ON SOME BK SPACES

BRUNO DE MALAFOSSE

We characterize the spaces $s_\alpha(\Delta)$, $s_\alpha^\circ(\Delta)$, and $s_\alpha^{(c)}(\Delta)$ and we deal with some sets generalizing the well-known sets $w_0(\lambda)$, $w_\infty(\lambda)$, $w(\lambda)$, $c_0(\lambda)$, $c_\infty(\lambda)$, and $c(\lambda)$.

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1. Notations and preliminary results. For a given infinite matrix $A = (a_{nm})_{n,m \geq 1}$, the operators A_n, for any integer $n \geq 1$, are defined by

$$A_n(X) = \sum_{m=1}^\infty a_{nm} x_m, \tag{1.1}$$

where $X = (x_n)_{n \geq 1}$ is the series intervening in the second member being convergent. So, we are led to the study of the infinite linear system

$$A_n(X) = b_n, \quad n = 1, 2, \ldots, \tag{1.2}$$

where $B = (b_n)_{n \geq 1}$ is a one-column matrix and X the unknown, see [2, 3, 5, 6, 7, 9]. Equation (1.2) can be written in the form $AX = B$, where $AX = (A_n(X))_{n \geq 1}$. In this paper, we will also consider A as an operator from a sequence space into another sequence space.

A Banach space E of complex sequences with the norm $\|\ \|_E$ is a BK space if each projection $P_n : X \to P_n X$ is continuous. A BK space E is said to have AK (see [8]) if for every $B = (b_n)_{n \geq 1}$, $B = \sum_{n=1}^\infty b_m e_m$, that is,

$$\left\| \sum_{m=N+1}^\infty b_m e_m \right\|_E \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{1.3}$$

We shall write s, c, c_0, and l_∞ for the sets of all complex, convergent sequences, sequences convergent to zero, and bounded sequences, respectively. We shall write cs and l_1 for the sets of convergent and absolutely convergent series, respectively. We will use the set

$$U^{+*} = \{(u_n)_{n \geq 1} \in s \mid u_n > 0 \ \forall n\}. \tag{1.4}$$

Using Wilansky's notations [12], we define, for any sequence $\alpha = (\alpha_n)_{n \geq 1} \in U^{+*}$ and for any set of sequences E, the set

$$\alpha * E = \left\{ (x_n)_{n \geq 1} \in s \mid \left(\frac{x_n}{\alpha_n} \right)_n \in E \right\}. \tag{1.5}$$

Writing

$$\alpha * E = \begin{cases} s_\alpha^\circ & \text{if } E = c_0, \\ s_\alpha^{(c)} & \text{if } E = c, \\ s_\alpha & \text{if } E = l_\infty, \end{cases} \tag{1.6}$$

we have for instance

$$\alpha * c_0 = s_\alpha^\circ = \{(x_n)_{n \geq 1} \in s \mid x_n = o(\alpha_n) \ n \longrightarrow \infty\}. \tag{1.7}$$

Each of the spaces $\alpha * E$, where $E \in \{c_0, c, l_\infty\}$, is a BK space normed by

$$\|X\|_{s_\alpha} = \sup_{n \geq 1} \left(\frac{|x_n|}{\alpha_n} \right), \tag{1.8}$$

and s_α° has AK.

Now, let $\alpha = (\alpha_n)_{n \geq 1}$ and $\beta = (\beta_n)_{n \geq 1} \in U^{+*}$. We shall write $S_{\alpha,\beta}$ for the set of infinite matrices $A = (a_{nm})_{n,m \geq 1}$ such that

$$(a_{nm}\alpha_m)_{m \geq 1} \in l^1 \quad \forall n \geq 1, \qquad \sum_{m=1}^\infty |a_{nm}| \alpha_m = O(\beta_n) \quad (n \longrightarrow \infty). \tag{1.9}$$

The set $S_{\alpha,\beta}$ is a Banach space with the norm

$$\|A\|_{S_{\alpha,\beta}} = \sup_{n \geq 1} \left(\sum_{m=1}^\infty |a_{nm}| \frac{\alpha_m}{\beta_n} \right). \tag{1.10}$$

Let E and F be any subsets of s. When A maps E into F, we write $A \in (E, F)$, see [10]. So, for every $X \in E$, $AX \in F$ ($AX \in F$ will mean that for each $n \geq 1$, the series defined by $y_n = \sum_{m=1}^\infty a_{nm} x_m$ is convergent and $(y_n)_{n \geq 1} \in F$). It has been proved in [8] that $A \in (s_\alpha, s_\beta)$ if and only if $A \in S_{\alpha,\beta}$. So, we can write $(s_\alpha, s_\beta) = S_{\alpha,\beta}$.

When $s_\alpha = s_\beta$, we obtain the unital Banach algebra $S_{\alpha,\beta} = S_\alpha$, (see [2, 3, 9]) normed by $\|A\|_{S_\alpha} = \|A\|_{S_{\alpha,\alpha}}$.

We also have $A \in (s_\alpha, s_\alpha)$ if and only if $A \in S_\alpha$. If $\|I - A\|_{S_\alpha} < 1$, we say that $A \in \Gamma_\alpha$. The set S_α being a unital algebra, we have the useful result: if $A \in \Gamma_\alpha$, A is bijective from s_α into itself.

If $\alpha = (r^n)_{n \geq 1}$, then Γ_α, S_α, s_α, s_α°, and $s_\alpha^{(c)}$ are replaced by Γ_r, S_r, s_r, s_r°, and $s_r^{(c)}$, respectively, (see [2, 3, 5, 6, 7, 8, 9]). When $r = 1$, we obtain $s_1 = l_\infty$, $s_1^\circ = c_0$, and $s_1^{(c)} = c$, and putting $e = (1,1,\ldots)$, we have $S_1 = S_e$. It is well known, see [10], that

$$(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1. \qquad (1.11)$$

We write $e_n = (0,\ldots,1,\ldots)$ (where 1 is in the nth position).

For any subset E of s, we put

$$AE = \{Y \in s \mid \exists X \in E \ Y = AX\}. \qquad (1.12)$$

If F is a subset of s, we denote

$$F(A) = F_A = \{X \in s \mid Y = AX \in F\}. \qquad (1.13)$$

We can see that $F(A) = A^{-1}F$.

2. Sets $s_\alpha(\Delta)$, $s_\alpha^\circ(\Delta)$, and $s_\alpha^{(c)}(\Delta)$. In this section, we will give necessary and sufficient conditions permitting us to write the sets $s_\alpha(\Delta)$, $s_\alpha^\circ(\Delta)$, and $s_\alpha^{(c)}(\Delta)$ by means of the spaces s_ξ, s_ξ°, or $s_\xi^{(c)}$. For this, we need to study the sequence $C(\alpha)\alpha$.

2.1. Properties of the sequence $C(\alpha)\alpha$. Here, we will deal with the operators represented by $C(\lambda)$ and $\Delta(\lambda)$, see [2, 5, 7, 8, 9].

