gu),\delta^r(u,Fu,gu)\})\\&= \phi(\delta^r(Fu,gu,u)),\end{aligned} \tag{3.19}$$

which is possible only when $Fu = \{u\} = gu$ since $\phi \in \Phi$. Thus u is a common fixed point of F and g.

Next, suppose that F is (F/g)-orbitally continuous on X. Then we have

$$\lim_n Fgx_n = \lim_n gFx_n = \lim_n FFx_n = Fu = \{z\}. \tag{3.20}$$

We will show that z is a common fixed point of F and g. Since $F(X) \subseteq g(X)$, there is a point $v \in X$ such that $Fu = gv = z$. We will show that $Fv = gv = \{z\}$. By (3.2),

$$\begin{aligned}&\delta^r(Fv,gv,Fv)\\&\quad = \lim_n \delta^r(Fv,Fv,FFx_n)\\&\quad \le \lim_n \phi(\max\{\rho^r(gv,gv,gFx_n),\delta^r(Fv,gv,gFx_n),\delta^r(gv,Fv,gFx_n)\})\\&\quad = \phi(\max\{\delta^r(gv,gv,gv),\delta^r(Fv,gv,z)\}),\end{aligned} \tag{3.21}$$

that is,

$$\delta^r(Fv,gv,z) \le \phi(\delta^r(Fv,gv,z)), \tag{3.22}$$

which implies that $Fv = gv = \{z\}$ since $\phi \in \Phi$.

Since F and g are limit coincidentally commuting, they are coincidentally commuting on X. Therefore, we have $Fgv = gFv$. Now, proceeding with the arguments as in Case 1, it is proved that z is a common fixed point of F and g.

To prove the uniqueness, let $z^* \ (\neq z)$ be another common fixed point of F and g. Then by (3.2),

$$\begin{aligned}\rho^r(z,z,z^*) &= \delta^r(Fz,Fz,Fz^*)\\&\le \phi(\max\{\rho^r(gz,gz,gz^*),\delta^r(Fz,Fz,gz^*),\\&\qquad \delta^r(gz,Fz,gz^*),\delta^r(gz,Fz,gz^*)\})\\&= \phi(\rho^r(z,z,z^*)),\end{aligned} \tag{3.23}$$

which is a contradiction. Hence $z = z^*$. Then F and g have a unique common fixed point $z \in X$ with $Fz = \{z\} = gz$.

Finally, suppose that g is continuous at the common fixed point z of F and g. Then we will prove that F is also continuous at z. Let $\{z_n\}$ be any sequence in X converging to the common fixed point z. Since g is continuous on X, we have

$$\lim_{m,n}\rho(z_m,z_n,z)=0\Longrightarrow\lim_{m,n}\rho(gz_m,gz_n,gz)=0. \tag{3.24}$$

From (1.2), it follows that

$$\kappa(Fz_m,Fz_n,Fz)\le\delta(Fz_m,Fz_n,Fz). \tag{3.25}$$

Now,

$$\begin{aligned}\delta^r&(Fz_m,Fz_n,Fz)\\&\le\phi(\max\{\rho^r(gz_m,gz_n,gz),\delta^r(Fz_m,Fz_n,gz),\delta^r(gz_m,Fz_m,gz),\\&\qquad\delta^r(gz_n,Fz_n,gz),\delta^r(gz_m,Fz_n,gz),\delta^r(gz_n,Fz_m,gz)\}).\end{aligned} \tag{3.26}$$

Therefore,

$$\begin{aligned}\lim_{m,n}&\delta^r(Fz_m,Fz_n,Fz)\\&\le\lim_{m,n}\phi(\max\{\rho^r(gz_m,gz_n,gz),\delta^r(Fz_m,Fz_n,Fz),\delta^r(gz_m,Fz_m,z),\\&\qquad\delta^r(gz_n,Fz_n,z),\delta^r(gz_m,Fz_n,z),\delta^r(gz_n,Fz_m,z)\})\\&=\phi\Big(\max\Big\{0,\lim_{m,n}\delta^r(Fz_m,Fz_n,Fz),\lim_m\delta^r(z,Fz_m,z),\lim_n\delta^r(z,Fz_n,z)\Big\}\Big)\\&=\phi\Big(\max\Big\{\lim_m\delta^r(z,Fz_m,z),\lim_n\delta^r(z,Fz_n,z)\Big\}\Big).\end{aligned} \tag{3.27}$$

