

MODULUS OF SMOOTHNESS AND THEOREMS CONCERNING APPROXIMATION ON COMPACT GROUPS

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We consider the generalized shift operator defined by $(\text{Sh}_u f)(g) = \int_G f(tut^{-1}g)dt$ on a compact group G , and by using this operator, we define “spherical” modulus of smoothness. So, we prove Stechkin and Jackson-type theorems.

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1. Introduction. In this paper, we prove some theorems on absolutely convergent Fourier series in the metric space $L_2(G)$, where G is a compact group. The algebra of absolutely convergent Fourier series is a subject matter about which a good deal, although far from everything, is known (see [5, page 328]). Like many branches of harmonic analysis on T and R , the theory of absolutely convergent Fourier series is a fruitful source of questions about the corresponding entity for compact groups. By using some absolute convergence theorems of the classical Fourier series, (see [1, 11]), a generalized form of Stechkin [6] and Szasz theorem [1, 11] of the Fourier series on compact groups is obtained. Thus, we solve open problems formulated in [5, page 366] (see also [3, Chapter I, page 9]).

2. Preliminaries and notation. Now, we explain some of the notation and terminologies used throughout the paper.

Let G be a compact group with a dual space \hat{G} , $d\mathbf{g}$ denote the Haar measure on G normalized by the condition $\int_G d\mathbf{g} = 1$, and $\int_G f(\mathbf{g})d\mathbf{g}$ denote the Haar integral of a function f on G . Let U_α , $\alpha \in \hat{G}$ denotes the irreducible unitary representation of G in the finite dimensional Hilbert space V_α . We reserve the symbol d_α for the dimension of U_α . Thus, d_α is a positive integer. Also, we denote by χ_α and t_{ij}^α ($i, j = 1, 2, \dots, d_\alpha$), $\alpha \in \hat{G}$ the character and matrix elements (coordinate functions) of U_α , respectively.

Let $L_p(G)$ be the space of all functions f equipped with the norm

$$\|f\|_p = \left\{ \int_G |f(\mathbf{g})|^p d\mathbf{g} \right\}^{1/p}. \quad (2.1)$$

We write $\|\cdot\|_p$ instead of $\|\cdot\|_{L_p(G)}$, and $L_\infty = C$ is the corresponding space of continuous functions, and $\|f\| = \max\{|f(\mathbf{g})| : \mathbf{g} \in G\}$. As it is known (see [4])

or [10, page 99]), the space $L_2(G)$ can be decomposed into the sum

$$L_2(G) = \sum_{\alpha \in \hat{G}} \oplus H_\alpha, \quad (2.2)$$

where

$$H_\alpha = \{f \in C(G) : f(g) = \text{tr}(U_\alpha(g)C), C = \text{Hom}(V_\alpha, V_\alpha)\}. \quad (2.3)$$

This theorem is one of the most important results of the harmonic analysis on compact groups. The orthogonal projection $Y_\alpha : L_2(G) \rightarrow H_\alpha$ is given by

$$(Y_\alpha f)(g) = d_\alpha \int_G f(h) \chi_\alpha(gh^{-1}) dh, \quad (2.4)$$

where $(Y_\alpha f)(g)$ does not depend on the choice of a basis in L_2 . Carrying out this construction for every space H_α , $\alpha \in \hat{G}$, we obtain an orthonormal basis in L_2 consisting of the functions $\sqrt{d_\alpha} t_{ij}^\alpha$, $\alpha \in \hat{G}$, $1 \leq i, j \leq d_\alpha$. Any function $f \in L_2(G)$ can be expanded into a Fourier series with respect to this basis

$$f(g) = \sum_{\alpha \in \hat{G}} \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha t_{ij}^\alpha(g), \quad (2.5)$$

where the Fourier coefficients a_{ij}^α are defined by the following relations:

$$a_{ij}^\alpha = d_\alpha \int_G f(g) \overline{t_{ij}^\alpha(g)} dg, \quad (2.6)$$

such that $\overline{t_{ij}^\alpha(g)} = t_{ij}^\alpha(g^{-1})$, where g^{-1} is the inverse of g . Note that (2.5) is a convergent series in the mean and that the Parseval's equality

$$\int_G |f(g)|^2 dg = \sum_{\alpha \in \hat{G}} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} |a_{ij}^\alpha|^2 \quad (2.7)$$

holds. The aforementioned result of harmonic analysis on a compact group can be found, for example, in [4, 5, 7, 10].