Let

$$U = \{(u_n)_{n \geq 1} \in s \mid u_n \neq 0 \ \forall n\}. \qquad (2.1)$$

We define $C(\lambda) = (c_{nm})_{n,m \geq 1}$, for $\lambda = (\lambda_n)_{n \geq 1} \in U$, by

$$c_{nm} = \begin{cases} \dfrac{1}{\lambda_n} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases} \qquad (2.2)$$

It can be proved that the matrix $\Delta(\lambda) = (c'_{nm})_{n,m \geq 1}$, with

$$c'_{nm} = \begin{cases} \lambda_n & \text{if } m = n, \\ -\lambda_{n-1} & \text{if } m = n-1, \ n \geq 2, \\ 0 & \text{otherwise,} \end{cases} \qquad (2.3)$$

is the inverse of $C(\lambda)$, see [8]. If $\lambda = e$, we get the well-known operator of first difference represented by $\Delta(e) = \Delta$ and it is usually written $\Sigma = C(e)$. Note that $\Delta = \Sigma^{-1}$, and Δ and Σ belong to any given space S_R with $R > 1$.

We use the following sets:

$$\hat{C} = \left\{ \alpha \in U^{+*} \mid C(\alpha)\alpha = \left(\frac{1}{\alpha_n} \left(\sum_{k=1}^{n} \alpha_k \right) \right)_{n \geq 1} \in c \right\},$$

$$\widehat{C_1} = \{ \alpha \in U^{+*} \mid C(\alpha)\alpha \in s_1 = l_\infty \}, \tag{2.4}$$

$$\Gamma = \left\{ \alpha \in U^{+*} \mid \overline{\lim_{n \to \infty}} \left(\frac{\alpha_{n-1}}{\alpha_n} \right) < 1 \right\}.$$

Note that $\Delta \in \Gamma_\alpha$ implies $\alpha \in \Gamma$. It can be easily seen that $\alpha \in \Gamma$ if and only if there is an integer $q \geq 1$ such that

$$\gamma_q(\alpha) = \sup_{n \geq q+1} \left(\frac{\alpha_{n-1}}{\alpha_n} \right) < 1. \tag{2.5}$$

See [7].

In order to express the following results, we will denote by $[C(\alpha)\alpha]_n$ (instead of $[C(\alpha)]_n(\alpha)$) the nth coordinate of $C(\alpha)\alpha$. We get the following proposition.

PROPOSITION 2.1. *Let $\alpha \in U^{+*}$. Then*
(i) *$\alpha_{n-1}/\alpha_n \to 0$ if and only if $[C(\alpha)\alpha]_n \to 1$;*
(ii) (a) *$\alpha \in \hat{C}$ implies that $(\alpha_{n-1}/\alpha_n)_{n \geq 1} \in c$,*
　　 (b) *$[C(\alpha)\alpha]_n \to l$ implies that $\alpha_{n-1}/\alpha_n \to 1 - 1/l$;*
(iii) *if $\alpha \in \widehat{C_1}$, there are $K > 0$ and $\gamma > 1$ such that*

$$\alpha_n \geq K\gamma^n \quad \forall n; \tag{2.6}$$

(iv) *the condition $\alpha \in \Gamma$ implies that $\alpha \in \widehat{C_1}$ and there exists a real $b > 0$ such that*

$$[C(\alpha)\alpha]_n \leq \frac{1}{1-\chi} + b\chi^n \quad \text{for } n \geq q+1, \ \chi = \gamma_q(\alpha) \in \,]0,1[. \tag{2.7}$$

PROOF. (i) Assume that $\alpha_{n-1}/\alpha_n \to 0$. Then there is an integer N such that

$$n \geq N+1 \Longrightarrow \frac{\alpha_{n-1}}{\alpha_n} \leq \frac{1}{2}. \tag{2.8}$$

So, there exists a real $K > 0$ such that $\alpha_n \geq K2^n$ for all n and

$$\frac{\alpha_k}{\alpha_n} = \frac{\alpha_k}{\alpha_{k+1}} \cdots \frac{\alpha_{n-1}}{\alpha_n} \leq \left(\frac{1}{2} \right)^{n-k} \quad \text{for } N \leq k \leq n-1. \tag{2.9}$$

Then

$$\frac{1}{\alpha_n}\left(\sum_{k=1}^{n-1}\alpha_k\right) = \frac{1}{\alpha_n}\left(\sum_{k=1}^{N-1}\alpha_k\right) + \sum_{k=N}^{n-1}\frac{\alpha_k}{\alpha_n}$$

$$\leq \frac{1}{K2^n}\left(\sum_{k=1}^{N-1}\alpha_k\right) + \sum_{k=N}^{n-1}\left(\frac{1}{2}\right)^{n-k};$$

(2.10)

and since

$$\sum_{k=N}^{n-1}\left(\frac{1}{2}\right)^{n-k} = 1 - \left(\frac{1}{2}\right)^{n-N} \longrightarrow 1 \quad (n \longrightarrow \infty),$$

(2.11)

we deduce that

$$\frac{1}{\alpha_n}\left(\sum_{k=1}^{n-1}\alpha_k\right) = O(1), \qquad ([C(\alpha)\alpha]_n) \in l_\infty.$$

(2.12)

Using the identity

$$[C(\alpha)\alpha]_n = \frac{\alpha_1 + \cdots + \alpha_{n-1}}{\alpha_{n-1}}\frac{\alpha_{n-1}}{\alpha_n} + 1$$

$$= [C(\alpha)\alpha]_{n-1}\left(\frac{\alpha_{n-1}}{\alpha_n}\right) + 1,$$

(2.13)

we get $[C(\alpha)\alpha]_n \to 1$. This proves the necessity.

Conversely, if $[C(\alpha)\alpha]_n \to 1$, then

$$\frac{\alpha_{n-1}}{\alpha_n} = \frac{[C(\alpha)\alpha]_n - 1}{[C(\alpha)\alpha]_{n-1}} \longrightarrow 0.$$

(2.14)

Assertion (ii) is a direct consequence of identity (2.14).

(iii) We put $\Sigma_n = \sum_{k=1}^n \alpha_k$. Then for a real $M > 1$,

$$[C(\alpha)\alpha]_n = \frac{\Sigma_n}{\Sigma_n - \Sigma_{n-1}} \leq M \quad \forall n.$$

(2.15)

So, $\Sigma_n \geq (M/(M-1))\Sigma_{n-1}$ and $\Sigma_n \geq (M/(M-1))^{n-1}\alpha_1$ for all n. Therefore, from

$$\frac{\alpha_1}{\alpha_n}\left(\frac{M}{M-1}\right)^{n-1} \leq [C(\alpha)\alpha]_n = \frac{\Sigma_n}{\alpha_n} \leq M,$$

(2.16)

we conclude that $\alpha_n \geq K\gamma^n$ for all n, with $K = (M-1)\alpha_1/M^2$ and $\gamma = M/(M-1) > 1$.