But

$$\begin{aligned}\lim_m&\delta^r(z,Fz_m,z)\\&=\lim_m\delta^r(Fz,Fz,Fz_m)\\&\le\lim_m\phi(\max\{\rho^r(gz,gz,gz_m),\delta^r(Fz,Fz,gz_m),\delta^r(gz,Fz,gz_m)\})\\&=\phi(\max\{0,0,0\})\\&=0.\end{aligned} \tag{3.28}$$

Similarly, $\lim_n\delta^r(z,Fz_n,z)=0$. Substituting these estimates in (3.27) yields that

$$\lim_{m,n}\delta^r(Fz_m,Fz_n,Fz)=0 \tag{3.29}$$

or

$$\lim_{m,n}\delta(Fz_m,Fz_n,Fz)=0. \tag{3.30}$$

Now from (3.25), it follows that

$$\lim_{m,n}\kappa(Fz_m,Fz_n,Fz)=0, \tag{3.31}$$

and so F is continuous at the common fixed point z of F and g. This completes the proof. □

Letting $g = I$, the identity map on X and $r = 1$, in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let $F : X \to \mathrm{CB}(X)$ be a multivalued mapping satisfying*

$$\delta(Fx,Fy,Fz) \le \phi(\rho(x,y,z),\delta(Fx,Fy,z),\delta(x,Fx,z), \delta(y,Fy,z),\delta(x,Fy,z),\delta(y,Fx,z)) \tag{3.32}$$

for all $x,y,z \in X$, where $\phi \in \Phi$. Further if X is F-orbitally bounded and F-orbitally complete D-metric space, then F has a unique fixed point $u \in X$ such that $Fu = \{u\}$ and F is continuous at u.

Corollary 3.3. *Let $F : X \to \mathrm{CB}(X)$ be a multivalued mapping satisfying*

$$\delta(Fx,Fy,Fz) \le \lambda \max\{\rho(x,y,z),\delta(Fx,Fy,z),\delta(x,Fx,z), \delta(y,Fy,z),\delta(x,Fy,z),\delta(y,Fx,z)\} \tag{3.33}$$

for all $x,y,z \in X$, where $0 \le \lambda < 1$. Further if X is F-orbitally bounded and F-orbitally complete D-metric space, then F has a unique fixed point $u \in X$ such that $Fu = \{u\}$ and F is continuous at u.

Corollary 3.3 includes the following fixed point of Dhage [3] as a special case.

Corollary 3.4 (see [3]). *Let X be a bounded and complete D-metric space and let $F : X \to \mathrm{CB}(X)$ be a multivalued mapping satisfying*

$$\delta(Fx,Fy,Fz) \le \lambda\rho(x,y,z) \tag{3.34}$$

for all $x,y,z \in X$, where $0 \le \lambda < 1$. Then F has a unique fixed point $u \in X$ such that $Fu = \{u\}$ and F is continuous at u.

Corollary 3.5. *Let $f,g : X \to X$ be two mappings satisfying*

$$\rho^r(fx,fy,fz) \le \phi(\max\{\rho^r(gx,gy,gz),\rho^r(fx,fy,gz),\rho^r(gx,fx,gz), \rho^r(gy,fy,gz),\rho^r(gx,fy,gz),\rho^r(gy,fx,gz)\}) \tag{3.35}$$

for all $x,y,z \in X$, where $\phi \in \Phi$. Suppose that

(a) *$f(X) \subseteq g(X)$,*

(b) *$\{f,g\}$ is limit coincidentally commuting,*

(c) *f or g is continuous.*

Further if X is (f/g)-orbitally bounded and (f/g)-orbitally complete D-metric space, then f and g have a unique common fixed point $u \in X$. Moreover, if g is continuous at u, then f is also continuous at u.