We denote by Sh_u the generalized translation operator on compact group G defined by

$$\begin{aligned} (\text{Sh}_u f)(g) &= \int_G f(tut^{-1}g) dt, \\ (\triangle_u f)(g) &= f(g) - (\text{Sh}_u f)(g) = (E - \text{Sh}_u)f, \end{aligned} \quad (2.8)$$

where $u, g \in G$ and E is the identity operator. We set

$$\Delta_u^k f = \Delta_u \left(\Delta_u^{k-1} f \right) = (E - \text{Sh}_u)^k f = \sum_{i=0}^k (-1)^{k+i} C_k^i \text{Sh}_u^i f, \quad (2.9)$$

in which $\text{Sh}_u^0 f = f$ and $\text{Sh}_u(\text{Sh}_u^{i-1} f) = \text{Sh}_u^i f$, $i = 1, 2, \dots, k$ and $k \in \mathbb{N}$.

We note that α is a complicated index. Since \hat{G} is a countable set, there are only countably many $\alpha \in \hat{G}$ for which $\alpha_{ij}^\alpha \neq 0$ for some i and j ; enumerate them as $\{\alpha_0, \alpha_1, \dots, \alpha_n, \dots\}$. So, $d_{\alpha_0} < d_{\alpha_1} < d_{\alpha_2} < \dots < d_{\alpha_n} < \dots$. Because of that, the symbol “ $\alpha < n$ ” is interpreted as $\{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\} \subset \hat{G}$, and $\alpha \geq n$ denotes the set $\hat{G} \setminus (\alpha < n)$. Let d_α , as usual, be the dimension of U_α . For typographical convenience, we write d_n for the dimension of the representation U^{α_n} , $n = 1, 2, \dots$ (See [5, page 458].)

We denote by $E_n(f)_p$ the approximation of the function $f \in L_p(G)$ by “Spherical” polynomials of degree not greater than n :

$$E_n(f)_p = \inf \left\{ \|f - T_n\|_p : T_n \in \sum_{\alpha < n, \alpha \in \hat{G}} \oplus H_\alpha \right\}. \quad (2.10)$$

The sequence of best approximations $\{E_n(f)_p\}_{n=0}^\infty$ is a constructive characteristic of the function f . In the capacity of structural characteristic of the function f on a compact group G , we define its Spherical modulus of smoothness of order k by

$$\omega_k(f; \tau)_p = \sup \left\{ \|(E - \text{Sh}_u)^k f\|_p : u \in W_\tau \right\}, \quad (2.11)$$

where W_τ is a neighborhood of e in G . In other words,

$$W_\tau = \{u : \rho(u, e) < \tau, u \in G\}, \quad (2.12)$$

where ρ is a pseudometric on G and τ is any positive real number. It is easy to show the following properties of $\omega_k(f, \tau)_p$:

- (a) $\lim_{\tau \rightarrow 0} \omega_k(f, \tau)_p = 0$;
- (b) $\omega_k(f, \tau)_p$ is a continuous monotonically increasing function with respect to τ ;
- (c) $\omega_k(f_1 + f_2, \tau)_p \leq \omega_k(f_1, \tau)_p + \omega_k(f_2, \tau)_p$;
- (d) $\omega_{k+l}(f, \tau)_p \leq 2^l \omega_k(f, \tau)_p$, $l = 1, 2, \dots$.

3. Main results. We need the following simple but useful lemma.

LEMMA 3.1. *The following equality holds for all $u, g \in G$:*

$$(\text{Sh}_u t_{ij}^\alpha)(g) = \frac{\chi_\alpha(u)}{d_\alpha} t_{ij}^\alpha(g). \quad (3.1)$$

PROOF. Using the orthogonality relations and other formulas for matrix elements $t_{ij}^\alpha(g)$ (see [7, page 189]), we have

$$\begin{aligned} \int_G t_{ij}^\alpha(tut^{-1}g)dt &= \sum_{p=1}^{d_\alpha} \sum_{q=1}^{d_\alpha} t_{qp}^\alpha(u) t_{ij}^\alpha(g) \int_G t_{iq}^\alpha(t) \overline{t_{qp}^\alpha(t)} dt \\ &= \frac{1}{d_\alpha} \sum_{p=1}^{d_\alpha} t_{pp}^\alpha(u) t_{ij}^\alpha(g) = \frac{1}{d_\alpha} \chi_\alpha(u) t_{ij}^\alpha(g). \end{aligned} \quad (3.2)$$

This proves the lemma. \square

The following formula is the particular event of the above lemma:

$$\int_G \chi_\alpha(tut^{-1}g)dt = \frac{\chi_\alpha(u)\chi_\alpha(g)}{d_\alpha}. \quad (3.3)$$

It can be called a Weyl formula.

We note that the expansion (2.5) is connected with the expansion

$$f(g) = \sum_{\alpha \in G} Y_\alpha(f)(g), \quad Y_\alpha \in H_\alpha, \quad (3.4)$$

which is defined by (2.4), that is, by the equality

$$Y_\alpha(f)(g) = \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha t_{ij}^\alpha(g). \quad (3.5)$$

Thus, the coefficients a_{ij}^α are defined by (2.6). Using Lemma 3.1 and the definition of Y_α , we obtain

$$\begin{aligned} Y_\alpha(\text{Sh}_u f)(g) &= \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha \int_G t_{ij}^\alpha(tut^{-1}g)dt \\ &= \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha \frac{\chi_\alpha(u)}{d_\alpha} t_{ij}^\alpha(g) \\ &= \frac{\chi_\alpha(u)}{d_\alpha} Y_\alpha(f)(g). \end{aligned} \quad (3.6)$$

The following are simple facts with frequent usage: if $f \in L_p$, then

- (1) $\|\text{Sh}_u f\|_p \leq \|f\|_p$;
- (2) $\|f - \text{Sh}_u f\|_p \rightarrow 0$ as $u \rightarrow e$;
- (3) $(Y_\alpha(\text{Sh}_u f))(g) = (\chi_\alpha(u)/\chi_\alpha(e))(Y_\alpha f)(g)$ for all $\alpha \in \hat{G}$.

We note that $\chi_\alpha(e) = d_\alpha$.

THEOREM 3.2. *If $f \in L_2$ and f is not constant, then*

$$E_n(f)_2 \leq \sqrt{\frac{d_n}{d_n - 2k}} \omega_k\left(f; \frac{1}{n}\right)_2, \quad n = 1, 2, \dots \quad (3.7)$$

PROOF. Let $f \in L_2$ and $S_n(f, g)$ denote the n th partial sum of the Fourier series (2.5), that is,

$$S_n(f, g) = \sum_{\alpha < n} \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha t_{ij}^\alpha(g) = \sum_{p=0}^n \sum_{i,j=1}^{d_{\alpha p}} a_{ij}^{\alpha p} t_{ij}^{\alpha p}(g). \quad (3.8)$$

Using Parseval's equality for the compact group G , we have

$$E_n^2(f)_2 = \|f - S_n(f)\|_2^2 = \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} |a_{ij}^\alpha|^2. \quad (3.9)$$

Using (3), it is not hard to see that

$$(Y_\alpha(\Delta^k f))(g) = \left(1 - \frac{\chi_\alpha(u)}{d_\alpha}\right)^k (Y_\alpha f)(g), \quad \alpha \in \hat{G}. \quad (3.10)$$