(iv) If $\alpha \in \Gamma$, then there is an integer $q \geq 1$ for which

$$k \geq q + 1 \Longrightarrow \frac{\alpha_{k-1}}{\alpha_k} \leq \chi < 1 \quad \text{with } \chi = y_q(\alpha). \tag{2.17}$$

So, there is a real $M' > 0$ for which

$$\alpha_n \geq \frac{M'}{\chi^n} \quad \forall n \geq q + 1. \tag{2.18}$$

Writing $\sigma_{nq} = 1/\alpha_n (\sum_{k=1}^q \alpha_k)$ and $d_n = [C(\alpha)\alpha]_n - \sigma_{nq}$, we get

$$
\begin{aligned}
d_n &= \frac{1}{\alpha_n}\left(\sum_{k=q+1}^n \alpha_k\right) = 1 + \sum_{j=q+1}^{n-1}\left(\prod_{k=1}^{n-j} \frac{\alpha_{n-k}}{\alpha_{n-k+1}}\right) \\
&\leq \sum_{j=q+1}^n \chi^{n-j} \leq \frac{1}{1-\chi}.
\end{aligned}
\tag{2.19}
$$

And using (2.18), we get

$$\sigma_{nq} \leq \frac{1}{M'}\chi^n\left(\sum_{k=1}^q \alpha_k\right). \tag{2.20}$$

So

$$[C(\alpha)\alpha]_n \leq a + b\chi^n \tag{2.21}$$

with $a = 1/(1-\chi)$ and $b = (1/M')(\sum_{k=1}^q \alpha_k)$. $\square$

REMARK 2.2. Note that $\alpha \in \widehat{C_1}$ does not imply that $\alpha \in \Gamma$.

2.2. New properties of the operator represented by Δ. Throughout this paper, we will denote by D_ξ the infinite diagonal matrix $(\xi_n \delta_{nm})_{n,m \geq 1}$ for any given sequence $\xi = (\xi_n)_{n \geq 1}$. Now, we require some lemmas.

LEMMA 2.3. *The condition* $\Delta \in (s_\alpha^\circ, s_\alpha^\circ)$ *is equivalent to* $\Delta_\alpha = D_{1/\alpha}\Delta D_\alpha \in (c_0, c_0)$ *and* $\Delta \in (s_\alpha^{(c)}, s_\alpha^{(c)})$ *implies* $\Delta_\alpha = D_{1/\alpha}\Delta D_\alpha \in (c, c)$.

PROOF. First, D_α is bijective from c_0 into s_α°. In fact, the equation $D_\alpha X = B$, for every $B = (b_n)_n \in s_\alpha^\circ$, admits a unique solution $X = D_{1/\alpha}B = (b_n/\alpha_n)_n \in c_0$. Suppose now that $\Delta \in (s_\alpha^\circ, s_\alpha^\circ)$. Then for every $X \in c_0$, we get successively $X' = D_\alpha X \in s_\alpha^\circ$, $\Delta X' \in s_\alpha^\circ$, and $\Delta_\alpha = D_{1/\alpha}\Delta D_\alpha \in (c_0, c_0)$. Conversely, assume that $\Delta_\alpha \in (c_0, c_0)$ and let $X \in s_\alpha^\circ$. Then $X = D_\alpha X'$ with $X' \in c_0$. So, $\Delta X \in D_\alpha c_0 = s_\alpha^\circ$ and $\Delta \in (s_\alpha^\circ, s_\alpha^\circ)$. By a similar reasoning, we get $\Delta \in (s_\alpha^{(c)}, s_\alpha^{(c)}) \Rightarrow \Delta_\alpha \in (c, c)$.
                                                                                       $\square$

We need to recall here the following well-known results given in [12].

LEMMA 2.4. *The condition $A \in (c, c)$ is equivalent to the following conditions:*
  (i) $A \in S_1$;
 (ii) $(a_{nm})_{n \geq 1} \in c$ *for each $m \geq 1$;*
(iii) $(\sum_{m=1}^{\infty} a_{nm})_{n \geq 1} \in c$.

If for any given sequence $X = (x_n)_n \in c$, with $\lim_n x_n = l$, $A_n(X)$ is convergent for all n and $\lim_n A_n(X) = l$, it is written that

$$\lim X = A - \lim X, \tag{2.22}$$

and A is called a Toeplitz matrix. We also have the next result.

LEMMA 2.5. *The operator $A \in (c, c)$ is a Toeplitz matrix if and only if*
  (i) $A \in S_1$;
 (ii) $\lim_n a_{nm} = 0$ *for each $m \geq 1$;*
(iii) $\lim_n(\sum_{m=1}^{\infty} a_{nm}) = 1$.

Now, we can assert the following theorem.

THEOREM 2.6. *We have successively*
  (i) $s_\alpha(\Delta) = s_\alpha$ *if and only if $\alpha \in \widehat{C}_1$;*
 (ii) $s_\alpha^\circ(\Delta) = s_\alpha^\circ$ *if and only if $\alpha \in \widehat{C}_1$;*
(iii) $s_\alpha^{(c)}(\Delta) = s_\alpha^{(c)}$ *if and only if $\alpha \in \widehat{C}$;*
(iv) $\Delta_\alpha = D_{1/\alpha} \Delta D_\alpha$ *is bijective from c into itself with $\lim X = \Delta_\alpha - \lim X$ if and only if*

$$\frac{\alpha_{n-1}}{\alpha_n} \longrightarrow 0. \tag{2.23}$$

PROOF. (i) We have $s_\alpha(\Delta) = s_\alpha$ if and only if $\Delta, \Sigma \in (s_\alpha, s_\alpha)$. This means that $\Delta, \Sigma \in S_\alpha$, that is,

$$\|\Delta\|_{S_\alpha} = \sup_{n \geq 1}\left(1 + \frac{\alpha_{n-1}}{\alpha_n}\right) < \infty, \qquad \|\Sigma\|_{S_\alpha} = \sup_{n \geq 1}[C(\alpha)\alpha]_n < \infty. \tag{2.24}$$

Since $0 < \alpha_{n-1}/\alpha_n \leq [C(\alpha)\alpha]_n$, we deduce that $\Delta, \Sigma \in S_\alpha$ if and only if $\|\Sigma\|_{S_\alpha} < \infty$, that is, $\alpha \in \widehat{C}_1$.

(ii) From Lemma 2.3, if $s_\alpha^\circ(\Delta) = s_\alpha^\circ$, then $\Delta_\alpha = D_{1/\alpha}\Delta D_\alpha \in (c_0, c_0)$. So, $\Delta_\alpha \in (c_0, l_\infty) = S_1$ and since $\Delta_\alpha = (d_{nm})_{n,m \geq 1}$ with

$$d_{nm} = \begin{cases} 1 & \text{if } m = n, \\ -\dfrac{\alpha_{n-1}}{\alpha_n} & \text{if } m = n-1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.25}$$

we deduce that $\alpha_{n-1}/\alpha_n = O(1)$, $n \to \infty$. Further, $s_\alpha^\circ(\Delta) = s_\alpha^\circ$ implies $\Sigma_\alpha = D_{1/\alpha} \Sigma D_\alpha \in (c_0, c_0)$ and $\Sigma_\alpha \in (c_0, l_\infty) = S_1$. Since $\Sigma_\alpha = (\sigma_{nm})_{n,m \geq 1}$ with

$$\sigma_{nm} = \begin{cases} \dfrac{\alpha_m}{\alpha_n} & \text{if } m \leq n, \\ 0 & \text{if } m > n, \end{cases} \tag{2.26}$$

we deduce that

$$\sup_{n \geq 1} \left(\frac{1}{\alpha_n} \left(\sum_{k=1}^n \alpha_k \right) \right) < \infty, \tag{2.27}$$