Remark 3.6. Note that Corollary 3.5 includes the class of pairs of fixed-point mappings of Dhage [7] characterized by the inequality

$$\begin{aligned}\rho^r(fx,fy,fz)\\ &\le \phi(\max\{\rho^r(gx,gy,gz),\rho^r(gx,fx,gz),\\ &\qquad \rho^r(gy,fy,gz),\rho^r(gx,fy,gz),\rho^r(gy,fx,gz)\})\end{aligned} \tag{3.36}$$

for all $x,y,z\in X$ and $\phi\in\Phi$.

Corollary 3.7. *Let $f,g:X\to X$ be two mappings satisfying for some positive real numbers p, q, and r,*

$$\begin{aligned}\rho^r(f^px,f^py,f^pz)\\ &\le \phi(\max\{\rho^r(g^qx,g^qy,g^qz),\rho^r(f^px,f^py,g^qz),\\ &\qquad \rho^r(g^qx,f^px,g^qz),\rho^r(g^qy,f^py,g^qz),\\ &\qquad \rho^r(g^qx,f^py,g^qz),\rho^r(g^qy,f^px,g^qz)\})\end{aligned} \tag{3.37}$$

for all $x,y,z\in X$, where $\phi\in\Phi$. Suppose that

(a) *$f^p(X)\subseteq g^q(X)$,*
(b) *$\{f,g\}$ is commuting,*
(c) *f or g is continuous.*

Further if X is an (f^p/g^q)-orbitally bounded and (f^p/g^q)-orbitally complete D-metric space, then f and g have a unique common fixed point $u\in X$. Moreover, if g is continuous at u, then f^p is also continuous at u.

Proof. Let $S=f^p$ and $T=g^q$. Then by Corollary 3.5, S and T have a unique common fixed point $u\in X$, that is, $Su=f^pu=u=g^qu=Tu$. Now by commutativity of f and g, we obtain

$$fu=f(f^pu)=f^p(fu),\qquad fu=f(g^qu)=g^q(fu). \tag{3.38}$$

This shows that fu is again a common fixed point of f^p and g^q. By the uniqueness of u, we have $fu=u$. Similarly it is proved that $gu=u$. Thus f and g have a unique common fixed point $u\in X$. Further if g is continuous on X, g^q is continuous on X and by application of Corollary 3.5 yields that f^p is continuous at u. This completes the proof. □

Corollary 3.7 includes the class of pairs of fixed-point mappings of Dhage [7] characterized by the inequality

$$\begin{aligned}\rho^r(f^px,f^py,f^pz)\\ &\le \phi(\max\{\rho^r(g^qx,g^qy,g^qy),\rho^r(g^qx,f^px,g^qz),\\ &\qquad \rho^r(g^qy,f^py,g^qz),\rho^r(g^qx,f^py,g^qz),\rho^r(g^qy,f^px,g^qz)\})\end{aligned} \tag{3.39}$$

for all $x,y,z\in X$ and $\phi\in\Phi$.

Corollary 3.8. *Let f be a self-map of a D-metric space X satisfying*

$$\rho(fx,fy,fz) \le \lambda \max\{\rho(x,y,z),\rho(fx,fy,z),\rho(x,fx,z),\rho(y,fy,z),\rho(x,fy,z),\rho(y,fx,z)\} \tag{3.40}$$

for all $x,y,z \in X$, where $0 \le \lambda < 1$. Further if X is f-orbitally bounded and f-orbitally complete, then f has a unique fixed point $u \in X$ and f is continuous at u.

Corollary 3.9. *Let f be a self-map of a D-metric space X satisfying, for some positive real number p,*

$$\rho(f^px,f^py,f^pz) \le \lambda \max\{\rho(x,y,z),\rho(f^px,f^py,z),\rho(x,f^px,z),\rho(y,f^py,z),\rho(x,f^py,z),\rho(y,f^px,z)\} \tag{3.41}$$

for all $x,y,z \in X$, where $0 \le \lambda < 1$. Further if X is f-orbitally bounded and f-orbitally complete, then f has a unique fixed point $u \in X$, f^p is continuous, and f is f-orbitally continuous at u.