Consequently, $(\Delta^k f)(g) = \sum_{\alpha \in \hat{G}} (1 - \chi_\alpha(u)/d_\alpha)^k a_{ij}^\alpha t_{ij}^\alpha$. By another application of Parseval's equality, we obtain

$$\begin{aligned} \|\Delta_u^k f\|_2^2 &= \sum_{\alpha \in \hat{G}} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left|1 - \frac{\chi_\alpha(u)}{d_\alpha}\right|^{2k} |a_{ij}^\alpha|^2 \geq \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left|1 - \frac{\chi_\alpha(u)}{d_\alpha}\right|^{2k} |a_{ij}^\alpha|^2 \\ &= \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left(1 - \frac{2\operatorname{Re} \chi_\alpha(u)}{d_\alpha} + \frac{|\chi_\alpha(u)|^2}{d_\alpha^2}\right)^k |a_{ij}^\alpha|^2. \end{aligned} \quad (3.11)$$

Now, using Bernolli's inequality $(1+x)^k \geq 1+kx$ for $x \geq -1$, we obtain

$$\|\Delta_u^k f\|_2^2 \geq \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left(1 - \frac{2k\operatorname{Re} \chi_\alpha(u)}{d_\alpha} + \frac{k|\chi_\alpha(u)|^2}{d_\alpha^2}\right) |a_{ij}^\alpha|^2. \quad (3.12)$$

Consequently,

$$\|\Delta_u^k f\|_2^2 \geq \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} |a_{ij}^\alpha|^2 - \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \frac{2k\operatorname{Re} \chi_\alpha(u)}{d_\alpha} |a_{ij}^\alpha|^2; \quad (3.13)$$

therefore,

$$E_n^2(f)_2 \leq \|\Delta_u^k f\|_2^2 + 2k \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \frac{\operatorname{Re} \chi_\alpha(u)}{d_\alpha} |a_{ij}^\alpha|^2. \quad (3.14)$$

Let Φ_{W_τ} be a nonnegative integrable function vanishing outside W_τ and satisfying the condition $\int_G \Phi_{W_\tau}(g) dg = 1$. For example, we can take $\Phi_{W_\tau} = \xi_{W_\tau} / \mu(W_\tau)$, where $\mu(W_\tau)$ is the Haar measure of W_τ and ξ_{W_τ} is the characteristic function of W_τ . Multiplying both sides of (3.14) by $\Phi_{W_{1/n}}$, and integrating with respect to u on G , and using the equality $\int_G |\chi_\alpha|^2 dg = 1$ (see [7, page 195]), we obtain

$$\begin{aligned} \int_G E_n^2(f)_2 \Phi_{W_{1/n}}(u) du &\leq \int_G \left\| \Delta_u^k f \right\|_2^2 \Phi_{W_{1/n}} du \\ &\quad + 2k \sum_{\alpha \geq n} \frac{1}{d_\alpha^2} \sum_{i,j=1}^{d_\alpha} \left| a_{ij}^\alpha \right|^2 \int_G |\chi_\alpha(u)| \Phi_{W_{1/n}}(u) du \quad (3.15) \\ &\leq \sup \left\| \Delta_u^k f \right\|_2^2 + \frac{2k}{d_n} \sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} \left| a_{ij}^\alpha \right|^2. \end{aligned}$$

Therefore, it is not hard to see that

$$E_n^2(f)_2 \leq \omega_k^2 \left(f, \frac{1}{n} \right)_2 + \frac{2k}{d_n} E_n^2(f)_2. \quad (3.16)$$

Finally, we obtain

$$E_n(f)_2 \leq \sqrt{\frac{d_n}{d_n - 2k}} \omega_k \left(f, \frac{1}{n} \right)_2, \quad (3.17)$$

which proves the theorem. \square

This theorem is given without proof in [8] for the case where $k = 1$.

We note that the matrix elements of unitary representations $t_{ij}^\alpha(g)$ satisfy the relations

$$\sum_{j=1}^{d_\alpha} t_{ij}^\alpha(g) \overline{t_{kj}^\alpha(g)} = \sum_{j=1}^{d_\alpha} t_{ij}^\alpha(g) \overline{t_{jk}^\alpha(g)} = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases} \quad (3.18)$$

In particular, we have

$$\sum_{j=1}^{d_\alpha} \left| t_{ij}^\alpha \right|^2 = 1 \Rightarrow \left| t_{ij}^\alpha(g) \right| \leq 1 \quad (3.19)$$

for all $\alpha \in \hat{G}$ and $i, j = 1, 2, \dots, d_\alpha$. Furthermore, it is obvious that $|a_{ij}^\alpha t_{ij}^\alpha(g)| \leq |a_{ij}^\alpha|$; therefore, according to the sufficient condition for absolutely convergent Fourier series on the group G , the series $\sum_{\alpha \in \hat{G}} \sum_{i,j=1}^\alpha |a_{ij}^\alpha|$ is convergent. Let $A(G) := \{f : \sum_{\alpha \in \hat{G}} \sum_{i,j=1}^\alpha |a_{ij}^\alpha| < +\infty\}$. Using Theorem 3.2, and repeating the proof of analogous theorems (see [1, Chapter IX] or [6, Chapter II]) with some changes, we obtain the following theorems.

THEOREM 3.3. *If $f(g) \in L_2(G)$, then*

$$\sum_{n=1}^{\infty} \frac{\omega_k(f, 1/n)_2}{\sqrt{n}} < +\infty \Rightarrow f(g) \in A(G). \quad (3.20)$$

This theorem is analogous to the Szasz theorem of the classical Fourier series in the case where $k = 1$ and $G = T$.

THEOREM 3.4. *If $f(g) \in L_2(G)$, then*

$$\sum_{n=1}^{\infty} \frac{E_n(f)_2}{\sqrt{n}} < +\infty \Rightarrow f(g) \in A(G). \quad (3.21)$$

This theorem is also analogous to a theorem in trigonometric case proved by Stechkin [9].

4. Applications to compact group $SU(2)$. The group $SU(2)$ consists of unimodular unitary matrices of the second order, that is, matrices of the form

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1. \quad (4.1)$$

Therefore, each element u of $SU(2)$ is uniquely determined by a pair of complex numbers α and β such that $|\alpha|^2 + |\beta|^2 = 1$. We have (see [5]) the relation “ $(\alpha, \beta) \mapsto (\phi, \theta, \psi)$,” where $\alpha\beta \neq 0$, $|\alpha|^2 + |\beta|^2 = 1$, and the parameters ϕ , θ , and ψ are called Euler angles defined by

$$|\alpha| = \cos \frac{\theta}{2}; \quad \text{Arg } \alpha = \frac{\phi + \psi}{2}; \quad \text{Arg } \beta = \frac{\phi - \psi}{2}. \quad (4.2)$$

Let ϕ , θ , and ψ satisfy the conditions

$$0 \leq \phi < 2\pi, \quad 0 \leq \theta < \pi, \quad -2\pi \leq \psi < 2\pi. \quad (4.3)$$

Also, we know that the dimension of the representation T^l of $SU(2)$ is equal to $2l + 1$, where $l = 0, 1/2, 1, \dots$ and the matrix elements of T^l for group $SU(2)$ are defined by

$$t_{mn}^l(u) = e^{-(n\psi + m\phi)} P_{mn}^l(\cos \theta) i^{(m-n)}. \quad (4.4)$$

Expressing $t_{mn}^l(u)$ in terms of $P_{mn}^l(\cos \theta)$, we arrive at the following conclusion:

Any function $f(\phi, \theta, \psi)$, $0 \leq \phi < 2\pi$, $0 \leq \theta < \pi$, and $-2\pi \leq \psi < 2\pi$ belonging to the space $L^2(SU(2))$ such that

$$\int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^\pi |f(\phi, \theta, \psi)|^2 \sin \theta d\theta d\phi d\psi < \infty \quad (4.5)$$

can be expanded into the mean-convergent series

$$f(\phi, \theta, \psi) = \sum_l \sum_{m=-l}^l \sum_{n=-l}^l \alpha_{mn}^l e^{-i(m\phi+n\psi)} P_{mn}^l(\cos \theta), \quad (4.6)$$

where

$$\alpha_{mn}^l = \frac{2l+1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^\pi f(\phi, \theta, \psi) e^{i(m\phi+n\psi)} P_{mn}^l(\cos \theta) \sin \theta d\theta d\phi d\psi. \quad (4.7)$$

In addition, we obtain from Parseval's equality that

$$\sum_l \sum_{m=-l}^l \sum_{n=-l}^l \frac{1}{2l+1} |\alpha_{mn}^l|^2 = \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^\pi |f(\phi, \theta, \psi)|^2 \sin \theta d\theta d\phi d\psi. \quad (4.8)$$

Using [Theorem 3.2](#), we obtain the following theorem.

THEOREM 4.1. *If $f(\phi, \theta, \psi) \in L_2(\text{SU}(2))$, then*

$$E_n(f)_2 \leq \sqrt{1 + \frac{2}{n-1}} \omega_k\left(f, \frac{1}{n}\right)_2, \quad (4.9)$$

$$\left\{ \sum_{l \geq n} \sum_{m=-l}^l \sum_{n=-l}^l \frac{1}{2l+1} |\alpha_{mn}^l|^2 \right\}^{1/2} \leq \sqrt{1 + \frac{2}{n-1}} \omega_k\left(f, \frac{1}{n}\right)_2.$$

Using the relation between the polynomial $P_n^{(\alpha, \beta)}(z)$ and $P_{mn}^l(z)$, we conclude that

$$P_{mn}^l(z) = 2^{-m} \left[\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!} \right]^{1/2} (1-z)^{(m-n)/2} (1+z)^{(m+n)/2} P_{l-m}^{(m-n, m+n)}. \quad (4.10)$$

The Jacobi polynomials obtained here are characterized by the condition that α and β are integers and $n + \alpha + \beta \in Z_+$.

Now, we consider the following case.

Let $L_2^{(\alpha, \beta)}[-1, 1]$ be the Hilbert space of the functions f defined on the segment $[-1, 1]$ with the scalar product

$$(f_1, f_2) = \int_{-1}^1 f_1(x) \overline{f_2(x)} (1-x)^\alpha (1+x)^\beta dx; \quad (4.11)$$

then, any function f in this space is expanded into the mean-convergent series

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \hat{P}_n^{(\alpha, \beta)}(x), \quad (4.12)$$

where the polynomials $\hat{P}_n^{(\alpha, \beta)}(x)$ are given by

$$\hat{P}_k^{(\alpha, \beta)}(x) = 2^{-(\alpha+\beta+1)/2} \left[\frac{k!(k+\alpha+\beta)!(\alpha+\beta+2k+1)}{(k+\alpha)!(k+\beta)!} \right]^{1/2} P_k^{(\alpha, \beta)}(x), \quad (4.13)$$

$$\alpha_n = \int_{-1}^1 f(x) \hat{P}_n^{(\alpha, \beta)}(x) (1-x)^\alpha (1+x)^\beta dx. \quad (4.14)$$

The Parseval's equality

$$\int_{-1}^1 |f(x)|^2 (1-x)^\alpha (1+x)^\beta dx = \sum_{n=0}^{\infty} |\alpha|^2 \quad (4.15)$$

holds. The formulas (4.12), (4.14), and (4.15) are proved for integral nonnegative values of α and β . We can show that they are valid for arbitrary real values of α and β exceeding -1 . Finally, we reach the following theorem.

THEOREM 4.2. *If $f(x) \in L_2[-1, 1]$, then the following hold for Jacobi series:*

$$\begin{aligned} E_n(f)_2 &\leq \sqrt{1 + \frac{2}{n-1}} \omega_k\left(f, \frac{1}{n}\right)_2, \\ \left\{ \sum_{l=n}^{\infty} |\alpha_l|^2 \right\}^{1/2} &\leq \sqrt{1 + \frac{2}{n-1}} \omega_k\left(f, \frac{1}{n}\right)_2. \end{aligned} \quad (4.16)$$

NOTE. For the ideas similar to this paper we refer to [2] and its references.

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