that is, $\alpha \in \widehat{C_1}$. Conversely, assume that $\alpha \in \widehat{C_1}$. First, $\Delta \in (s_\alpha^\circ, s_\alpha^\circ)$. Indeed, from the inequality

$$\frac{\alpha_{n-1}}{\alpha_n} \leq \sup_{n \geq 1} \left([C(\alpha)\alpha]_n \right) < \infty, \tag{2.28}$$

we deduce that for every $X \in s_\alpha^\circ$, $x_n/\alpha_n = o(1)$,

$$\frac{x_n - x_{n-1}}{\alpha_n} = \frac{x_n}{\alpha_n} - \frac{x_{n-1}}{\alpha_{n-1}} \frac{\alpha_{n-1}}{\alpha_n} = o(1) \tag{2.29}$$

and $\Delta X \in s_\alpha^\circ$. Further, take $B = (b_n)_{n \geq 1} \in s_\alpha^\circ$. Then there exists $v = (v_n)_{n \geq 1} \in c_0$ such that $b_n = \alpha_n v_n$. We must prove that the equation $\Delta X = B$ admits a unique solution in the space s_α°. First, we obtain

$$X = \Sigma B = \left(\sum_{k=1}^n \alpha_k v_k \right)_{n \geq 1}. \tag{2.30}$$

In order to show that $X = (x_n)_{n \geq 1} \in s_\alpha^\circ$, we will consider any given $\varepsilon > 0$. From Proposition 2.1(iii), the condition $\alpha \in \widehat{C_1}$ implies that $\alpha_n \to \infty$. So, there exists an integer N such that

$$S_n = \frac{1}{\alpha_n} \left| \sum_{k=1}^N \alpha_k v_k \right| \leq \frac{\varepsilon}{2} \quad \text{for } n \geq N,$$

$$\sup_{n \geq N+1} \left(|v_k| \right) \leq \frac{\varepsilon}{2 \sup_{n \geq 1} \left([C(\alpha)\alpha]_n \right)}. \tag{2.31}$$

Writing $R_n = 1/\alpha_n |\sum_{k=N+1}^n \alpha_k v_k|$, we conclude that

$$R_n \leq \left(\sup_{N+1 \leq k \leq n} \left(|v_k| \right) \right) [C(\alpha)\alpha]_n \leq \frac{\varepsilon}{2}. \tag{2.32}$$

Finally, we obtain

$$
\frac{|x_n|}{\alpha_n} = \left| \frac{1}{\alpha_n} \left(\sum_{k=1}^{N} \alpha_k v_k \right) + \frac{1}{\alpha_n} \left(\sum_{k=N+1}^{n} \alpha_k v_k \right) \right| \tag{2.33}
$$
$$
\leq S_n + R_n \leq \varepsilon \quad \text{for } n \geq N,
$$

and $X \in s_\alpha^0$.

(iii) As above, $s_\alpha^{(c)}(\Delta) = s_\alpha^{(c)}$ if and only if $\Delta_\alpha, \Sigma_\alpha \in (c,c)$; and from Lemma 2.4, we have $\Delta_\alpha \in (c,c)$ if and only if $(\alpha_{n-1}/\alpha_n)_n \in c$. In fact, we have $\Delta_\alpha \in S_1$ and $\sum_{m=1}^{n} d_{nm} = 1 + \alpha_{n-1}/\alpha_n$ tends to a limit as $n \to \infty$. Afterwards, $\Sigma_\alpha \in (c,c)$ is equivalent to

  (a) $\Sigma_\alpha \in S_1$, that is, $\alpha \in \widehat{C_1}$;
  (b) $\lim_n (\alpha_m/\alpha_n) = 0$ for all $m \geq 1$;
  (c) $\alpha \in \hat{C}$.

From Proposition 2.1(iii), (c) implies that α_n tends to infinity, so (c) implies (a) and (b). Finally, from Proposition 2.1(ii), we conclude that $\alpha \in \hat{C}$ implies $(\alpha_{n-1}/\alpha_n)_n \in c$. This completes the proof of (iii).

(iv) From Lemma 2.5, it can be easily verified that $\Delta_\alpha \in (c,c)$ and $\lim X = \Delta_\alpha - \lim X$ if and only if $\alpha_{n-1}/\alpha_n \to 0$. We conclude, using (iii), since $\alpha_{n-1}/\alpha_n = o(1)$ implies that $\alpha \in \hat{C}$. □

REMARK 2.7. In Theorem 2.6(iv), we see that $\Sigma_\alpha \in (c,c)$ and $\lim X = \Sigma_\alpha - \lim X$ if and only if $\alpha_{n-1}/\alpha_n \to 0$. In fact, we must have for each $m \geq 1$, $\sigma_{nm} = \alpha_m/\alpha_n = o(1)$ $(n \to \infty)$ and

$$
\lim_n \left(\sum_{m=1}^{n} \sigma_{nm} \right) = \lim_n \left(1 + \sum_{m=1}^{n-1} \frac{\alpha_m}{\alpha_n} \right) = 1, \tag{2.34}
$$

and from Proposition 2.1(i), the previous property is satisfied if and only if $\alpha_{n-1}/\alpha_n \to 0$.

REMARK 2.8. It can be seen that the condition $(\alpha_{n-1}/\alpha_n)_n \in c$ does not imply that $\alpha \in \widehat{C_1}$. It is enough to consider $C(e)e = (n)_n \notin c_0$.

The next corollary is a direct consequence of the previous results.

COROLLARY 2.9. *Consider the following properties:*
  (i) $\alpha \in \hat{C}$;
  (ii) $s_\alpha^{(c)}(\Delta) = s_\alpha^{(c)}$;
  (iii) $\alpha \in \Gamma$;
  (iv) $\alpha \in \widehat{C_1}$;
  (v) $s_\alpha(\Delta) = s_\alpha$;
  (vi) $s_\alpha^\circ(\Delta) = s_\alpha^\circ$.
Then (i)⟺(ii)⟹(iii)⟹(iv)⟺(v)⟺(vi).

We obtain from the precedent the following corollary.

COROLLARY 2.10. (i) $(s_\alpha - s_\alpha^\circ)(\Delta) = s_\alpha - s_\alpha^\circ$ *if and only if* $\alpha \in \widehat{C_1}$,

(ii) $(s_\alpha^{(c)} - s_\alpha^\circ)(\Delta) = s_\alpha^{(c)} - s_\alpha^\circ$ *if and only if* $\alpha \in \widehat{C}$,

(iii) $\alpha \in \widehat{C}$ *implies* $(s_\alpha - s_\alpha^{(c)})(\Delta) = s_\alpha - s_\alpha^{(c)}$.

PROOF. (i) If Δ is bijective from $s_\alpha - s_\alpha^\circ$ into itself, then for every $B \in s_\alpha - s_\alpha^\circ$, we have $X = \Sigma B \in s_\alpha - s_\alpha^\circ$. Since $\alpha \in s_\alpha - s_\alpha^\circ$, we conclude that $\Sigma \alpha \in s_\alpha$, that is, $C(\alpha)\alpha \in l_\infty$. Conversely, from Theorem 2.6(i) and (ii), it can be easily seen that Δ is bijective from s_α to s_α and from s_α° to s_α°, since $\alpha \in \widehat{C_1}$. So, Δ is bijective from $s_\alpha - s_\alpha^\circ$ to $s_\alpha - s_\alpha^\circ$.