Note that Corollaries 3.8 and 3.9 include the fixed-point theorems of Rhoades [12] and Dhage [9] for the mappings characterized by the inequalities

$$\rho(fx,fy,fz) \le \lambda \max\{\rho(x,y,z),\rho(x,fx,z),\rho(y,fy,z),\rho(x,fy,z),\rho(y,fx,z)\}, \tag{3.42}$$

$$\rho(f^px,f^py,f^pz) \le \lambda \max\{\rho(x,y,z),\rho(x,f^px,z),\rho(y,f^py,z),\rho(x,f^py,z),\rho(y,f^px,z)\}, \tag{3.43}$$

for all $x,y,z \in X$ and $0 \le \lambda < 1$.

3.2. Coincidentally commuting mappings. The coincidentally commuting mappings require some stronger condition than limit coincidentally commuting mappings and a good number of mathematicians have studied them on metric and D-metric spaces for the existence of their common fixed point. See [5, 11] and the references therein. The novelty of the fixed-point theorems for these coincidentally commuting mappings lies in the fact that here we do not require any of the maps under consideration to be continuous. Below, we prove a result in this direction and derive some interesting corollaries.

Theorem 3.10. *Let X be a D-metric space and let $F : X \to \mathrm{CB}(X)$ and $g : X \to X$ be two mappings satisfying (3.2). Further suppose that*

(a) *$F(X) \subseteq g(X)$,*
(b) *$g(X)$ is bounded and complete,*
(c) *$\{F,g\}$ is coincidentally commuting.*

Then F and g have a unique common fixed point $u \in X$ such that $Fu = \{u\} = gu$. Moreover, if g is continuous at u, then F is also continuous at u in the Kasubai D-metric on X.

Proof. Let $x \in X$ be arbitrary and define a sequence $\{y_n\} \subset X$ by (3.3). Clearly the sequence $\{y_n\}$ is well defined since $F(X) \subseteq g(X)$. Further we note that $\{y_n\} \subseteq g(X)$. We prove the conclusion of the theorem in two cases.

Case 1. Suppose that $y_r = y_{r+1}$ for some $r \in \mathbb{N}$. Then proceeding with the arguments similar to Case 1 of the proof of Theorem 3.1, it is proved that $y_r = u$ is a common fixed point of F and g such that $Fu = \{u\} = gu$.

Case 2. Assume that $y_n \neq y_{n+1}$ for each $n \in \mathbb{N}$. Then following Case 2 of the proof of Theorem 3.1, it is shown that $\{y_n\}$ is a D-Cauchy sequence. Since $g(X)$ is complete, there is a point $z \in g(X)$ such that $\lim_n y_n = z = \lim_n gx_n$. We will show that $\lim_n Fx_n = \{z\}$.

Now,

$$\begin{aligned}
&\lim_n \delta^r(Fx_n, Fx_n, z)\\
&\quad = \lim_n \delta^r(Fx_n, Fx_n, y_{n+1})\\
&\quad \leq \lim_n \delta^r(Fx_n, Fx_n, Fx_n)\\
&\quad \leq \lim_n \phi(\max\{\rho^r(gx_n, gx_n, gx_n), \delta^r(Fx_n, Fx_n, gx_n), \delta^r(gx_n, Fx_n, gx_n)\})\\
&\quad = \phi\Big(\max\Big\{0, \lim_n \delta^r(Fx_n, Fx_n, z)\Big\}\Big)\\
&\quad = \phi\Big(\lim_n \delta^r(Fx_n, Fx_n, z)\Big),
\end{aligned} \tag{3.44}$$

which gives that $\lim_n Fx_n = \{z\}$.

Since $z \in g(X)$, there is a point $u \in X$ such that $gu = u$. We will show that $Fu = \{z\} = gu$. Now,

$$\begin{aligned}
&\delta^r(Fu, z, z)\\
&\quad = \lim_n \delta^r(Fu, Fx_n, Fx_n)\\
&\quad = \lim_n \delta^r(Fx_n, Fx_n, Fu)\\
&\quad \leq \lim_n \phi(\max\{\rho^r(gu, gx_n, gx_n), \delta^r(Fx_n, Fx_n, gu), \delta^r(gx_n, Fx_n, gu)\})\\
&\quad = \phi(\max\{0,0,0\})\\
&\quad = \phi(0)\\
&\quad = 0
\end{aligned} \tag{3.45}$$

and so $Fu = gu = \{z\}$. Thus u is a coincidence point of F and g. The rest of the proof is similar to Case 2 of the proof of Theorem 3.1. We omitted the details.

