

ASYMPTOTIC ALMOST PERIODICITY OF C -SEMGROUPS

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Let $\{T(t)\}_{t \geq 0}$ be a C -semigroup on a Banach space X with generator A . We will investigate the asymptotic almost periodicity of $\{T(t)\}$ via the Hille-Yosida space of its generator.

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1. Introduction. Motivated by the abstract Cauchy problem

$$\frac{d}{dt}u(t) = Au(t) \quad (t \geq 0), \quad u(0) = x, \quad (1.1)$$

a generalization of strongly continuous semigroups, C -semigroups, has recently received much attention (see [6, 7, 17]). The operator A generating a C -semigroup leads to (1.1) having a unique solution, whenever $x = Cy$ for some y in the domain of A . It is well known that the class of operators that generate C -semigroups is much larger than the class of operators that generate strongly continuous semigroups.

On the other hand, the asymptotic almost periodicity of C_0 -semigroup has been studied systematically in [2, 4, 8, 11, 12, 14, 15, 16]. It was shown that when A generates an asymptotically almost periodic C_0 -semigroup on a Banach space X , then X can be decomposed into the direct sum of two subspaces X_a and X_s , and the mild solutions with initial values taken from X_a are almost periodic and thus can be extended to the whole line while the mild solutions from X_s are vanishing at infinity.

In this paper, we will discuss the asymptotic almost periodicity of C -semigroups. We show that if A generates an asymptotically almost periodic C -semigroup, then the range of C has an analogous decomposition. The technique we use here is the Hille-Yosida space for A , which is a maximal imbedded subspace of X such that the part of A on this subspace generates a C_0 -semigroup. The crucial facts are that a mild solution of the abstract Cauchy problem is asymptotically almost periodic in the Hille-Yosida space if and only if it is asymptotically almost periodic in X , and the mild solution is vanishing at infinity in the Hille-Yosida space if and only if it is vanishing in X .

Under suitable spectral conditions, we obtain a theorem for asymptotic almost periodicity of C -semigroups that is more easily testified (Theorem 3.7).

At last, we give a theorem for asymptotically almost periodic integrated semi-groups (Theorem 3.9).

Although there are papers devoted to asymptotic properties of individual solution [1, 2, 3, 4], our main concern is some kind of global property.

Throughout the paper, all operators are linear. We write $D(A)$ for the domain of an operator A , $R(A)$ for the range, and $\rho(A)$ for the resolvent set; X will always be a Banach space, and the space of all bounded linear operators on X will be denoted by $B(X)$ while C will always be a bounded, injective operator on X . Finally, \mathbb{R}^+ will be the half-line $[0, +\infty)$, and \mathbb{C}^+ will be the half-plane $\{z \mid z = \tau + iw, \tau, w \in \mathbb{R} \text{ and } \tau > 0\}$.

2. Preliminaries. We start with the definitions and properties of C -semi-groups.

DEFINITION 2.1. A family $\{T(t)\}_{t \geq 0} \subset B(X)$ is a C -semigroup if

- (a) $T(0) = C$,
- (b) the map $t \mapsto T(t)x$, from $[0, +\infty)$ into X , is continuous for all $x \in X$,
- (c) $CT(t+s) = T(t)T(s)$.

The generator of $\{T(t)\}_{t \geq 0}$, A is defined by

$$Ax = C^{-1} \left[\lim_{t \downarrow 0} \frac{1}{t} (T(t)x - Cx) \right] \quad (2.1)$$

with

$$D(A) = \left\{ x \in X \mid \lim_{t \downarrow 0} \frac{1}{t} (T(t)x - Cx) \text{ exists and is in } R(C) \right\}. \quad (2.2)$$

The complex number λ is in $\rho_C(A)$, the C -resolvent set of A , if $(\lambda - A)$ is injective and $R(C) \subseteq R(\lambda - A)$. And we denote by $\sigma_C(A)$ the set of all points in complex plane which are not in the C -resolvent of A .

For the basic properties of C -semigroups and their generators, we refer to [7]. Next, we introduce the Hille-Yosida space for an operator.

DEFINITION 2.2. Suppose that A has no eigenvalues in $(0, \infty)$ and is a closed linear operator. The Hille-Yosida space for A , Z_0 is defined by $Z_0 = \{x \in X \mid \text{The Cauchy problem (1.1) has a bounded uniformly continuous mild solution } u(\cdot, x) \}$ with

$$\|x\|_{Z_0} = \sup \{ \|u(t, x)\| : t \geq 0 \} \quad \text{for } x \in Z_0. \quad (2.3)$$

LEMMA 2.3 (see [7]). Suppose that A generates a strongly uniformly continuous bounded C -semigroup $\{T(t)\}_{t \geq 0}$ on X . Then, the Hille-Yosida space for

A, Z_0 is a Banach space and

$$Z_0 = \{x \mid t \mapsto C^{-1}T(t)x \text{ is uniformly continuous and bounded}\} \quad (2.4)$$

with

$$\|x\|_{Z_0} = \sup \{\|C^{-1}T(t)x\| : t \geq 0\}. \quad (2.5)$$

Moreover, $Z_0 \hookrightarrow X$, that is, Z_0 can be continuously imbedded in X , $A|_{Z_0}$ generates a contraction C_0 -semigroup $S(t) = C^{-1}T(t)$ on Z_0 , and $T(t)x = S(t)Cx$ for all $x \in X$.

Let $J = \mathbb{R}^+$ or \mathbb{R} . The spaces of all bounded continuous functions from J into X will be denoted by $C_b(J, X)$, $C_0(\mathbb{R}^+, X)$ will designate the set of those $\varphi \in C_b(\mathbb{R}^+, X)$ that vanish at infinity on \mathbb{R}^+ , and we will hereafter assume that each of these spaces is equipped with the supremum norm. Moreover, for $f \in C_b(J, X)$, $w \in J$, we put $f_w(t) = f(t + w)$ ($t \in J$) and let $H(f) = \{f_w : w \in J\}$ denote the set of all translates of f .

DEFINITION 2.4. (a) A function $f \in C(J, X)$ is said to be *almost periodic* if for every $\varepsilon > 0$, there exists $l > 0$ such that every subinterval of J of length l contains, at least, one τ satisfying $\|f(t + \tau) - f(t)\| \leq \varepsilon$ for $t \in J$. The space of all almost periodic functions will be denoted by $AP(J, X)$.

(b) A function $f \in C_b(\mathbb{R}^+, X)$ is said to be *asymptotically almost periodic* if for every $\varepsilon > 0$, there exist $l > 0$ and $M > 0$ such that every subinterval of \mathbb{R}^+ of length l contains, at least, one τ satisfying $\|f(t + \tau) - f(t)\| \leq \varepsilon$ for all $t \geq M$. The space of all asymptotically almost periodic functions will be denoted by $AAP(\mathbb{R}^+, X)$.

LEMMA 2.5 (see [10, 15, 16]). *For a function $f \in C(\mathbb{R}^+, X)$, the following statements are equivalent:*

- (a) $f \in AAP(\mathbb{R}^+, X)$;
- (b) *there exist uniquely determined functions $g \in AP(\mathbb{R}, X)$ and $\varphi \in C_0(\mathbb{R}^+, X)$ such that $f = g|_{\mathbb{R}^+} + \varphi$;*
- (c) $H(f)$ is relatively compact in $C_b(\mathbb{R}^+, X)$.

DEFINITION 2.6. Let $\{F(t)\}_{t \in J} \subseteq B(X)$ be a strongly continuous operator family.

- (a) $F(t)$ ($t \in J$) is *almost periodic* if $F(\cdot)x$ is almost periodic for every $x \in X$.
- (b) $F(t)$ ($t \in \mathbb{R}^+$) is *asymptotically almost periodic* if $F(\cdot)x$ is asymptotically almost periodic for every $x \in X$.

3. Main results. In order to characterize the asymptotic almost periodicity of C -semigroups, we need the following result.