(ii) Suppose that Δ is bijective from $s_\alpha^{(c)} - s_\alpha^\circ$ into itself. Reasoning as above, we have $\alpha \in s_\alpha^{(c)} - s_\alpha^\circ$ and $\Sigma \alpha \in s_\alpha^{(c)}$, so $D_{1/\alpha}\Sigma\alpha = C(\alpha)\alpha \in c$. Conversely, using Theorem 2.6(i) and (iii), we see that Δ bijective from $s_\alpha^{(c)}$ to $s_\alpha^{(c)}$ and from s_α° to s_α° since $\alpha \in \widehat{C}$ and $\widehat{C} \subset \widehat{C_1}$. So, $(s_\alpha^{(c)} - s_\alpha^\circ)(\Delta) = s_\alpha^{(c)} - s_\alpha^\circ$.

Similarly, (iii) comes from the fact that Δ is bijective from $s_\alpha^{(c)}$ into itself and from s_α into itself, since $\alpha \in \widehat{C}$.   □

REMARK 2.11. Assume that $\lim_{n \to \infty}[C(\alpha)\alpha]_n = l$. Then

$$\frac{x_n}{\alpha_n} \longrightarrow L \text{ implies } \frac{x_n - x_{n-1}}{\alpha_n} \longrightarrow \frac{L}{l}. \tag{2.35}$$

Indeed, from Proposition 2.1(ii) (b), $\alpha_{n-1}/\alpha_n \to 1 - (1/l)$ and

$$\frac{x_n - x_{n-1}}{\alpha_n} = \frac{x_n}{\alpha_n} - \frac{x_{n-1}}{\alpha_{n-1}}\frac{\alpha_{n-1}}{\alpha_n} \longrightarrow L - L\left(1 - \frac{1}{l}\right) = \frac{L}{l}. \tag{2.36}$$

3. Generalization to the sets $s_r(\Delta^h)$ and $s_\alpha(\Delta^h)$ for h real. In this section, we consider the operator Δ^h, where h is a real, and give among other things a necessary and sufficient condition to have $s_\alpha(\Delta^h) = s_\alpha$.

First, recall that we can associate to any power series $f(z) = \sum_{k=0}^\infty a_k z^k$, defined in the open disk $|z| < R$, the upper triangular infinite matrix $A = \varphi(f) \in \bigcup_{0 < r < R} S_r$ defined by

$$\varphi(f) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdot \\ & a_0 & a_1 & \cdot \\ 0 & & a_0 & \cdot \\ & & & \cdot \end{pmatrix} \tag{3.1}$$

(see [3, 4, 5]). Practically, we will write $\varphi[f(z)]$ instead of $\varphi(f)$. We have the following lemma.

LEMMA 3.1. (i) *The map* $\varphi : f \to A$ *is an isomorphism from the algebra of the power series defined in* $|z| < R$ *into the algebra of the corresponding matrices* $\bar{A}$.

(ii) *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, with $a_0 \neq 0$, and assume that $1/f(z) = \sum_{k=0}^{\infty} a_k' z^k$ admits $R' > 0$ as radius of convergence. Then*

$$\varphi\left(\frac{1}{f}\right) = [\varphi(f)]^{-1} \in \bigcup_{0 < r < R'} s_r. \tag{3.2}$$

Now, for $h \in R - N$, we define (see [13])

$$\binom{-h+k-1}{k} = \frac{-h(-h+1)\cdots(-h+k-1)}{k!} \quad \text{if } k > 0,$$

$$\binom{-h+k-1}{k} = 1 \quad \text{if } k = 0, \tag{3.3}$$

and putting $\Delta^+ = \Delta^t$, we get for any $h \in R$,

$$(\Delta^+)^h = \varphi[(1-z)^h] = \varphi\left[\sum_{k=0}^{\infty} \binom{-h+k-1}{k} z^k\right] \quad \text{for } |z| < 1. \tag{3.4}$$

Then if $\Delta^h = (\tau_{nm})_{n,m}$,

$$\tau_{nm} = \begin{cases} \binom{-h+n-m-1}{n-m} & \text{if } m \leq n, \\ 0 & \text{if } m > n. \end{cases} \tag{3.5}$$

Using the isomorphism φ, we get the following proposition.

PROPOSITION 3.2 (see [5]). (i) *The operator represented by Δ is bijective from s_r into itself for every $r > 1$, and Δ^+ is bijective from s_r into itself for all r, $0 < r < 1$.*

(ii) *The operator Δ^+ is surjective and not injective from s_r into itself for all $r > 1$.*

(iii) *For all $r \neq 1$ and for every integer $\mu \geq 1$, $(\Delta^+)^h s_r = s_r$.*

(iv) *We have successively*

(α) *if h is a real greater than 0 and $h \notin N$, then Δ^h maps s_r into itself when $r \geq 1$, but not for $0 < r < 1$; if $-1 < h < 0$, then Δ^h maps s_r into itself when $r > 1$, but not for $r = 1$;*

(β) *if $h > 0$ and $h \notin N$, then $(\Delta^+)^h$ maps s_r into itself when $0 < r \leq 1$, but not if $r > 1$; if $-1 < h < 0$, then $(\Delta^+)^h$ maps s_r into itself for $0 < r < 1$, but not for $r = 1$.*

(v) *Let h be any given integer ≥ 1, Then*

$$A \in (s_r(\Delta^h), s_r) \iff \sup_{n \geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| r^{m-n}\right) < \infty \quad \forall r > 1,$$

$$A \in \left(s_r(\Delta^+)^h, s_r\right) \iff \sup_{n \geq 1}\left(\sum_{m=1}^{\infty} |a_{nm}| r^{m-n}\right) < \infty \quad \forall r \in \,]0,1[. \tag{3.6}$$

(vi) *For every integer $h \geq 1$,*

$$s_1 \subset s_1(\Delta^h) \subset s_{(n^h)_{n \geq 1}} \subset \bigcap_{r>1} s_r. \tag{3.7}$$

(vii) *If $h > 0$ and $h \notin N$, then q is the greatest integer strictly less than $(h+1)$. For all $r > 1$,*

$$\mathrm{Ker}\left((\Delta^+)^h\right) \bigcap s_r = \mathrm{span}\,(V_1, V_2, \dots, V_q), \tag{3.8}$$

where

$$V_1 = e^t, \qquad V_2 = (A_1^1, A_2^1, \dots)^t,$$
$$V_3 = (0, A_2^2, A_3^2, \dots)^t, \dots, \qquad V_q = \left(0, 0, \dots, A_{q-1}^{q-1}, A_q^{q-1}, \dots, A_n^{q-1}, \dots\right)^t; \tag{3.9}$$

$A_i^j = i!/(i-j)!$, *with $0 \leq j \leq i$, being the number of permutations of i things taken j at a time.*

We give here an extension of the previous results, where s_r is replaced by s_α.