□

As a consequence of Theorem 3.10, we obtain the following corollaries.

Corollary 3.11. *Let $f,g : X \to X$ be two mappings satisfying (3.35). Suppose that*
(a) $f(X) \subseteq g(X)$,
(b) $g(X)$ *is bounded and complete,*
(c) $\{f,g\}$ *is coincidentally commuting.*
Then f and g have a unique common fixed point u and if g is continuous at u, then f is also continuous at u.

Corollary 3.12. *Let X be a D-metric space and let $f,g : X \to X$ be two mappings satisfying*

$$\begin{aligned}&\rho(fx,fy,fz)\\&\quad\le \lambda\max\{\rho(gx,gy,gz),\rho(fx,fy,gz),\rho(gx,fx,gz),\\&\qquad\qquad \rho(gy,fy,gz),\rho(gx,fy,gz),\rho(gy,fx,gz)\}\end{aligned} \tag{3.46}$$

for all $x,y,z \in X$, where $0 \le \lambda < 1$. Further suppose that hypotheses (a), (b), and (c) of Corollary 3.11 hold. Then f and g have a unique common fixed point $u \in X$ and if g is continuous at u, then f is also continuous at u.

Corollary 3.12 includes a common fixed-point theorem of Dhage [5] for the mappings f and g on a D-metric space characterized by the inequality

$$\begin{aligned}&\rho(fx,fy,gz)\\&\quad\le \lambda\max\{\rho(gx,gy,gz),\rho(gx,fx,gz),\\&\qquad\qquad \rho(gy,fy,gz),\rho(gx,fy,gz),\rho(gy,fx,gz)\}\end{aligned} \tag{3.47}$$

for all $x,y,z \in X$ and $0 \le \lambda < 1$.

Corollary 3.13. *Let X be a D-metric space and let $f,g : X \to X$ be two mappings satisfying (3.37). Further suppose that*
(a) $f^p(X) \subseteq g^q(X)$,
(b) $g^p(X)$ *is bounded and complete,*
(c) $\{f,g\}$ *is commuting.*
Then f and g have a unique common fixed point u and if g^q is continuous at u, then f^p is also continuous at u.

Notice that Corollary 3.13 includes a class of common fixed-point mappings f and g on a D-metric space X characterized by the inequality

$$\begin{aligned}&\rho(f^px,f^py,f^pz)\\&\quad\le \lambda\max\{\rho(g^qx,g^qy,g^qz),\rho(g^qx,f^px,g^qz),\\&\qquad\qquad \rho(g^qy,f^py,g^qz),\rho(g^qx,f^py,g^qz),\rho(g^qy,f^px,g^qz)\}\end{aligned} \tag{3.48}$$

for all $x,y,z \in X$ and $0 \le \lambda < 1$. See [5].

4. Weak commuting mappings in compact D-metric spaces. In this section, we prove some common fixed-point theorems for the pairs of singlevalued and multivalued coincidentally commuting mappings on a D-metric space satisfying a contraction condition more general than (4.3). But in this case the D-metric space under consideration is required to satisfy a stronger condition of compactness and the mappings under consideration are required to satisfy the continuity condition on the D-metric spaces. Our results of this section generalize some earlier known fixed-point theorems such as those of Dhage [9] and Rhoades [12] for single maps as well as for a pair of maps on D-metric spaces.

Theorem 4.1. *Let X be a compact D-metric space and let $F : X \to \mathrm{CB}(X)$ and $g : X \to X$ be two continuous mappings satisfying, for some positive real number r,*

$$\begin{aligned}
&\delta^r(Fx,Fy,Fz)\\
&\quad< \max\{\rho^r(gx,gy,gz),\delta^r(Fx,Fy,gz),\delta^r(gx,Fx,gz),\\
&\qquad\delta^r(gy,Fy,gz),\delta^r(gx,Fy,gz),\delta^r(gy,Fx,gz)\}
\end{aligned}\tag{4.1}$$

for all $x,y,z \in X$ for which the right-hand side is not zero. Further suppose that

(a) *$F(X) \subseteq g(X)$,*

(b) *$\{F,g\}$ is limit coincidentally commuting.*

Then F and g have a unique common fixed point $u \in X$ such that $Fu = \{u\} = gu$.