LEMMA 3.1. Assume that $\{T(t)\}_{t \geq 0}$ is a C -semigroup on X generated by A and A has no eigenvalues in $(0, \infty)$. Suppose that $T(\cdot)x : \mathbb{R}^+ \rightarrow X$ is asymptotically almost periodic for some $x \in X$. Then,

- (a) there exist $y \in X$ and $\varphi \in C_0(\mathbb{R}^+, X)$ such that, for all $t \geq 0$, $T(t)y \in R(C)$, $C^{-1}T(\cdot)y \in \text{AP}(\mathbb{R}^+, X)$, and

$$T(t)x = C^{-1}T(t)y + \varphi(t); \quad (3.1)$$

- (b) if $\{T(t)\}$ is strongly uniformly continuous and bounded, then there exist $y, z \in Z_0$ such that $S(\cdot)y \in \text{AP}(\mathbb{R}^+, Z_0)$, $S(\cdot)z \in C_0(\mathbb{R}^+, Z_0)$, and

$$T(t)x = S(t)y + S(t)z \quad \forall t \geq 0, \quad (3.2)$$

where Z_0 is the Hille-Yosida space for A and $\{S(t)\}$ is the C_0 -semigroup generated by $A|_{Z_0}$.

PROOF. (a) By Lemma 2.5, there exist uniquely determined functions $h \in \text{AP}(\mathbb{R}, X)$ and $\varphi \in C_0(\mathbb{R}^+, X)$ such that $T(\cdot)x = h|_{\mathbb{R}^+} + \varphi$, and, from the proof of Lemma 2.5 (see [14]), we know that there exists $0 < t_n \rightarrow \infty$ such that

$$h(t) = \lim_{n \rightarrow \infty} T(t + t_n)x \quad \forall t \geq 0. \quad (3.3)$$

Let $y = h(0)$, then we have $\lim_{n \rightarrow \infty} T(t_n)x = y$. Therefore, for each $t \geq 0$,

$$Ch(t) = \lim_{n \rightarrow \infty} CT(t + t_n)x = T(t) \lim_{n \rightarrow \infty} T(t_n)x = T(t)y, \quad (3.4)$$

which implies $T(t)y \in R(C)$ for all $t \geq 0$ and $C^{-1}T(\cdot)y = h|_{\mathbb{R}^+} \in \text{AP}(\mathbb{R}^+, X)$, so that

$$T(t)x = C^{-1}T(t)y + \varphi(t) \quad \forall t \geq 0. \quad (3.5)$$

(b) Let y and $\varphi(t)$ be the same as in (a). Since $C^{-1}T(\cdot)y = h|_{\mathbb{R}^+} \in \text{AP}(\mathbb{R}^+, X)$, we have that $C^{-1}T(t)y$ is uniformly continuous and bounded in $[0, +\infty)$, and so it follows that $y \in Z_0$ by Lemma 2.3. Setting $z = Cx - y$, then obviously, $z \in Z_0$. Hence, $S(t)y = C^{-1}T(t)y$ and

$$S(t)z = C^{-1}T(t)z = C^{-1}T(t)Cx - C^{-1}T(t)y = T(t)x - C^{-1}T(t)y = \varphi(t) \quad (3.6)$$

so that

$$T(t)x = S(t)y + S(t)z \quad \forall t \geq 0. \quad (3.7)$$

Next, we show that $S(\cdot)y \in \text{AP}(\mathbb{R}^+, Z_0)$ and $S(\cdot)z \in C_0(\mathbb{R}^+, Z_0)$. By the definition of almost periodicity, we know, for every $\varepsilon > 0$, there exists $l > 0$ such that every subinterval of \mathbb{R}^+ of length l contains, at least, one τ satisfying

$$\sup_{t \geq 0} \|S(t + \tau)y - S(t)y\| < \varepsilon, \quad (3.8)$$

hence,

$$\begin{aligned}
& \sup_{t \geq 0} \|S(t + \tau)y - S(t)y\|_{Z_0} \\
&= \sup_{t \geq 0, s \geq 0} \|C^{-1}T(s)S(t + \tau)y - C^{-1}T(s)S(t)y\| \\
&= \sup_{t \geq 0, s \geq 0} \|C^{-1}T(s)C^{-1}T(t + \tau)y - C^{-1}T(s)C^{-1}T(t)y\| \\
&= \sup_{t \geq 0, s \geq 0} \|C^{-1}T(t + s + \tau)y - C^{-1}T(t + s)y\| \\
&= \sup_{t \geq 0, s \geq 0} \|S(t + s + \tau)y - S(t + s)y\| \\
&= \sup_{t \geq 0} \|S(t + \tau)y - S(t)y\| < \varepsilon,
\end{aligned} \tag{3.9}$$

which yields $S(t)y \in \text{AP}(\mathbb{R}^+, Z_0)$. Also,

$$\|S(t)z\|_{Z_0} = \sup_{s \geq 0} \|C^{-1}T(s)S(t)z\| = \sup_{s \geq 0} \|S(t + s)z\| = \sup_{s \geq t} \|S(s)z\| \tag{3.10}$$

converges to 0 as $t \rightarrow +\infty$ since $S(t)z \in C_0(\mathbb{R}^+, X)$, and this yields $S(t)z \in C_0(\mathbb{R}^+, Z_0)$. \square

Now, we can prove the following theorem which characterizes the asymptotic almost periodicity of C -semigroups.

THEOREM 3.2. *Let $\{T(t)\}$ be a C -semigroup generated by A on X . Then, $\{T(t)\}$ is asymptotically almost periodic if and only if $R(C) \subseteq X_{0a} + X_{0s}$, where $X_{0a} = \{x \mid x \in Z_0, S(t)x \in \text{AP}(\mathbb{R}^+, Z_0)\}$ and $X_{0s} = \{x \mid x \in Z_0, S(t)x \in C_0(\mathbb{R}^+, Z_0)\}$, where Z_0 is the Hille-Yosida space for A , $\{S(t)\}$ is the C_0 -semigroup generated by $A|_{Z_0}$.*

PROOF. Necessity. First, it follows from [17, Lemma 1.6.(a)] and the uniform boundedness theorem that $\{T(t)\}$ is bounded and strongly uniformly continuous. By Lemma 3.1(b), for every $x \in X$, there exist $y, z \in Z_0$ such that $S(\cdot)y \in \text{AP}(\mathbb{R}^+, Z_0)$, $S(\cdot)z \in C_0(\mathbb{R}^+, Z_0)$, and $T(t)x = S(t)y + S(t)z$ for all $t \geq 0$. Choosing $t = 0$, we obtain $Cx = y + z$, that is, $R(C) \subseteq X_{0a} + X_{0s}$.

Sufficiency. Since $R(C) \subseteq X_{0a} + X_{0s}$, for any $x \in X$, there exist $y \in X_{0a}$ and $z \in X_{0s}$ such that $Cx = y + z$. Hence, by Lemma 2.3, we have $T(t)x = S(t)Cx = S(t)y + S(t)z$ while $S(t)y \in \text{AP}(\mathbb{R}^+, Z_0)$ and $S(t)z \in C_0(\mathbb{R}^+, Z_0)$. Since Z_0 is continuously imbedded in X , it follows that $T(t)x$ is asymptotically almost periodic by Lemma 2.5, that is, $\{T(t)\}$ is an asymptotically almost periodic C -semigroup. \square

From the proof of Lemma 3.1(b), we know that for $y, z \in Z_0$, $S(t)y \in \text{AP}(\mathbb{R}^+, X)$ and $S(t)z \in C_0(\mathbb{R}^+, X)$ if and only if $S(t)y \in \text{AP}(\mathbb{R}^+, Z_0)$ and $S(t)z \in C_0(\mathbb{R}^+, Z_0)$, respectively. Hence, we have the following corollary.

COROLLARY 3.3. *Let $\{T(t)\}_{t \geq 0}$ be a C -semigroup generated by A on X . Then, $\{T(t)\}$ is asymptotically almost periodic if and only if for any $x \in X$, there*

exist $y, z \in Z_0$ such that $Cx = y + z$, $C^{-1}T(t)y \in \text{AP}(\mathbb{R}^+, X)$, and $C^{-1}T(t)z \in C_0(\mathbb{R}^+, X)$, where Z_0 is the Hille-Yosida spaces for A .

In the case of $\overline{R(C)} = X$, we have the following theorem.

THEOREM 3.4. *Assume that $\{T(t)\}_{t \geq 0}$ is a C -semigroup on X and $\overline{R(C)} = X$. Then, $\{T(t)\}$ is asymptotically almost periodic if and only if $R(C) \subseteq X_a + X_s$, where $X_a = \{x \mid x \in X, T(t)x \in \text{AP}(\mathbb{R}^+, X)\}$ and $X_s = \{x \mid x \in X, T(t)x \in C_0(\mathbb{R}^+, X)\}$.*

PROOF. Necessity. By Lemma 3.1(a), for every $x \in X$, there exist $y \in X$ and $\varphi \in C_0(\mathbb{R}^+, X)$ such that for all $t \geq 0$, $T(t)y \in R(C)$, $C^{-1}T(\cdot)y \in \text{AP}(\mathbb{R}^+, X)$, and

$$T(t)x = C^{-1}T(t)y + \varphi(t). \quad (3.11)$$

It follows that $T(t)Cx = CT(t)x = T(t)y + C\varphi(t)$. Setting $z = Cx - y$, then $T(t)z = T(t)Cx - T(t)y = C\varphi(t) \in C_0(\mathbb{R}^+, X)$; on the other hand, $T(\cdot)y \in \text{AP}(\mathbb{R}^+, X)$ since C is bounded. So, we have $R(C) \subseteq X_a + X_s$.

Sufficiency follows from the fact that $R(C)$ is dense in X and $\text{AAP}(\mathbb{R}^+, X)$ is closed in the space of all bounded uniformly continuous functions from \mathbb{R}^+ to X . \square

The following result clarifies the relations between the generator of an asymptotically almost periodic C -semigroup and of an asymptotically almost periodic C_0 -semigroup.

THEOREM 3.5. *Suppose that A is closed and has no eigenvalues in $(0, \infty)$, and assume that $C^{-1}AC = A$. Then, A generates an asymptotically almost periodic C -semigroup if and only if $R(C) \subseteq Z_{\text{aap}} \equiv (Z_0)_a + (Z_0)_s$, where $(Z_0)_a = \{x \mid x \in X, \text{the Cauchy problem (1.1) has an almost periodic mild solution } u(\cdot, x)\}$ and $(Z_0)_s = \{x \mid x \in X, \text{the Cauchy problem (1.1) has a mild solution } u(\cdot, x) \in C_0(\mathbb{R}^+, X)\}$. And Z_{aap} is the maximal continuously imbedded subspace on which A generates an asymptotically almost periodic C_0 -semigroup.*

PROOF. Necessity holds by Lemma 3.1 and the relations between C -semigroup and solutions of the corresponding Cauchy problem [7, Theorem 3.13].

Sufficiency. By Definition 2.2, we know that both $(Z_0)_a$ and $(Z_0)_s$ are contained in Z_0 ; so $R(C) \subseteq Z_0$. Thus, by [7, Theorem 5.17] and [9, Corollary 3.14], A generates a bounded C -semigroup; since all mild solutions with initial data taken from $R(C)$ are asymptotically almost periodic, so is the C -semigroup.

Now, suppose that $Y \hookrightarrow X$ and $A|_Y$ generates a contraction asymptotically almost periodic C_0 -semigroup, then $Y \hookrightarrow Z_0$ since Z_0 is maximal (cf. [7, Theorem 5.5]). So, $Y \hookrightarrow Z_{\text{aap}}$ follows from the fact that the asymptotic almost periodicity of the mild solution of the abstract Cauchy problem in Z_0 is equivalent to the same property in X , and the mild solution $u(t)$ converges to 0 as $t \rightarrow \infty$ in X is equivalent to $u(t) \rightarrow 0$ in Z_0 . \square

REMARK 3.6. (a) $(Z_0)_a$ is the maximal subspace on which A generates an almost periodic C_0 -semigroup.

(b) From the proof of [Lemma 3.1](#) and [Theorems 3.2](#) and [3.5](#), it is clear that $(Z_0)_a = X_{0a}$, $(Z_0)_s = X_{0s}$.

When $\sigma_C(A) \cap i\mathbb{R}$ is countable, we obtain a result which is more easily testified for asymptotic almost periodicity of C -semigroups.

Let $f : \mathbb{R}^+ \rightarrow X$ be strongly measurable, and let \tilde{f} be the Laplace transform of f ,

$$\tilde{f}(z) = \int_0^{+\infty} e^{-tz} f(t) dt. \quad (3.12)$$

We assume that $\tilde{f}(z)$ exists for all z in \mathbb{C}^+ , so \tilde{f} is holomorphic in \mathbb{C}^+ (usually, f will be bounded). A point $\lambda = i\eta$ in $i\mathbb{R}$ is said to be a regular point for \tilde{f} if there is an open neighborhood U of λ in \mathbb{C} and a holomorphic function $g : U \rightarrow X$ such that $g(z) = \tilde{f}(z)$ whenever $z \in U \cap \mathbb{C}^+$. The singular set E of \tilde{f} is the set of all points of $i\mathbb{R}$ which are not regular points.

THEOREM 3.7. *Let $\{T(t)\}_{t \geq 0}$ be a C -semigroup generated by A in X , and let $\sigma_C(A) \cap i\mathbb{R}$ be countable. Then, the following assertions are equivalent:*

- (a) $\{T(t)\}$ is asymptotically almost periodic;
- (b) $\{T(t)\}$ is bounded, strongly uniformly continuous and, for every $r \in \sigma_C(A) \cap i\mathbb{R}$, $x \in X$, $\lim_{\lambda \rightarrow 0} \lambda \int_0^{+\infty} e^{-(\lambda+ir)t} T(t+s)x dt$ exists uniformly for $s \geq 0$.

PROOF. (a) \Rightarrow (b). It follows from the properties of asymptotically almost periodic functions (cf. [\[4\]](#)).

(b) \Rightarrow (a). Given $x \in X$, let $f(t) = T(t)x$, and then we have that $f(t)$ is bounded, uniformly continuous and $\tilde{f}(\lambda) = (\lambda - A)^{-1}Cx$ ($\operatorname{Re} \lambda > 0$). Let E be the singular set of \tilde{f} in $i\mathbb{R}$, then $E \subseteq \sigma_C(A) \cap i\mathbb{R}$, and then it follows that E is countable by the assumption. Moreover, for each $ir \in \sigma_C(A) \cap i\mathbb{R}$,

$$\lim_{\lambda \rightarrow 0} \lambda \tilde{f}_s(\lambda + ir) = \lim_{\lambda \rightarrow 0} \lambda \int_0^{+\infty} e^{-(\lambda+ir)t} T(t+s)x dt \quad (3.13)$$

exists uniformly for $s \geq 0$, where $f_s(t) = f(s+t)$. Therefore, $f(t) = T(t)x$ is asymptotically almost periodic by [\[5, Theorem 4.1\]](#); so, $\{T(t)\}$ is asymptotically almost periodic. \square

REMARK 3.8. The result of [Theorem 3.7](#) can be deduced directly from [\[8, Theorem 4\]](#) with the assumption on $\sigma_C(A)$ replaced by that on $\sigma(A)$; while with the aid of [\[5, Theorem 4.1\]](#), the result can be improved.

We end this paper with a theorem for integrated semigroups, see [\[13\]](#) for the definitions and basic properties of integrated semigroups.

THEOREM 3.9. *Suppose that A generates a bounded n -times integrated semigroup $\{T(t)\}_{t \geq 0}$ and $\sigma(A) \cap i\mathbb{R}$ is at most countable. Then, the following assertions are equivalent:*

- (a) $\{T(t)\}$ is asymptotically almost periodic;
- (b) for every $r \in \sigma(A) \cap i\mathbb{R}$, $x \in X$, the limit

$$\lim_{\lambda \rightarrow 0} \lambda \int_0^{+\infty} e^{-(\lambda + ir)t} T(t+s)x \, dt \quad (3.14)$$

exists uniformly for $s \geq 0$.

PROOF. We only need to show (b) \Rightarrow (a).

We first recall that for bounded integrated semigroup $\{T(t)\}$, we have

$$(\lambda - A)^{-1}x = \lambda^n \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt \quad (3.15)$$

for $\operatorname{Re} \lambda > 0$, that is,

$$\int_0^{+\infty} e^{-\lambda t} T(t)x \, dt = \frac{1}{\lambda^n} (\lambda - A)^{-1}x \quad (3.16)$$

for $\operatorname{Re} \lambda > 0$, so that $\tilde{T}(\lambda) := \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt$ can be extended holomorphically to a connected open neighborhood V of $(i\mathbb{R} \setminus \sigma(A)) \setminus \{0\}$, hence the singular set of $\tilde{T}(\lambda)$ in $i\mathbb{R}$ is contained in $(\sigma(A) \cap i\mathbb{R}) \cup \{0\}$. By our assumption and [5, Theorem 4.1], we derive (a) from (b). \square

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This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
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