PROPOSITION 3.3. *Let h be a real greater than 0. The condition $s_\alpha(\Delta^h) = s_\alpha$ is equivalent to*

$$\gamma_n(h) = \frac{1}{\alpha_n} \left[\sum_{k=1}^{n-1} \binom{h+n-k-1}{n-k} \alpha_k \right] = O(1) \quad (n \longrightarrow \infty). \tag{3.10}$$

PROOF. The operator Δ^h is bijective from s_α into itself if and only if Δ^h, $\Sigma^h \in (s_\alpha, s_\alpha)$. We have $\Delta^h \in (s_\alpha, s_\alpha)$ if and only if

$$D_{1/\alpha} \Delta^h D_\alpha \in S_1, \tag{3.11}$$

and using (3.5), we deduce that $\Delta^h \in (s_\alpha, s_\alpha)$ if and only if

$$\frac{1}{\alpha_n} \sum_{k=1}^{n} \left| \binom{-h+n-k-1}{n-k} \right| \alpha_k = O(1). \tag{3.12}$$

Further, $(\Sigma^t)^h = \varphi[(1-z)^{-h}]$, where

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} \binom{h+n-1}{n} z^n \quad \text{with } |z| < 1. \tag{3.13}$$

So, $D_{1/\alpha}\Sigma^h D_\alpha \in S_1$ if and only if (3.10) holds. Finally, since $h > 0$, we have

$$\left| \binom{-h+n-k-1}{n-k} \right| \le \binom{h+n-k-1}{n-k} \quad \text{for } k = 1, 2, \ldots, n-1, \qquad (3.14)$$

and we conclude since (3.10) implies (3.12). □

We deduce immediately the next result.

COROLLARY 3.4. *Let h be an integer greater than or equal to 1. The following properties are equivalent:*
  (i) $\alpha \in \widehat{C_1}$;
  (ii) $s_\alpha(\Delta) = s_\alpha$;
  (iii) $s_\alpha(\Delta^h) = s_\alpha$;
  (iv) $C(\alpha)(\Sigma^{h-1}\alpha) \in l_\infty$.

PROOF. From the proof of Proposition 3.3, $s_\alpha(\Delta^h) = s_\alpha$ is equivalent to $D_{1/\alpha}\Sigma^h D_\alpha = C(\alpha)\Sigma^{h-1}D_\alpha \in S_1$, that is, $C(\alpha)(\Sigma^{h-1}\alpha) \in l_\infty$. So, (iii) and (iv) are equivalent. It remains to prove that (ii)⇔(iii). If $s_\alpha(\Delta) = s_\alpha$, Δ and consequently Δ^h are bijective from s_α into itself and condition (iii) holds. Conversely, assume that $s_\alpha(\Delta^h) = s_\alpha$ holds. Then (3.10) holds, and since

$$\binom{h+n-k-1}{n-k} \ge 1 \quad \text{for } k = 1, 2, \ldots, n-1, \qquad (3.15)$$

we deduce that

$$[C(\alpha)\alpha]_n \le \gamma_n(h) = O(1), \quad n \longrightarrow \infty. \qquad (3.16)$$

So, (i) holds and (ii) is satisfied. □

4. Generalization of well-known sets. In this section, we see that under some conditions, the spaces $\widetilde{w_\alpha}(\lambda)$, $\widetilde{w_\alpha^\circ}(\lambda)$, $\widetilde{w_\alpha^*}(\lambda)$, $\widetilde{c_\alpha}(\lambda,\mu)$, $\widetilde{c_\alpha^\circ}(\lambda,\mu)$, and $\widetilde{c_\alpha^*}(\lambda,\mu)$ can be written by means of the sets s_ξ or s_ξ°.

4.1. Sets $\widetilde{w_\alpha}(\lambda)$, $\widetilde{w_\alpha^\circ}(\lambda)$, and $\widetilde{w_\alpha^*}(\lambda)$. We recall some definitions and properties of some spaces. For every sequence $X = (x_n)_n$, we define $|X| = (|x_n|)_n$ and

$$\begin{aligned}
\widetilde{w_\alpha}(\lambda) &= \{X \in s \mid C(\lambda)(|X|) \in s_\alpha\}, \\
\widetilde{w_\alpha^\circ}(\lambda) &= \{X \in s \mid C(\lambda)(|X|) \in s_\alpha^\circ\}, \\
\widetilde{w_\alpha^*}(\lambda) &= \{X \in s \mid X - le^t \in \widetilde{w_\alpha^\circ}(\lambda) \text{ for some } l \in C\}.
\end{aligned} \qquad (4.1)$$

For instance, we see that

$$\widetilde{w_\alpha}(\lambda) = \left\{ X = (x_n)_n \in s \mid \sup_{n \geq 1} \left(\frac{1}{|\lambda_n| \, \alpha_n} \sum_{k=1}^{n} |x_k| \right) < \infty \right\}. \tag{4.2}$$

If there exist $A, B > 0$ such that $A < \alpha_n < B$ for all n, we get the well-known spaces $\widetilde{w_\alpha}(\lambda) = w_\infty(\lambda)$, $\widetilde{w_\alpha^\circ}(\lambda) = w_0(\lambda)$, and $\widetilde{w_\alpha^*}(\lambda) = w(\lambda)$ (see [12]). It has been proved that if λ is a strictly increasing sequence of reals tending to infinity, $w_0(\lambda)$ and $w_\infty(\lambda)$ are BK spaces and $w_0(\lambda)$ has AK, with respect to the norm

$$\|X\| = \big\| C(\lambda)(|X|) \big\|_{l^\infty} = \sup_n \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} |x_k| \right) \tag{4.3}$$

(see [1]).

We have the next result.

THEOREM 4.1. *Let α and λ be any sequences of U^{+*}.*

(i) *Consider the following properties:*

(a) $\alpha_{n-1}\lambda_{n-1}/\alpha_n\lambda_n \to 0$;

(b) $s_\alpha^{(c)}(C(\lambda)) = s_{\alpha\lambda}^{(c)}$;

(c) $\alpha\lambda \in \widehat{C_1}$;

(d) $\widetilde{w_\alpha}(\lambda) = s_{\alpha\lambda}$;

(e) $\widetilde{w_\alpha^\circ}(\lambda) = s_{\alpha\lambda}^\circ$;

(f) $\widetilde{w_\alpha^*}(\lambda) = s_{\alpha\lambda}^\circ$.

Then (a)$\Rightarrow$(b), (c)$\Leftrightarrow$(d), and (c)$\Rightarrow$(e) and (f).

(ii) *If $\alpha\lambda \in \widehat{C_1}$, $\widetilde{w_\alpha}(\lambda)$, $\widetilde{w_\alpha^\circ}(\lambda)$, and $\widetilde{w_\alpha^*}(\lambda)$ are BK spaces with respect to the norm*

$$\|X\|_{s_{\alpha\lambda}} = \sup_{n \geq 1} \left(\frac{|x_n|}{\alpha_n\lambda_n} \right), \tag{4.4}$$

and $\widetilde{w_\alpha^\circ}(\lambda) = \widetilde{w_\alpha^*}(\lambda)$ has AK.