Proof. From inequality (4.3), it follows that if F and g have a common fixed point $u \in X$, then it is unique and $Fu = \{u\} = gu$. Since X is compact and δ is continuous, both sides of inequality (4.1) are bounded on X. Now, there are two cases.

Case 1. Suppose that the right-hand side of (4.1) is zero for some $x,y,z \in X$. Then, we have

$$Fx = gx = gz, \qquad Fy = gy = gz. \tag{4.2}$$

Now, proceeding with the arguments similar to Case 1 of the proof of Theorem 3.1, it is proved that $u = Fx = gx$ is a common fixed point of F and g and so it is unique.

Case 2. Suppose that the right-hand side of inequality (4.1) is not zero for all $x,y,z \in X$. Define a mapping $T : X\times X\times X \to (0,\infty)$ by

$$T(x,y,z) = \frac{\delta^r(Fx,Fy,Fz)}{M(x,y,z)}, \tag{4.3}$$

where

$$\begin{aligned}
M(x,y,z) = \max\{&\rho^r(gx,gy,gz),\delta^r(Fx,Fy,gz),\delta^r(gx,Fx,gz),\\
&\delta^r(gy,Fy,gz),\delta^r(gx,Fy,gz),\delta^r(gy,Fx,gz)\}.
\end{aligned}\tag{4.4}$$

Clearly, the function T is well defined since $M(x,y,z) \neq 0$ for all $x,y,z \in X$. Since F and g are continuous, from the compactness of X it follows that the function T attains its maximum on X^3 at some point $u,v,w \in X$. Call the value c. It is clear from (4.1) that $0 < c < 1$. By the definition of c, we have $T(x,y,z) \leq c$ for all $x,y,z \in X$. This further, in view of (4.3), implies that

$$\begin{aligned}\delta^r(Fx,Fy,Fz) & \\ &\leq cM(x,y,z) \\ &= c\max\{\rho^r(gx,gy,gz),\delta^r(Fx,Fy,Fz),\delta^r(gx,Fx,gx), \\ &\qquad \delta^r(gy,Fy,gz),\delta^r(gx,Fy,gz),\delta^r(gy,Fx,gz)\}\end{aligned} \tag{4.5}$$

for all $x,y,z \in X$.

As X is compact, it is complete and $g(X)$ is bounded in view of the continuity of g on X. Now, the desired conclusion follows by an application of Theorem 3.1. This completes the proof. □

Now we derive some interesting corollaries.

Corollary 4.2. *Let X be a compact D-metric space and let $F : X \to \mathrm{CB}(X)$ be a continuous mapping satisfying*

$$\begin{aligned}\delta(Fx,Fy,Fz) < \max\{&\rho(x,y,z),\delta(Fx,Fy,z),\delta(x,Fx,z), \\ &\delta(y,Fy,z),\delta(x,Fy,z),\delta(y,Fx,z)\}\end{aligned} \tag{4.6}$$

for all $x,y,z \in X$ for which the right-hand side is not zero. Then F has a unique fixed point $u \in X$ such that $Fu = \{u\}$.

Proof. The proof follows by letting $g = I$ in Theorem 4.1, where I is the identity map on X. □

Corollary 4.3 (see [3]). *Let X be a compact D-metric space and let $F : X \to \mathrm{CB}(X)$ be a continuous mapping satisfying*

$$\delta(Fx,Fy,Fz) < \rho(x,y,z) \tag{4.7}$$

for all $x,y,z \in X$ for which $\rho(x,y,z) \neq 0$. Then F has a unique fixed point $u \in X$ such that $Fu = \{u\}$.

