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ON CERTAIN QUASI-COMPLEMENTED AND COMPLEMENTED BANACH ALGEBRAS

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ABSTRACT. In this paper, we continue the study of quasi-complemented algebras and complemented algebras. The former are generalizations of the latter and were introduced in [4] and studied in [4] and [11]. Some results are proved.

KEY WORDS AND PHRASES. *Quasi-complemented and complemented Banach algebras.*

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1. INTRODUCTION.

Quasi-complemented algebras, which are generalizations of complemented algebras, were introduced in [4] and studied in [4] and [11]. In this paper, we continue the study of these two classes of algebras.

In Section 3, we introduce the concept of continuous quasi-complementor on a semi-simple annihilator Banach algebra. This is similar to the concept of continuous complementor given by Alexander in [1]. Let A be a simple annihilator Banach algebra such that $x \in cl_A(xA)$ for all x in A. If A is infinite dimensional, we show that every quasi-complementor on A is continuous. This result is not true if A is finite dimensional. In this case, we obtain that a quasi-complementor q on A is continuous if and only if the set E_q of all q-projections is closed and bounded in A. By using these results, we give a characterization of continuous quasi-complementors (Theorem 3.4).

Section 4 is devoted to the study of uniformly continuous quasi-complementors. Let A be a semi-simple annihilator Banach algebra in which $x \in cl_A(xA)$ for all x in A and q a quasi-complementor on A. Suppose that A has no minimal left ideals of dimension less than three. Then we show that A is a dense subalgebra of some dual B*-algebra B and $R^q = \ell(R)^* \cap A$ for all closed right ideals R of A. Also every continuous complementor on A is uniformly continuous.

2. NOTATION AND PRELIMINARIES.

For any subset S in an algebra A, let $\ell_A(S)$ and $r_A(S)$ denote the left and right annihilators of S in A, respectively. Let A be a Banach algebra. Then A is called an annihilator algebra, if for every closed left ideal J and for every closed right ideal R, we have $r_A(J) = (0)$ if and only if $J = A$ and $\ell_A(R) = (0)$ if and only if $R = A$. If $\ell_A(r_A(J)) = J$ and $r_A(\ell_A(R)) = R$, then A is called a dual algebra.

Let A be a Banach algebra which is a subalgebra of a Banach algebra B. For each subset S of A, $cl(S)$ (resp. $cl_A(S)$) will denote the closure of S in B (resp. A). Also $\ell(S)$ and $r(S)$ (resp. $\ell_A(S)$ and $r_A(S)$) denote the left and right annihilators of S in B (resp. A). We write $||\cdot||$ for the norm on A and $|\cdot|$ for the norm on B.

Let A be a Banach algebra and let L_r be the set of all closed right ideals in A. Following [4], we shall say that A is a (right) quasi-complemented algebra if there exists a mapping $q : R \to R^q$ of L_r into itself having the following properties:

$$R \cap R^q = (0) \qquad (R \in L_r) ; \qquad (2.1)$$

$$(R^q)^q = R \qquad (R \in L_r) ; \qquad (2.2)$$

$$\text{if } R_1 \supset R_2, \text{ then } R_2^q \supset R_1^q \qquad (R_1, R_2 \in L_r). \qquad (2.3)$$

The mapping q is called a (right) quasi-complementor on A. We know that $R + R^q$ is always dense in A, $A^q = (0)$ and $(0)^q = A$ (see [4]). Hence $R^q = (0)$ if and only if $R = A$.

A quasi-complemented algebra A is called a (right) complemented algebra if it satisfies:

$$R + R^q = A \qquad (R \in L_r). \qquad (2.4)$$

In this case, the mapping q is called a (right) complementor on A (see [6, p. 651, Definition 1]).

Let A be a semi-simple Banach algebra with a quasi-complementor q. A minimal idempotent f in A is called a q-projection if $(fA)^q = (1 - f)A$. The set of all q-projection in A is denoted by E_q. By Lemma 3.1 in [11], every non-zero right ideal of A contains a q-projection.

In this paper, all algebras and linear spaces under consideration are over the complex field. Definitions not explicitly given are taken from Rickart's book [5].

We end the section with two new examples of complemented and quasi-complemented algebras.

EXAMPLE 1. Let A be a dual B*-algebra and Φ a symmetric norming function. Then the algebra $A_\Phi^{(0)}$ given in [10, p. 293] is a complemented algebra with the complementor $q : R \to \ell_{A_\Phi^{(0)}}(R)^*$. (Theorem 3.4 in [11]).

EXAMPLE 2. Let G be an infinite compact group with the Haar measure and A the algebra of all continuous functions on G, normed by the maximum of the absolute value and $L_1(G)$ the group algebra. It is well known that A and $L_1(G)$ are dual A*-algebras which are not two-sided ideals of their completions in an auxiliary norm. It is easy to see that the mapping $q : R \to \ell_A(R)^*$ (resp. $R \to \ell_{L_1(G)}(R)^*$) is a quasi-complementor on A (resp. $L_1(G)$). However, by Theorem 3.4 in [11], q is not a complementor.

3. CONTINUOUS QUASI-COMPLEMENTORS.

Let A be a semi-simple annihilator Banach algebra with a quasi-complementor q and M_A the set of all minimal right ideals of A. For each $R \in M_A$, by Lemma 3.1 in [11], $R = fA$ for some q-projection f in A. Therefore, $R + R^q = fA + (1 - f)A$. Let P_R be the projection on R along R^q. Then P_R is continuous.

DEFINITION. Suppose $a_n \in A$ with $a_nA \in M_A$ $(n = 0, 1, 2, \ldots)$. A quasi-complementor q on A is said to be continuous if whenever a_n converges to a_0, then P_{a_nA} converges to P_{a_0A} uniformly on any minimal left ideal of A.

REMARK. This is similar to the definition of continuous complementor introduced by Alexander (see [1, p. 387, Definition]).

Let A be a semi-simple annihilator quasi-complemented Banach algebra such that $x \in cl_A(xA)$ for all x in A and $\{I_\lambda : \lambda \in \Lambda\}$ the family of all minimal closed two-sided ideals of A. Define q_λ by $R^{q_\lambda} = R^q \cap I_\lambda$ for all closed right ideals R of I_λ. Then by [4, p. 144, Theorem 3.6] A is the direct topological sum of $\{I_\lambda : \lambda \in \Lambda\}$ and q_λ is a quasi-complementor on I_λ. Let H_λ be a minimal left ideal of I_λ. Then H_λ is a Hilbert space under some equivalent inner product norm by [4, p. 145, Lemma 4.2]. Let B_λ be the algebra of all completely continuous linear operators on H_λ.

Then by the proof of [4, p. 146, Theorem 4.3], I_λ is a dense subalgebra of B such that $||\cdot||$ majorizes $|\cdot|$ on I_λ. By the proof of [8, p. 442, Lemma 5.1], B_λ and I_λ have the same socle.

LEMMA 3.1. A quasi-complementor q on A is continuous if and only if each q_λ is continuous.

PROOF. Let $R \in M_A$ with $R \subset I_{\lambda_0}$ for some $\lambda_0 \in \Lambda$. Then $R = fA$, where f is a q-projection in I_{λ_0}. Hence, for all x in A, $P_R(x) = fx$. If $\lambda \neq \lambda_0$, then $I_{\lambda_0} I_\lambda = (0)$ and so $P_R(x) = 0$ for all x in I_λ. Using this fact and the proof of [1, p. 387, Theorem 2.2], we can show that q is continuous if and only if each q_λ is continuous.

The following result is a generalization of [3, p. 471, Theorem 6.8].

LEMMA 3.2. Let A be a simple annihilator Banach algebra in which $x \in cl_A(xA)$ for all x in A. If A is infinite dimensional, then every quasi-complementor q on A is continuous.

PROOF. Let H be a minimal left ideal of A. As observed before, H is a Hilbert space under some equivalent inner product and A is a dense dual subalgebra of B, the algebra of all completely continuous linear operators on H. Also $||\cdot||$ majorizes $|\cdot|$ on A and H is a minimal left ideal of B. Then by [4, p. 148, Theorem 5.4], q can be extended to a quasi-complementor p on B; $M^p = cl([M \cap A]^q)$ for all closed right ideals M of B. We show that $M^p = \ell(M)^*$. In fact, let $S(M)$ be the smallest closed subspace of H that contains the range $x(H)$ for all x in M. Since $||\cdot||$ and $|\cdot|$ are equivalent on H, it follows from [4, p. 145, Lemma 4.1] that

$$S(M) = M \cap H = MH = (M \cap A) \cap H = (M \cap A)H. \tag{3.1}$$

Therefore, we have

$$S(M^p) = M^pH = cl([M \cap A]^q) \cap H = [M \cap A]^q \cap H. \tag{3.2}$$

(see [4, p. 148] for the last equality). By the proof of [4, p. 145, Lemma 4.2], $M \cap A = cl_A((M \cap A)HA)$. Since A is infinite dimensional, by [4, p. 145, Theorem 4.2 (iii)] and (3.1)

$$S(M)^{\perp} = [cl_A(S(M)A)]^q \cap H = [cl_A((M \cap A)HA))]^q \cap H$$
$$= [M \cap A]^q \cap H.$$

Therefore, by (3.2), $S(M)^{\perp} = S(M^p)$. Hence it follows from [3, p. 464, Lemma 4.1] and [3, p. 465, Theorem 4.2] that $M^p = \ell(M)^*$. In particular, p is continuous by [1, p. 388, Theorem 2.4].

Suppose $a_nA \in M_A$ $(n = 0, 1, 2, \ldots)$ with $a_n \to a_0$ in $||\cdot||$. Hence $a_n \to a_0$ in $|\cdot|$. Let L be a minimal left ideal of A. Then L is a minimal left ideal of B and $||\cdot||$ and $|\cdot|$ are equivalent on L; also $a_nA = a_nB$ for all n. Let f_n be a (unique) q-projection contained in a_nA. Then $P_{a_nA}(x) = f_nx$ for all x in A. Since p is continuous, P_{a_nA} converges to P_{a_0A} uniformly on L in $|\cdot|$ and hence in $||\cdot||$. Therefore q is continuous and this completes the proof.

Let A be a semi-simple annihilator quasi-complemented Banach algebra such that $x \in cl_A(xA)$ for all x in A which is a dense subalgebra of a B^*-algebra B. Suppose $||\cdot||$ majorizes $|\cdot|$ on A. By [8, p. 442, Lemma 5.1], the set E of all hermitian minimal idempotents of B is contained in the socle of A and so $E \subset A$. Let E_q be the set of all q-projections in A. For each $e \in E$, by [4, p. 149, Lemma 6.4], there exists a unique element $Q(e) \in E_q$ such that $Q(e)A = eA$; the mapping $Q : e \to Q(e)$ is a one - one mapping from E onto E_q and is called the q-derived mapping (see [3] and [4]).

As shown in [3, p. 475], Lemma 3.2 is not true in general, if the algebra A is finite dimensional. In this case, we have the following result:

LEMMA 3.3. Let A be a simple finite dimensional annihilator Banach algebra with a quasi-complementor q and E_q the set of all q-projections in A. Then q is continuous if and only if E_q is a closed and bounded subset of A.

PROOF. By [4, p. 143, Corollary 3.2], q is a complementor on A. Let H be a minimal left ideal of A. Then H is a Hilbert space and A can be taken as the B*-algebra of all linear operators on H. Let Q be the q-derived mapping. By [1, p. 388, Theorem 2.4], Q is continuous if and only if q is continuous. Now Lemma 3.3 follows from Lemma 4.1 in [11].

We have the main result of this section.

THEOREM 3.4. Let A be a semi-simple annihilator quasi-complemented Banach algebra such that $x \in cl_A(xA)$ for all x in A and let $\Lambda_0 = \{\lambda \in \Lambda : I_\lambda$ is finite dimensional$\}$. Then a quasi-complementor q on A is continuous if and only if E_q^λ is closed and bounded for each $\lambda \in \Lambda_0$, where E_q^λ is the set of all q-projections in I_λ.

PROOF. This follows from Lemma 3.1, 3.2 and 3.3.

4. UNIFORMLY CONTINUOUS QUASI-COMPLEMENTORS.

In this section, we assume that A is a semi-simple annihilator Banach algebra with a quasi-complementor q such that $x \in cl_A(xA)$ for all x in A. Once again, M_A will be the set of all minimal right ideals of A and E_q the set of all q-projections in A. Also let I_λ, H_λ, q_λ and B_λ be as in §3. The norm on B_λ is denoted by $|\cdot|$.

DEFINITION. A quasi-complementor q on A is said to be uniformly continuous if $\{P_{fA} : f \in E_q\}$ is closed and bounded with respect to $||P_{fA}||$, the operator bound norm of P_{fA}.

REMARK. A uniformly continuous quasi-complementor q is continuous. In fact, by Theorem 3.4, we can assume that A is simple and finite dimensional.

Let H be a minimal left ideal of A. By the proof of Lemma 3.3, A can be taken as the B*-algebra of all linear operators on H. Then by [7, p. 259, Theorem 4], E_q is bounded. Since $||f|| = \sup\{||fh|| : h \in H \text{ and } ||h|| \leq 1\}$, we have $||P_{fA}|| = ||f||$ for all $f \in E_q$. It follows now that E_q is closed. Hence by Theorem 3.4, q is continuous.

If u and v are elements of a Hilbert space H, $u \otimes v$ will denote the operator on H defined by the relation $(U \otimes v)(h) = (h, v)u$ for all h in H.

THEOREM 4.1. Let A be a semi-simple annihilator Banach algebra with a uniformly continuous quasi-complementor q in which $x \in cl_A(xA)$ for all x in A. Suppose that A has no minimal left ideals of dimension less than three. Then A is a dense subalgebra of some dual B*-algebra B and $R^q = \ell(R)^* \cap A$ for all closed right ideals R of A.

PROOF. We know that q is continuous and so is q_λ $(\lambda \in \Lambda)$. By [4, p. 148, Theorem 5.4], q_λ induces a quasi-complementor p_λ on B_λ. If H_λ is finite dimensional, then by [4, p. 143, Corollary 3.2], q_λ is a complementor and so by the proof of Theorem 4.3 in [11], p_λ has the form $J_\lambda^{p_\lambda} = \ell(J_\lambda)^*$ for all closed right ideals J_λ in B_λ. If H_λ is infinite dimensional, this is also true by the proof of Lemma 3.2.

We show that there exists a constant M such that

$$||h|| \leq |h| \leq M||h|| \qquad (h \in H_\lambda,\ \lambda \in \Lambda). \tag{4.1}$$

We follow the argument in [1, p. 393, Lemma 4.3]. It can be assumed that

$$||h|| \leq |h| \not\geq \sqrt{2}||h|| \qquad (h \in H_\lambda,\ \lambda \in \Lambda). \tag{4.2}$$

Suppose (4.1) does not hold. Then there exists x_n in H_n such that $||x_n|| = 1$ and $|x_n| = k_n > n$. By (4.2), we can find z_n in H_n such that $||z_n|| = 1$, $|z_n| \leq \sqrt{2}$. Write $z_n = \alpha_n x_n + x_n'$ with $\alpha_n \in C$, $x_n' \in H_n$ and $(x_n, x_n') = 0$. Put $y_n = k_n^{-1}x_n + x_n'$ and $f_n = (y_n \otimes y_n)/(y_n, y_n)$. Then $f_n \in E_q$ and

$$||P_{f_n A}(x_n)|| = ||\frac{y_n \otimes y_n}{(y_n, y_n)} x_n|| = \frac{|(x_n, y_n)|}{(y_n, y_n)} ||y_n|| \to \infty.$$

Hence $\{||P_{f_n A}||\}$ is unbounded and this contradicts the uniform continuity of q. Therefore (4.1) holds. Now by using the argument in Theorem 4.3, in [11], we can complete the proof.

Theorem 4.1 shows that there is essentially one type of uniformly continuous quasi-complementors on A.

The following result generalizes [4, p. 153, Theorem 7.6].

COROLLARY 4.2. Let A and B be as in Theorem 4.1. Then q is a complementor on A if and only if A is a left ideal of B.

PROOF. This follows from Theorem 4.1 and Theorem 3.4 in [11].

On the other hand, if q is a complementor, then we have:

THEOREM 4.3. Let A be a semi-simple annihilator Banach algebra such that A has no minimal left ideal of dimension less than three. Then every continuous complementor q on A is uniformly continuous.

PROOF. By [6, p. 655, Theorem 4], A is the direct topological sum of its minimal closed two-sided ideals $\{I_\lambda : \lambda \in \Lambda\}$ each of which is a complemented and dual algebra. Let q_λ, H_λ and B_λ be as before and $|\cdot|$ the norm on B_λ. By [1, p. 390, Theorem 3.2], q_λ induces a complementor p_λ on B_λ and by [1, p. 391, Theorem 3.3], p_λ has the form $J_\lambda^{p_\lambda} = \ell(J_\lambda)^*$ for all closed right ideals J_λ in B_λ. By [1, p. 393, Lemma 4.3], there exists a constant M such that

$$||h|| \leq |h| \leq M||h|| \qquad (h \in H_\lambda,\ \lambda \in \Lambda). \tag{4.3}$$

Let B be the $B^*(\infty)$-sum of $\{B_\lambda : \lambda \in \Lambda\}$. Then B is a dual B^*-algebra and E_q coincides with the set of all hermitian minimal idempotents in B. Since A is a left ideal of B, it is well-known that there exists a constant k such that $||ba|| \leq k|b| \quad ||a||$ for all b in B and a in A. Then

$||P_{fA}(x)|| = ||fx|| \leq k|f| \quad ||x|| = k||x||$ for all x in A and f in E_q. Hence $\{P_{fA} : f \in E_q\}$ is bounded. It remains to show that it is closed. Let $\{P_{f_n A}\}$ be a Cauchy sequence, where $f_n \in E_q$. We show that, for m and n large enough, f_m and f_n are contained in the same minimal closed two-sided ideal. Suppose this is not so. Then there exists some minimal closed two-sided ideal I_{λ_n} of A such that $f_n \in I_{\lambda_n}$, but $f_m \notin I_{\lambda_n}$. Let H_{λ_n} be the minimal left ideal in I_{λ_n}. Since $|f_n| = 1$, we can choose $h_n \in H_{\lambda_n}$ such that $|f_n h_n| > 1/2$ with $|h_n| = 1$. Since $f_m I_{\lambda_n} = (0)$, by (4.3) we have

$$1/2 < |f_n h_n| = |f_n h_n - f_m h_n| \leq M||f_n h_n - f_m h_n||$$

$$\leq M||P_{f_n A} - P_{f_m A}|| \; |h_n| = M||P_{f_n A} - P_{f_m A}||.$$

But $\{P_{f_n A}\}$ is a Cauchy sequence; a contradiction. Therefore, we can assume that f_m and f_n belong to the same I_{λ_n}. Hence,

$$|f_n - f_m| = \sup \{|(f_n - f_m)h| : h \in H_{\lambda_n} \text{ and } |h| \leq 1\}$$

$$\leq M||P_{f_n A} - P_{f_m A}||$$

and so $\{f_n\}$ is a Cauchy sequence in $|\cdot|$. Since E_q is closed in $|\cdot|$ by Theorem 4.2, in [11], $f_n \to f$ in $|\cdot|$ for some f in E_q. Since

$$||(P_{f_n A} - P_{fA})(x)|| = ||f_n x - fx|| \leq k|f_n - f| \quad ||x||$$

for all x in A, $P_{f_n A} \to P_{fA}$ and so $\{P_{fA} : f \in E_q\}$ is closed. This completes the proof.

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