PROOF. (i) First, we prove that (a)$\Rightarrow$(b). We have

$$s_\alpha^{(c)}(C(\lambda)) = \Delta(\lambda)s_\alpha^{(c)} = \Delta D_\lambda s_\alpha^{(c)} = \Delta s_{\alpha\lambda}^{(c)}, \tag{4.5}$$

and from Proposition 2.1(i) and Theorem 2.6(iii), we get successively $\alpha\lambda \in \widehat{C}$, $\Delta s_{\alpha\lambda}^{(c)} = s_{\alpha\lambda}^{(c)}$, and (b) holds.

(c)$\Leftrightarrow$(d). Assume that (c) holds. Then

$$\widetilde{w_\alpha}(\lambda) = \{ X \mid |X| \in \Delta(\lambda)s_\alpha \}. \tag{4.6}$$

Since $\Delta(\lambda) = \Delta D_\lambda$, we get $\Delta(\lambda)s_\alpha = \Delta s_{\alpha\lambda}$. Now, using (c), we see that Δ is bijective from $s_{\alpha\lambda}$ into itself and $w_\alpha(\lambda) = s_{\alpha\lambda}$. Conversely, assume that $w_\alpha(\lambda) = s_{\alpha\lambda}$. Then $\alpha\lambda \in s_{\alpha\lambda}$ implies that $C(\lambda)(\alpha\lambda) \in s_\alpha$, and since $D_{1/\alpha}C(\lambda)(\alpha\lambda) \in s_1 = l_\infty$, we conclude that $C(\alpha\lambda)(\alpha\lambda) \in l_\infty$. The proof of (c)⇒(e) follows on the same lines of the proof of (c)⇒(d) replacing $s_{\alpha\lambda}$ by $s_{\alpha\lambda}^\circ$.

We prove that (c) implies (f). Take $X \in \widetilde{w_\alpha^*}(\lambda)$. There is a complex number l such that

$$C(\lambda)(|X - le^t|) \in s_\alpha^\circ. \tag{4.7}$$

So

$$|X - le^t| \in \Delta(\lambda)s_\alpha^\circ = \Delta s_{\alpha\lambda}^\circ, \tag{4.8}$$

and from Theorem 2.6(ii), $\Delta s_{\alpha\lambda}^\circ = s_{\alpha\lambda}^\circ$. Now, since (c) holds, we deduce from Proposition 2.1(iii) that $\alpha_n\lambda_n \to \infty$ and $le^t \in s_{\alpha\lambda}^\circ$. We conclude that $X \in \widetilde{w_\alpha^*}(\lambda)$ if and only if $X \in le^t + s_{\alpha\lambda}^\circ = s_{\alpha\lambda}^\circ$.

Assertion (ii) is a direct consequence of (i). $\square$

4.2. Sets $\widetilde{c_\alpha}(\lambda,\mu)$, $\widetilde{c_\alpha^\circ}(\lambda,\mu)$, and $\widetilde{c_\alpha^*}(\lambda,\mu)$. Let $\alpha = (\alpha_n)_n \in U^{+*}$ be a given sequence, we consider now for $\lambda \in U$, $\mu \in s$ the space

$$\widetilde{c_\alpha}(\lambda,\mu) = (w_\alpha(\lambda))_{\Delta(\mu)} = \{X \in s \mid \Delta(\mu)X \in w_\alpha(\lambda)\}. \tag{4.9}$$

It is easy to see that

$$\widetilde{c_\alpha}(\lambda,\mu) = \{X \in s \mid C(\lambda)(|\Delta(\mu)X|) \in s_\alpha\}, \tag{4.10}$$

that is,

$$\widetilde{c_\alpha}(\lambda,\mu) = \left\{X = (x_n)_n \in s \mid \sup_{n\geq 2}\left(\frac{1}{|\lambda_n|\alpha_n}\sum_{k=2}^n |\mu_k x_k - \mu_{k-1}x_{k-1}|\right) < \infty\right\}, \tag{4.11}$$

see [1]. Similarly, we define the following sets:

$$\begin{aligned}\widetilde{c_\alpha^\circ}(\lambda,\mu) &= \{X \in s \mid C(\lambda)(|\Delta(\mu)X|) \in s_\alpha^\circ\}, \\ \widetilde{c_\alpha^*}(\lambda,\mu) &= \{X \in s \mid X - le^t \in \widetilde{c_\alpha^\circ}(\lambda,\mu) \text{ for some } l \in C\}.\end{aligned} \tag{4.12}$$

Recall that if $\lambda = \mu$, it is written that $c_0(\lambda) = (w_0(\lambda))_{\Delta(\lambda)}$,

$$c(\lambda) = \{X \in s \mid X - le^t \in c_0(\lambda) \text{ for some } l \in C\}, \tag{4.13}$$

and $c_\infty(\lambda) = (w_\infty(\lambda))_{\Delta(\lambda)}$, see [11]. It can be easily seen that

$$c_0(\lambda) = \widetilde{c_e^\circ}(\lambda,\lambda), \qquad c_\infty(\lambda) = \widetilde{c_e}(\lambda,\lambda), \qquad c(\lambda) = \widetilde{c_e^*}(\lambda,\lambda). \tag{4.14}$$

These sets of sequences are called strongly convergent to 0, strongly convergent, and strongly bounded. If $\lambda \in U^{+*}$ is a sequence strictly increasing to infinity, $c(\lambda)$ is a Banach space with respect to

$$\|X\|_{c_\infty(\lambda)} = \sup_{n \geq 1} \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} |\lambda_k x_k - \lambda_{k-1} x_{k-1}| \right) \tag{4.15}$$

with the convention $x_0 = 0$. Each of the spaces $c_0(\lambda)$, $c(\lambda)$, and $c_\infty(\lambda)$ is a BK space, relatively to the previous norm (see [1]). The set $c_0(\lambda)$ has AK and every $X \in c(\lambda)$ has a unique representation given by

$$X = le^t + \sum_{k=1}^{\infty} (x_k - l)e_k^t, \tag{4.16}$$

where $X - le^t \in c_0$. The scalar l is called the strong $c(\lambda)$-limit of the sequence X.

We obtain the next result.

THEOREM 4.2. *Let α, λ, and μ be sequences of U^{+*}.*

(i) *Consider the following properties:*

(a) $\alpha\lambda \in \widehat{C}_1$;

(b) $\widetilde{c_\alpha}(\lambda,\mu) = s_{\alpha(\lambda/\mu)}$;

(c) $\widetilde{c_\alpha^\circ}(\lambda,\mu) = s_{\alpha(\lambda/\mu)}^\circ$;

(d) $\widetilde{c_\alpha^*}(\lambda,\mu) = \{X \in s \mid X - le^t \in s_{\alpha(\lambda/\mu)}^\circ \text{ for some } l \in C\}$.

Then (a)$\Leftrightarrow$(b) and (a)$\Rightarrow$(c) and (d).

(ii) *If $\alpha\lambda \in \widehat{C}_1$, then $\widetilde{c_\alpha^\circ}(\lambda,\mu)$, $\widetilde{c_\alpha^\circ}(\lambda,\mu)$, and $\widetilde{c_\alpha^*}(\lambda,\mu)$ are BK spaces with respect to the norm*

$$\|X\|_{s_{\alpha(\lambda/\mu)}} = \sup_{n \geq 1} \left(\mu_n \frac{|x_n|}{\alpha_n \lambda_n} \right). \tag{4.17}$$

The set $\widetilde{c_\alpha^\circ}(\lambda,\mu)$ has AK and every $X \in \widetilde{c_\alpha^*}(\lambda,\mu)$ has a unique representation given by (4.16), where $X - le^t \in s_{\alpha(\lambda/\mu)}^\circ$.