Corollary 4.4. *Let X be a compact D-metric space and let $f,g : X \to X$ be two continuous mappings satisfying*

$$\begin{aligned}\rho(fx,fy,fz) < \max\{&\rho(gx,gy,gz),\rho(fx,fy,gz),\rho(gx,fx,gz), \\ &\rho(gy,fy,gz),\rho(gx,fy,gz),\rho(gy,fx,gz)\}\end{aligned} \tag{4.8}$$

for all $x,y,z \in X$ for which the right-hand side is not zero. Further suppose that

(a) $f(X) \subseteq g(X)$,

(b) $\{f,g\}$ *is limit coincidentally commuting.*

Then f and g have a unique common fixed point.

Proof. The proof follows by letting $F = \{f\}$, a single-valued mapping in Theorem 4.1. □

Corollary 4.5. *Let X be a compact D-metric space and let $f : X \to X$ be a continuous mapping satisfying*

$$\rho(fx,fy,fz) < \max\{\rho(x,y,z),\rho(fx,fy,z),\rho(x,fx,z),\\ \rho(y,fy,z),\rho(x,fy,z),\rho(y,fx,z)\} \tag{4.9}$$

for all $x,y,z \in X$ for which the right-hand side is not zero. Then f has a unique fixed point.

Proof. The conclusion follows by letting $g = I$ in Corollary 4.4, where I is the identity map on X. □

Note that Corollaries 4.4 and 4.5 include the fixed-point theorems of Dhage [5] and Rhoades [12] for the mappings f and g on a D-metric space X characterized by the inequalities

$$\rho(fx,fy,fz) < \max\{\rho(gx,gy,gz),\rho(gx,fx,gz),\\ \rho(gy,fy,gz),\rho(gx,fy,gz),\rho(gy,fx,gz)\}, \tag{4.10}$$

$$\rho(fx,fy,fz) < \max\{\rho(x,y,z),\rho(x,fx,z),\\ \rho(y,fy,z),\rho(x,fy,z),\rho(y,fx,z)\}, \tag{4.11}$$

respectively.

Theorem 4.6. *Let X be a D-metric space and let $F : X \to \mathrm{CB}(X)$, $g : X \to X$ be two continuous mappings satisfying (4.1). Suppose further that*

(a) $F(X) \subseteq g(X)$,

(b) $g(X)$ *is compact,*

(c) $\{f,g\}$ *is coincidentally commuting.*

Then F and g have a unique common fixed point $u \in X$ such that $Fu = \{u\} = gu$.

Proof. Let $A = g(X)$. Then A is a compact D-metric space and F and g define the maps $F : A \to \mathrm{CB}(A)$ and $g : A \to A$. Now, the desired conclusion follows by an application of Theorem 4.1. □

Corollary 4.7. *Let X be a D-metric space and let $f,g : X \to X$ be two continuous mappings satisfying (4.8). Further suppose that*

(a) $f(X) \subseteq g(X)$,
(b) $g(X)$ *is compact,*
(c) $\{f,g\}$ *is coincidentally commuting.*

Then f *and* g *have a unique common fixed point.*

5. Remarks and conclusion. It has been noted in [6, 10] that the fixed-point theorems for the limit coincidentally commuting mappings have some nice applications to approximation theory, and therefore it is of interest to discuss the fixed-point theorems for a wide class of coincidentally commuting mappings in a D-metric space. The terms "compatible" and "δ-compatible" have been used by Jungck and Rhoades [11] for limit coincidentally commuting and coincidentally commuting mappings, respectively, but our terminologies are natural and more informative than the previous one patterned after [4]. Further we note that a similar study can be made for coincidentally pseudocommuting mappings on a D-metric space and analogously for limit coincidentally pseudocommuting mappings. But in order to prove fixed-point theorems for these classes of weakly pseudocommuting mappings, we require a stronger contraction condition for the mappings F and g under consideration:

$$\begin{aligned}\delta^r(Fx,Fy,Fz) & \\ \leq \phi(\max\{&\rho^r(gx,gy,gz),D^r(Fx,Fy,gz),D^r(gx,Fx,gz),\\ &D^r(gy,Fy,gz),D^r(gx,Fy,gz),D^r(gy,Fx,gz)\}).\end{aligned} \tag{5.1}$$

Obviously, condition (5.1) implies condition (3.2) on a D-metric space X and hence the fixed-point theorems for weakly pseudocommuting mappings can be obtained very easily with appropriate modifications. Finally, we close this discussion with the following open question.

Open question. Can we prove fixed-point theorems for a class of multivalued mapping F on a D-metric space X satisfying the generalized contraction condition

$$\begin{aligned}\kappa(Fx,Fy,Fz) \leq \lambda\max\{&\rho(x,y,z),D(Fx,Fy,z),D(x,Fx,z),\\ &D(y,Fy,z),D(x,Fy,z),D(y,Fx,z)\}\end{aligned} \tag{5.2}$$

for all $x,y,z \in X$ and $0 \leq \lambda < 1$?

Acknowledgment. The authors are thankful to Professor B. E. Rhoades (USA) for giving some useful suggestions for the improvement of this paper.

References

[1] B. C. Dhage, *A study of some fixed point theorem*, Ph.D. thesis, Marathwada University, Aurangabad, India, 1984.
[2] ———, *Generalised metric spaces and mappings with fixed point*, Bull. Calcutta Math. Soc. **84** (1992), no. 4, 329–336.

[3] ______, *Generalized D-metric spaces and multi-valued contraction mappings*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **44** (1998), no. 1, 179–200.

[4] ______, *A common fixed point principle in D-metric spaces*, Bull. Calcutta Math. Soc. **91** (1999), no. 6, 475–480.

[5] ______, *On common fixed points of pairs of coincidentally commuting mappings in D-metric spaces*, Indian J. Pure Appl. Math. **30** (1999), no. 4, 395–406.

[6] ______, *Applications of some fixed point theorems in approximation theory*, Math. Sci. Res. Hot-Line **4** (2000), no. 9, 63–69.

[7] ______, *A common fixed point theorem for a pair of limit coincidentally commuting mappings in D-metric spaces*, Math. Sci. Res. Hot-Line **4** (2000), no. 2, 45–55.

[8] ______, *Generalized metric spaces and topological structure. I*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **46** (2000), no. 1, 3–24.

[9] ______, *On some fixed point theorems for contractive mapping in D-metric spaces*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **46** (2000), no. 1, 31–38.

[10] ______, *A result in best approximation theory*, Indian J. Pure Appl. Math. **32** (2001), no. 8, 1209–1216.

[11] G. Jungck and B. E. Rhoades, *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math. **29** (1998), no. 3, 227–238.

[12] B. E. Rhoades, *A fixed point theorem for generalized metric spaces*, Int. J. Math. Math. Sci. **19** (1996), no. 3, 457–460.

B. C. Dhage: Kasubai, Gurukul Colony, Ahmedpur 413515, Maharashtra, India
E-mail address: bcd20012001@yahoo.co.in

A. Jennifer Asha: Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai 600005, India

S. M. Kang: Department of Mathematics and Research Institute of Natural Science, Gyeongsang National University, Chinju 660-701, Korea
E-mail address: smkang@nongae.gsnu.ac.kr

Advances in Difference Equations

Special Issue on Boundary Value Problems on Time Scales

Call for Papers

The study of dynamic equations on a time scale goes back to its founder Stefan Hilger (1988), and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics; moreover, it often revels the reasons for the discrepancies between two theories.

In recent years, the study of dynamic equations has led to several important applications, for example, in the study of insect population models, neural network, heat transfer, and epidemic models. This special issue will contain new researches and survey articles on Boundary Value Problems on Time Scales. In particular, it will focus on the following topics:

- Existence, uniqueness, and multiplicity of solutions
- Comparison principles
- Variational methods
- Mathematical models
- Biological and medical applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/ade/guidelines.html. Authors should follow the Advances in Difference Equations manuscript format described at the journal site http://www.hindawi.com/journals/ade/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	April 1, 2009
First Round of Reviews	July 1, 2009
Publication Date	October 1, 2009

Lead Guest Editor

Alberto Cabada, Departamento de Análise Matemática, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain; alberto.cabada@usc.es

Guest Editor

Victoria Otero-Espinar, Departamento de Análise Matemática, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain; mvictoria.otero@usc.es

Hindawi Publishing Corporation
http://www.hindawi.com