PROOF. We show that (a)$\Rightarrow$(b). Take $X \in c_\alpha(\lambda,\mu)$. We have $\Delta(\mu)X \in w_\alpha(\lambda)$, which is equivalent to

$$X \in C(\mu)s_{\alpha\lambda} = D_{1/\mu}\Sigma s_{\alpha\lambda}, \tag{4.18}$$

and using Theorem 2.6(i), Δ and consequently Σ are bijective from $s_{\alpha\lambda}$ into itself. So, $\Sigma s_{\alpha\lambda} = s_{\alpha\lambda}$ and $X \in D_{1/\mu}\Sigma s_{\alpha\lambda} = s_{\alpha(\lambda/\mu)}$. We conclude that (b) holds. We prove that (b) implies (a). First, put $\widetilde{\alpha}_{\lambda,\mu} = ((-1)^n(\lambda_n/\mu_n)\alpha_n)_{n\geq 1}$. We have

$\widetilde{\alpha}_{\lambda,\mu} \in s_{\alpha(\lambda/\mu)} = \widetilde{c_\alpha}(\lambda,\mu) = s_{\alpha(\lambda/\mu)}$, and since $\Delta(\mu) = \Delta D_\mu$ and $D_\mu \widetilde{\alpha}_{\lambda,\mu} = ((-1)^n \lambda_n \alpha_n)_{n\geq 1}$, we get $|\Delta(\mu)\widetilde{\alpha}_{\lambda,\mu}| = (\xi_n)_{n\geq 1}$, with

$$\xi_n = \begin{cases} \lambda_1 \alpha_1 & \text{if } n = 1, \\ \lambda_{n-1}\alpha_{n-1} + \lambda_n \alpha_n & \text{if } n \geq 2. \end{cases} \tag{4.19}$$

From (b), we deduce that $\Sigma|\Delta(\mu)\widetilde{\alpha}_{\lambda,\mu}| \in s_{\alpha\lambda}$. This means that

$$C'_n = \frac{1}{\alpha_n \lambda_n}\left(\lambda_1 \alpha_1 + \sum_{k=2}^{n}(\lambda_{k-1}\alpha_{k-1} + \lambda_k \alpha_k)\right) = O(1), \quad n \longrightarrow \infty. \tag{4.20}$$

From the inequality

$$\left[C(\alpha\lambda)(\alpha\lambda)\right]_n \leq C'_n, \tag{4.21}$$

we obtain (a). The proof of (a)$\Rightarrow$(c) follows on the same lines of the proof of (a)$\Rightarrow$(b) with s_α replaced by s_α°.

We show that (a) implies (d). Take $X \in \widetilde{c_\alpha^*}(\lambda,\mu)$. There exists $l \in C$ such that

$$\Delta(\mu)(X - le^t) \in \widetilde{w_\alpha^\circ}(\lambda), \tag{4.22}$$

and from (c)$\Rightarrow$(e) in Theorem 4.1, we have $\widetilde{w_\alpha^\circ}(\lambda) = s_{\alpha\lambda}^\circ$. So

$$X - le^t \in C(\mu)s_{\alpha\lambda}^\circ = D_{1/\mu}\Sigma s_{\alpha\lambda}^\circ, \tag{4.23}$$

and from Theorem 2.6(ii), $\Sigma s_{\alpha\lambda}^\circ = s_{\alpha\lambda}^\circ$, and $D_{1/\mu}\Sigma s_{\alpha\lambda}^\circ = s_{\alpha(\lambda/\mu)}^\circ$, we conclude that $X \in \widetilde{c_\alpha^*}(\lambda,\mu)$ if and only if $X \in le^t + s_{\alpha(\lambda/\mu)}^\circ$ for some $l \in C$.

Assertion (ii) is a direct consequence of (i) and of the fact that for every $X \in \widetilde{c_\alpha^*}(\lambda)$, we have

$$\left\|X - le^t - \sum_{k=1}^{N}(x_k - l)e_k^t\right\|_{s_{\alpha(\lambda/\mu)}} = \sup_{n \geq N+1}\left(\mu_n \frac{|x_n - l|}{\alpha_n \lambda_n}\right) = o(1), \quad N \longrightarrow \infty. \tag{4.24}$$

$\square$

We deduce immediately the following corollary.

Corollary 4.3. *Assume that* α, λ, $\mu \in U^{+*}$.
(i) *If* $\alpha\lambda \in \widehat{C_1}$ *and* $\mu \in l_\infty$, *then*

$$\widetilde{c_\alpha^*}(\lambda,\mu) = s_{\alpha(\lambda/\mu)}^\circ. \tag{4.25}$$

(ii) *Then*

$$\lambda \in \Gamma \Longrightarrow \lambda \in \widehat{C}_1 \Longrightarrow c_0(\lambda) = s_\lambda^\circ, \qquad c_\infty(\lambda) = s_\lambda. \tag{4.26}$$

PROOF. (i) Since $\mu \in l_\infty$, we deduce, using Proposition 2.1(iii), that there are $K > 0$ and $\gamma > 1$ such that

$$\frac{\alpha_n \lambda_n}{\mu_n} \geq K\gamma^n \quad \forall n. \tag{4.27}$$

So, $le^t \in s_{\alpha(\lambda/\mu)}^\circ$ and (4.25) holds. (ii) comes from Theorem 4.2 since $\Gamma \subset \widehat{C}_1$. $\qquad \square$

EXAMPLE 4.4. We denote by $\tilde{e}$ the base of the natural system of logarithms. From the well-known Stirling formula, we have

$$\frac{n^{n+1/2}}{n!} \sim \tilde{e}^n \frac{1}{\sqrt{2\pi}}, \tag{4.28}$$

so $s_{(n^{n+(1/2)}/n!)_n} = s_{\tilde{e}}$. Further, $\lambda = (n^n/n!)_n \in \Gamma$ since

$$\frac{\lambda_{n-1}}{\lambda_n} = \tilde{e}^{-(n-1)\ln(1+1/(n-1))} \longrightarrow \frac{1}{\tilde{e}} < 1. \tag{4.29}$$

We conclude that

$$\widetilde{c_{\tilde{e}}^\circ}\left(\left(\frac{n^n}{n!}\right)_n, \left(\frac{1}{\sqrt{n}}\right)_n\right) = s_{\tilde{e}}^\circ, \qquad \widetilde{c_e}\left(\left(\frac{n^n}{n!}\right)_n, \left(\frac{1}{\sqrt{n}}\right)_n\right) = s_{\tilde{e}}. \tag{4.30}$$

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Bruno de Malafosse: Laboratoire de Mathematiques Appliquées Havrais, Institut Universitaire de Technologie Le Havre, Université du Havre, BP 4006, 76610 Le Havre, France

E-mail address: bdemalaf@europost.org

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The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

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Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor

Donal O'Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie