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THE BEHAVIOR OF SOLUTIONS OF NON-LINEAR DIFFERENTIAL EQUATIONS IN HILBERT SPACE. I

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ABSTRACT. This paper deals with the behavior of solutions of ordinary differential equations in a Hilbert Space. Under certain conditions, we obtain lower estimates or upper estimates (or both) for the norm of solutions of two kinds of equations. We also obtain results about the uniqueness and the quasi-uniqueness of the Cauchy problems of these equations. A method similar to that of Agmon-Nirenberg is used to study the uniqueness of the Cauchy problem for the non-degenerate linear case.

0. INTRODUCTION.

In this paper the behavior of the solutions of ordinary differential equations in a Hilbert space are studied.

In Section I we study two problems for two types of equations.

Let H be a Hilbert space with scalar product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$. Let us consider the equation

$$t \frac{du}{dt} = B\ (t,u(t)) \tag{0.1}$$

where $t \in I = (0,1]$. If $t \geq 0$, $u(t)$ has a derivative with respect to t and $u(t)$ is an element of H, in other words, $u(t) \in C^1 (I, H)$. $B(t,\cdot)$ is a non-linear map from H to H for every $t \in I$ with domain D_B a dense subset of H.

For this equation we study the behavior of its solutions as $t \to 0$. Under some conditions, we obtain lower estimates or upper estimates (or both) for the norm of solutions of equation (0.1). We also obtain results about the uniqueness and the quasiuniqueness of the Cauchy problem for this equation.

Note that equation (0.1) is not an equation of the normal type in the Cauchy-Kovelevsky sense.

Let us consider the equation

$$\frac{du}{dt} = B\ (t,u(t)) \tag{0.2}$$

where $t \in \bar{I} = [1,\infty)$. If $t \in \bar{I}$, $u(t)$ has a derivative with respect to t and $u(t)$ is an element of the Hilbert space H. For every $t \in \bar{I}$, $B(t,\cdot)$ is a non-linear map from H to H with domain D_B, a dense subset of H.

For this equation we study the behavior of its solutions as $t \to \infty$. Under some conditions we obtain lower estimates or upper estimates (or both) for the norm of solutions of equation (0.2). We also obtain results about the uniqueness and the quasiuniqueness of the Cauchy problem for (0.2). We do not require smoothness or a Lipshitz condition on $B(t,u)$.

In the first part of section 1 we study equation (0.1), in the second part, equation (0.2). In the third part we investigate graphs of $\|u(t)\|$ and the uniqueness of the Cauchy problem for equations (0.1) and (0.2) and in part 4 we study several examples of quasi-linear ordinary and partial differential equations.

Our results for equation (0.1) are like the results of Alinhac-Baouendi [1] and our estimates for (0.2) are similar to those of Agmon-Nirenbert [2,3]. In Section 1 for a special case of $B(t.u)$ it is possible to obtain these results utilizing only the first derivative of $u(t)$. However, the results of Agmon-Nirenberg for a linear case are more exact and deep. In Section 2 we use a method similar to Agmon-Nirenberg to obtain results concerning the quasiuniqueness for a special case of equations (0.1) and (0.2).

In Section 2 we study the same problems for special types of equations (0.1) and (0.2) where

$$B\ (t,\ u) = \frac{\partial}{\partial x}\,[K(u)\,\frac{\partial u}{\partial x}\,]\ . \tag{0.3}$$

Here $t \in T = (0,1]$, $x \in R$ or $x \in \Omega$, Ω compact, $u|_{\partial\Omega} = 0$, and $K(u)$ is a real-valued function from C^1. H is a Hilbert space $L_2(\Omega)$.

For this type of equation we study the behavior of its solutions as $t \to 0$. We obtain esimates which can be used for the study of uniqueness and the quasiuniqueness for Cauchy problem.

In part 1 of section 2 we study the case of $K(u)$ with $K(0) \neq 0$, in part 2 we study the equation (0.2) with the same $B(t,u)$ as in (0.3) and in part 3 the case $K(0) = 0$ is examined. In part 4 convexity of the norm of the solutions of these equations is studied. In addition we examine (0.1) when $B(t,u)$ has the following form

$$B(t,\ u) = \frac{\partial}{\partial x}\,[K(u)\,\frac{\partial u}{\partial x}] + A(t,u)$$

where $A(t,u)$ is a bounded operator. The main results of Section 2 are Theorem 2.5 for equation (0.1) and Theorem 2.6 for (0.2).

We use in Section 2 the method which was used first by Agmon-Nirenberg in the study of uniqueness of the Cauchy problem for non-degenerate linear case [1,3]. We use a modification of this method from [4].

We will use the following definition:

DEFINITION 1. Let us consider scalar function $f(t)$ in the interval $I = [0,1]$. A function $f(t)$ is called flat at the point $t = 0$ if for any $n \to 0$, $t^{-n}f(t) \to 0$ est $\to 0$.

DEFINITION 2. We say that the quasiuniqueness takes place for equation (0.1) if from flatness of $\|u(t)\|$, it follows that $u(t) = 0$.

1. STUDY OF DIFFERENTIAL EQUATIONS (0.1) and (0.2).

EQUATION (0.1)

Let us now consider the equation

$$t\,\frac{du}{dt} = B(t,u(t)) \tag{1.1}$$

under the same conditions on $B(t,u(t))$ as above.

THEOREM 1.1. Assume that the following condition is satisfied:

$$\|B(t,u(t))\| \leq C\ \|u(t)\| \tag{1.2}$$

for some constant $C > 0$. In this case, the domain of B is all Hilbert space H for each $t \in I$. Then for each solution $u(t)$ of (1.1) from $C^1(I,H)$ the following estimates hold with the same C as in (1.2):

$$\|u(t)\| > \|u(t_0)\| \left(\frac{t}{t_0}\right)^C \tag{1.3}$$

for each $t_0 \in I$ and $t < t_0$, and

$$\|u(t)\| \le \|u(t_0)\| \left(\frac{t}{t_0}\right)^{-C} \tag{1.4}$$

for each $t_0 \in I$ and $t < t_0$.

PROOF. From (1.1) after taking the scalar product with $u(t)$, we get

$$(t\frac{du}{dt}, u(t)) = (B(t,u(t)),u(t))$$

and

$$\text{Re } (t\frac{du}{dt}, u(t)) = \tfrac{1}{2}t\frac{d}{dt}(u(t),u(t)) .$$

From (1.2), we obtain

$$-C(u(t),u(t)) \le \text{Re } (B(t,u(t),u(t)) \le C(u(t),u(t)) .$$

If now $q(t) = (u(t),u(t))$, then for $q(t)$, we have the two inequalities

$$\tfrac{1}{2}t\frac{dq}{dt} \le Cq(t)$$

and

$$\tfrac{1}{2}t\frac{dq}{dt} \ge -Cq(t)$$

From the first of these inequalities after rewriting we obtain

$$t\dot{q} - 2Cq = \phi_1(t) \le 0 .$$

If $q(t) > 0$ in the interval (t_2,t_0), we have

$$\frac{t\dot{q}}{q} - 2C = \frac{\phi_1(t)}{q} \le 0 .$$

From the following equation

$$\frac{d}{dt}[\ln q(t)] = \frac{2C}{t} + \frac{\phi_1(1)}{tq(t)}$$

and

$$\ln q(t) - \ln q(t_0) = \int_{t_0}^{t}\frac{\phi_1(\tau)}{\tau q(\tau)}d\tau = 2C\ln\frac{t}{t_0} + \int_{t_0}^{t}\frac{\phi_1(\tau)}{\tau q(\tau)}d\tau ,$$

$$q(t) = q(t_0)e^{2C\ln(\frac{t}{t_0})}\cdot\exp\left(\int_{t_0}^{t}\frac{\phi_1(\tau)}{\tau q(\tau)}d\tau\right)$$

and since

$$e^{2C\ln(\frac{t}{t_0})} = \left(\frac{t}{t_0}\right)^{2C} ,$$

and $\phi_1(t) \le 0\ \tau$, $q(\tau) \ne 0$ and $t < t_0$, we have

$$\exp\left(\int_{t_0}^{t}\frac{\phi_1(\tau)}{\tau q(\tau)}d\tau\right) = \exp\left(-\int_{t_0}^{t_0}\frac{\phi_1(\tau)}{\tau q(\tau)}d\tau\right)$$

where $-\frac{\phi_1(\tau)}{\tau q(\tau)} \ge 0$ for each τ.

It follows that

$$- \int_{t}^{t_0} \frac{\phi_1(\tau)}{\tau q(\tau)} d\tau \geq 0 \quad \text{for each} \quad t \leq t_0$$

and

$$\exp\left(- \int_{t}^{t_0} \frac{\phi_1(\tau)}{\tau q(\tau)} d\tau\right) \geq 1 \quad \text{for each} \quad t < t_0 .$$

From this we obtain

$$q(t) \geq q(t_0) \left(\frac{t}{t_0}\right)^{2C}$$

and for $\|u(t)\|$ estimate (1.3) follows.

To show estimate (1.4), use the inequality

$$\tfrac{1}{2}\, t \frac{dq}{dt} \geq - Cq\,(t)$$

and define $\phi_2(t)$ by

$$t\dot{q} + 2Cq = \phi_2(t) \geq 0 .$$

Then

$$\frac{d}{dt}[\ln q(t)] = - \frac{2c}{t} + \frac{\phi_2(t)}{tq(t)}$$

and for $q(t)$

$$q(t) = q(t_0)C^{-2C \ln(\frac{t}{t_0})} \cdot \exp \int_{t_0}^{t_0} \frac{\phi_2(\tau)}{\tau q(\tau)} d\tau .$$

Since

$$e^{-2 \ln(\frac{t}{t_0})} = \left(\frac{t}{t_0}\right)^{-2C}$$

and

$$- \frac{\phi_2(\tau)}{\tau q(\tau)} \leq 0 \quad \text{for each} \quad \tau ,$$

we have

$$\exp\left(\int_{t_0}^{t} \frac{\phi_2(\tau)}{\tau q(\tau)} d\tau\right) = \exp\left(- \int_{t}^{t_0} \frac{\phi_2(\tau)}{\tau q(\tau)} d\tau\right) \leq 1$$

for each $t \leq t_0$.

From this we obtain

$$q(t) \leq q(t_0) \left(\frac{t}{t_0}\right)^{-2C}$$

and for $\|u(t)\|$ we have estimate (1.4).

REMARK 1.1. Our estimates (1.3)-(1.4) are exact and it is impossible to improve these. This may be seen by observing

i) that the following function $u(t) = u\,(t_0) \left(\frac{t}{t_0}\right)^{C}$ is a solution of equation (1.1) with $B(t,u(t)) = Cu(t)$ and $u(t)|_{t=t_0} = u(t_0)$, and

ii) that the following function $u(t) = u\,(t_0) \left(\frac{t}{t_0}\right)^{-C}$ is a solution of equation (1.1) with $B(t,u(t)) = -Cu(t)$ and $u(t)|_{t=t_0} = u(t_0)$.

THEOREM 1.2. Let us assume that $u(t)$ is a solution of equation (1.1) with condition (1.2). If $u(t)$ satisfies

$$t^{-C} \|u(t)\| \to 0 \quad \text{as} \quad t \to 0 .$$

with the same C as in (1.2), then $u(t) = 0$ in interval I.

PROOF. The proof follows immediately from our estimate (1.3).

DEFINITION 3. We say that C-uniqueness for equation (1.1) takes place at the point $t = 0$, if there exists constant $C \geq 0$ such that from the following condition

$$t^{-C} \|u(t)\| \to 0 \quad \text{as} \quad t \to 0 ,$$

we obtain that $u(t) = 0$ in the interval I. Recall that the classical uniqueness is C-uniqueness with $C=0$. In other words, it is possible to formulate our Theorem 1.2 in the following form:

THEOREM 1.2a. Under conditions of Theorem 1.1, C-uniqueness takes place at the point $t = 0$ for solutions of equation (1.1).

From the proof of Theorem 1.1, it is easy to see that it can also be used to obtain similar one-sided estimates for un-bounded $B(t,u(t))$. Namely, we have the following theorem.

THEOREM 1.3 i) If

$$\mathrm{Re}(B(t,u(t)),u(t)) \geq -C \|u(t)\|^2 \tag{1.5}$$

for some constant $C \geq 0$ for each $u(t)$ from a dense subset D_B of the Hilbert space H, then for each solution of equation (1.1) we have the following estimate with the same C as in condition (1.5):

$$\|u(t)\| \leq \|u(t_0)\| \left(\frac{t}{t_0}\right)^{-C} \quad \text{for} \quad t < t_0 . \tag{1.6}$$

ii) If

$$\mathrm{Re}(B(t,u(t)),u(t)) \leq C \|u(t)\|^2 \tag{1.7}$$

for some constant $C > 0$ for each $u(t)$ from a dense subset D_B of the Hilbert space H, then for each solution $u(t)$ of equation (1.1), the following estimate holds with the same C as in condition (1.7):

$$\|u(t)\| \geq \|u(t_0)\| \left(\frac{t}{t_0}\right)^{C} \quad \text{for} \quad t < t_C . \tag{1.8}$$

REMARK 1.2. Our estimates (1.6),(1.8) are exact and it is impossible to improve them (see Remark 1.1).

REMARK 1.3. Recall that we do not require smoothness of $B(t,u(t))$. In the same way as Theorem 1.2, we have from Theorem 1.3 that the following statements holds.

THEOREM 1.4. Let us assume that $u(t)$ is a solution of equation (1.1) with condition (1.7). If $u(t)$ satisfies

$$t^{-C} \|u(t)\| \to 0 \quad \text{as} \quad t \to 0 , \tag{1.9}$$

with same C as in (1.7), then $u(t) = 0$ in interval I. In other words, under condition (1.7) C-uniqueness takes place at the point $t = 0$ for solutions of equation (1.1).

EQUATION (0.2).
Let us consider the equation

$$\frac{du}{dt} = B(t,u(t)) \tag{1.10}$$

in the interval $\bar{I} = [1,+\infty)$ and with the same conditions on $B(t,u(t))$ as above. After the change

$$s = \epsilon^{-t} \tag{1.11}$$

we obtain from equation (1.10) the following equation

$$s \frac{du}{ds} = B(s,u(s)) \tag{1.12}$$

in the interval $(0,1]$.

The equation (1.12) is an equation of the same type as equation (1.1). Because of this, it is possible to rewrite Theorems 1.1-1.3 for this situation.

DEFINITION 4. We say that C-uniqueness for equation (1.10) takes place at the point $t = +\infty$, if there exists constant $C \geq 0$ such that from the condition

$$e^{Ct} \|u(t)\| \to 0 \quad \text{as} \quad t \to C$$

implies that $u(t) = 0$ in the interval $\bar{I}$. Recall that the classical uniqueness at the point $t = +\infty$ may be formulated as C-uniqueness with $C = 0$.

THEOREM 1.5. If the following condition is satisfied

$$\mathrm{Re}(B(t,u(t)),u(t)) \geq -C \|u(t)\|^2 \tag{1.13}$$

for some constant C, for each $u(t)$ from a dense subset D_B of the Hilbert space H, then for each solution of equation (1.10) we have the following estimate with the same C as in (1.13):

$$\|u(t)\| \leq \|u(t_0)\| e^{C(t-t_0)} \tag{1.14}$$

for each $t_0 \in \bar{I}$ and $t > t_0$.

PROOF. The proof follows from the proof of Theorem 1.1.

THEOREM 1.6. If the following condition is satisfied

$$\mathrm{Re}(B(t,u(t)),u(t)) \leq C \|u(t)\|^2 \tag{1.15}$$

for some constant C, for each $u(t)$ from the dense subset D_B of the Hilbert space H, then for each solution $u(t)$ of equation (1.10) the following estimate holds with the same C as in (1.15)

$$\|u(t)\| \geq \|u(t_0)\| \, e^{-C(t-t_0)} \, . \tag{1.16}$$

The proof follows from the proof of Theorem 1.3.

GRAPHS OF $\|u(t)\|$.

From proof of Theorem 1.1, it is easy to see that the following statement is true.

THEOREM 1.7. Let us assume that one of the following conditions is satisfied for each non-trivial $u(t)$:

$$\text{i)} \quad \|B(t,u(t))\| < C \, \|u(t)\|^2 \tag{1.17}$$

for some constant C and for each $u(t) \neq 0$ from the Hilbert space H, or

$$\text{ii)} \quad |(B(t,u(t))| < C \, \|u(t)\|^2 \tag{1.18}$$

for some constant C and for each $u(t) \neq 0$ from a dense subset D_B of the Hilbert space H, or

$$\text{iii)} \quad |Re(B(t,u(t)),u(t))| < C \, \|u(t)\|^2 \tag{1.19}$$

for some constant C and for each $u(t) \neq 0$ from a dense subset D_B of the Hilbert space H.

Then for each non-trivial solution $u(t)$ of equation (1.1) from $C^1(I,H)$ the following estimates hold:

$$\|u(t)\| > \|u(t_0)\| \left(\frac{t}{t_0}\right)^C \quad \text{for} \quad t < t_0 \, , \tag{1.20}$$

and

$$\|u(t)\| < \|u(t_0)\| \left(\frac{t}{t_0}\right)^{-C} \quad \text{for} \quad t < t_0 \, . \tag{1.21}$$

From (1.20) and (1.21) we have that the following functions

$$\text{i)} \quad \|u(t)\| \, t^{-C} \tag{1.22}$$

$$\text{ii)} \quad \|u(t)\| \, t^{C} \tag{1.23}$$

are strongly monotonic in the interval I. The function defined in (1.22) is decreasing and the functions in (1.23) is increasing in this interval.

Namely, from (1.20) we obtain that for each pair, $t, t_1 \in I$ with $t < t_1$, then

$$\|u(t)\| \, t^{-C} > \|u(t_1)\| t_1^{-C} \, . \tag{1.24}$$

In a similar way, we obtain from (1.21) that for each pair, $t, t_1 \in I$, with $t < t_1$, then

$$\|u(t)\| \, t^C < \|u(t_1)\| t_1^C . \tag{1.25}$$

From this, we obtain the following statement:

THEOREM 1.8. i) Under the conditions of Theorem 1.7 it follows that each non-trivial solution $u(t)$ of equation (1.1) satisfies $u(t) \neq 0$, $t \in I = (0,1]$.

ii) Let $u(t)$ be a solution of equation (1.1) under conditions of Theorem 1.7. If $u(t_0) = 0$ at a point $t_0 \in I = (0,1]$, then $u(t) = 0$ in the interval $I = (0,1]$.

PROOF. ii) follows from i), and i) follows from our estimates (1.20) and (1.21); in other words, from the monotonicity conditions (1.24) and (1.25).
In a similar way from Theorem 1.3 we obtain the following statement:

THEOREM 1.9 i) If the following condition is satisfied

$$\mathrm{Re}(B(t,u(t)),u(t)) > -C \|u(t)\|^2 \tag{1.26}$$

for some constant C, for each $u(t) \neq 0$ from a dense subset D_B of the Hilbert space H, then for each non-trivial solution $u(t)$ of equation (1.1) we have the following estimate with the same C as in (1.26):

$$\|u(t)\| < \|u(t_0)\| (\frac{t}{t_0})^{-C} \quad \text{for} \quad t < t_0 . \tag{1.27}$$

ii) If the following condition is satisfied

$$\mathrm{Re}(B(t,u(t)),u(t)) < C \|u(t)\| \tag{1.28}$$

for some constant C, for each $u(t) \neq 0$ from a dense subset D_B of the Hilbert space H, then for each non-trivial solution $u(t)$ of equation (1.1) we have the following estimate with the same C as in (1.28):

$$\|u(t)\| > \|u(t_0)\| (\frac{t}{t_0})^{C} \quad \text{for} \quad t < t_0 . \tag{1.29}$$

REMARK 1.4. From (1.27) and (1.29), we have that in the situation of Theorem 1.9 ii) the function (1.22) is strongly monotonic and in the situation of Theorem 1.9 i), the function (1.23) is strongly monotonic, since our estimates (1.27) and (1.29) are true for each pair t, t_0 in I, with $t < t_0$. From the montonicity of these functions, we obtain the following statement:

THEOREM 1.10 i) Under the conditions of Theorem 1.9, each non-trivial solution $u(t)$ of equation (1.1), satisfies $u(t) \neq 0$ for each $t \in (0,1]$.

ii) Let $u(t)$ be a solution of (1.1) under conditions of Theorem 1.9. If $u(t_0) = 0$ at a point $t_0 \in I = (0,1]$, then $u(t) = 0$ in the interval $I = (0,1]$.

PROOF. ii) follows from i), and i) follows from our estimates (1.27) and (1.29).

REMARK 1.5. From Theorems 1.7-1.9 we see that each non-trivial solution $u(t)$ under the conditions of those theorem has, at most, one point, $t = 0$, where $u(t)_{t=0} = 0$ or perhaps $u(t)_{t=0}$ is not defined. It is possible to rewrite the theorems of this section for equation (0.2).

EXAMPLES.

1. Let $u(t)$ be a vector. For $u(t)$, we have a system of ordinary differential equations in the form (1.1). In this case, H is a finite-dimension Hilbert space and $B(t,u(t))$ is bounded. In this situation, from Theorem 1.1, we have both estimates for $\|u(t)\|$. If for example, $u(t)$ is a solution of the equation

$$t \frac{du}{dt} = f(t,u), \; f(t,0) = 0 \;\text{ and }\; \|f(t,u)\| \le C \|u\| \tag{1.30}$$

with bounded $f(t,u)$ for each $t \in I$ and each $u \in R^1$, and if a solution $u(t)$ of this system satisfies the condition

$$t^{-C} \|u(t)\| \to 0 \;\text{ as }\; t \to 0 ,$$

then $u(t)$ is trivial.

In this situation, we have two estimates for each solution of (1.30).

$$\|u(t_0)\| (\frac{t}{t_0})^C \le \|u(t)\| \le \|u(t_0)\| (\frac{t}{t_0})^{-C} \;\text{ for }\; t < t_0 .$$

For each $\varepsilon > 0$, we also have the following estimates,

$$\|u(t_0)\| (\frac{t}{t_0})^{C+\varepsilon} < \|u(t)\| < \|u(t_0)\| (\frac{t}{t_0})^{-(C+\varepsilon)}$$

and from this we get uniqueness for (1.30) in the following sense:

If $u(t_0) = 0$ for $t_0 > 0$, then $u(t) = 0$ in I.
If $u(t_0) \ne 0$ for some $t_0 > 0$, then $u(t) \ne 0$ in $I = (0,1]$.

2. Consider the following equation

$$t \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} [K(u) \frac{\partial u}{\partial x}] + T(u) \tag{1.31}$$

in the domain $I \times \Omega^1$, $I(0,1]$, $\Omega \subset R^1$, with $u_{|\partial\Omega} = 0$ and $H = L_2(\Omega)$. $K(u)$ is a real valued continuous function and $T(u)$ is bounded in the following sense:

$$\|T(u)\| \le C \|u\|$$

i) If $K(u) \ge 0$, then we have the situation of Theorem 1.3 ii) and for each solution of equation (1.31) the following estimate holds,

$$\|u(t)\| \ge \|u(t_0)\| (\frac{t}{t_0})^C$$

for $t < t_0$ where C depends on T.

In this case, we have uniqueness for the Cauchy problem of equation (1.31) in the following sense:

If $u(t_0) = 0$ for $t_0 > 0$, then $u(t) \ne 0$ for each point t in the interval $I = (0,1]$.

This follows immediately from the estimate

$$\|u(t)\| > \|u(t_0)\|(\frac{t}{t_0})^{C + \varepsilon}$$

for $t < t_0$, and each $\varepsilon > 0$.

In this case we also have C-uniqueness at the point $t = 0$.

ii) If $K(u) \le 0$ then we have the situation of Theorem 1.3 i) and for each solution of equation (1.31) we have the following estimate

$$\|u(t)\| \le \|u(t_0)\|(\frac{t}{t_0})^{-C}$$

for $t < t_0$ where C depends on T in the sense

$$\|T(u)\| \le C \|u\| .$$

In this case we have uniqueness in the following sense: If $u(t_0) = 0$ for $t_0 > 0$, then $u(t) = 0$ in the interval $I = (0,1]$, and if $u(t_0) \ne 0$ for $t_0 > 0$, then $u(t) \ne 0$ for each point t in interval $I = (0,1]$.

3. Consider the following equation

$$t \frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [a_i(t,x,u(t,x,u(t,x)) \frac{\partial u}{\partial x_i}] + T(t,x,u(t,x)) \tag{1.32}$$

where $t \in I = (0,1]$, $x \in \Omega \subset R^n$ and $u_{|\partial\Omega} = 0$, $H = L_2(\Omega)$, $a_i(t,x,u)$ all continuous functions, and $b_i(t,x,u) = \mathrm{Re}\, a_i(t,x,u)$.

i) If $b_i(t,x,u) \ge 0$, $i = 1,\dots n$, and $\mathrm{Re}(T(t,u),u(t)) \le C \|u(t)\|^2$ for some constant C, then we have the situation of Theorem 1.3 ii) and for each solution of equation (1.32) we have the following estimate

$$\|u(t)\| \le \|u(t_0)\|(\frac{t}{t_0})^{C} \text{ for } t < t_0 .$$

In this case we have uniqueness for the Cauchy problem (1.32) in the following senses:

If $u(t_0) = 0$ for $t_0 > 0$, then $u(t) = 0$ in the interval $I = (0,1]$.

If $u(t_0) \ne 0$ for $t_0 > 0$, then $u(t) \ne 0$ for each point t in the interval $I = (0,1]$.

These results follow immediately from the estimate,

$$\|u(t)\| > \|u(t_0)\|(\frac{t}{t_0})^{C + \varepsilon}$$

for $t < t_0$ and $\varepsilon > 0$, and from the results of §3. In this case we have C-uniqueness at the point $t = 0$. If $C = 0$ we have the following estimate

$$\|u(t)\| \ge \|u(t_0)\| \text{ for } t < t_0$$

and if $-C > 0$ we have

$$\|u(t)\| \geq \|u(t_0)\|(\frac{t}{t_0})^C \quad \text{for } t > t_0 .$$

In this case each non-trivial solution of (1.32) is always increasing for $t \to 0$, $\|u(t)\| \to +\infty$ as $t \to 0$.

ii) If $b_i(t,x,u) \leq 0$, $i = 1,\dots n$, and $Re(T(t,u),u) \geq -C\|u(t)\|^2$ for some constant C, then we are in the situation of Theorem 1.3 i) and for each solution of equation (1.32) we have the following estimate:

$$\|u(t)\| \leq \|u(t_0)\|(\frac{t}{t_0})^{-C} .$$

In this case we have uniqueness of the Cauchy problem (1.32) in the following sense:

If the solution $u(t)$ of (1.32) is equal to zero at a point $t = t_0 > 0$, then it will be equal to zero at each point t in the interval I and if $u(t_0) \neq 0$, then $u(t) \neq 0$ in the interval $I = (0,1]$. If $C = 0$, then

$$\|u(t)\| \leq \|u(t_0)\|$$

for $t < t_0$ and each solution of equation (1.32) is bounded. If $C < 0$, then

$$\|u(t)\| \leq \|u(t_0)\|(\frac{t}{t_0})^{-C}$$

for $t < t_0$ and in this case, the solution of equation (1.32) is always decreasing for $t \to 0$, $\|u(t)\| \to 0$ as $t \to 0$. It is possible to rewrite these examples in form (0.2).

2. SPECIAL CASES OF EQUATIONS (0.1) and (0.2).

CASE $K(0) \neq 0$.

Consider the equation

$$t\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[-K(u)\frac{\partial u}{\partial x}\right] \tag{2.1}$$

where $t \in T = (0,1]$, $x \in R$ or $\Omega \subset R$ and $u_{|\partial\Omega} = 0$.

$K(u)$ is a real-valued function. If $K(u) < 0$ then the quasiuniqueness result follows from §1.4 of Chapter 1. Let $u(t)$ be a nontrivial solution of equation (2.1). After taking the scalar product with $u(t)$ we obtain from (2.1)

$$\begin{aligned}(t\frac{\partial u}{\partial t}, u) &= (\frac{\partial}{\partial x}\ K(u)\frac{\partial u}{\partial x}, u)\\ &= +\left(K(u)\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right) .\end{aligned}$$

If we set $q(t) = (u(t), u(t))$ we have for $q(t)$

$$t\,\dot{q}(t) = 2\,Re(t\frac{\partial u}{\partial t}, u) = +2\left(K(u)\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right) . \tag{2.2}$$

If $K(u)$ is differentiable we may differentiate this equation with respect to t. Letting $t\frac{\partial}{\partial t} = D$, we have

$$D_q^2 = +\, 2t \frac{\partial}{\partial t} (K(u) \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}) = +\, 2[(K_u'(u)D(u) \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}) + (K(u)t \frac{\partial^2 u}{\partial t \partial x}, \frac{\partial u}{\partial x})$$

$$+ (K(u) \frac{\partial u}{\partial x}, t \frac{\partial^2 u}{\partial t \partial x})] = +\, 2[(K_u'(u)\, Du \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}) + 2\, \mathrm{Re}(K(u) \frac{\partial u}{\partial x}, t \frac{\partial^2 u}{\partial t \partial x})] .$$

If u is twice continuously differentiable with respect to t and x, then

$$\frac{\partial^2 u}{\partial t \partial x} = \frac{\partial^2 u}{\partial x \partial t} \text{ and so } \mathrm{Re}\, (K(u) \frac{\partial u}{\partial x}, t \frac{\partial^2 u}{\partial t \partial x}) = \mathrm{Re}\, (K(u) \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x \partial t}) =$$

$$- \mathrm{Re}\, (\frac{\partial}{\partial x} K(u) \frac{\partial u}{\partial x}, t \frac{\partial u}{\partial t}) = (\frac{\partial}{\partial x} [-K(u) \frac{\partial u}{\partial x}], \frac{\partial}{\partial x} - K(u) \frac{\partial u}{\partial x}) = \| \frac{\partial}{\partial x} K(u) \frac{\partial u}{\partial x} \| =$$

$$\| \frac{\partial}{\partial x} [- K(u) \frac{\partial u}{\partial x}] \|^2 .$$

From this we have

$$D^2 q = 2(K_u'(u)Du \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}) + 4\, \| \frac{\partial}{\partial x} - K(u) \frac{\partial u}{\partial x} \|^2 .$$

If $T_1 = (t_1, t_0)$ is a subinterval in T, when $u(t) \neq 0$, then $q(t) \neq 0$ in T_1 and so

$$\frac{(Dq)^2}{q} = \frac{4(K(u) \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x})^2}{q} = \frac{4[- (\frac{\partial}{\partial x} K(u) \frac{\partial u}{\partial x}, u)]^2}{q}$$

and hence

$$D^2 q - \frac{(Dq)^2}{q} = 2(K_u'(u)Du \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}) + 4\, \| \frac{\partial}{\partial x}[- K(u) \frac{\partial u}{\partial x}] \|^2 -$$

$$\frac{4[(\frac{\partial}{\partial x} [- K(u) \frac{\partial u}{\partial x}], u)]^2}{q} .$$

For each u we have that

$$4\, \| \frac{\partial}{\partial x} [- K(u) \frac{\partial u}{\partial x}] \|^2 - \frac{4[(\frac{\partial}{\partial x} [- K(u) \frac{\partial u}{\partial x}], \frac{\partial u}{\partial x}]^2}{q} \geq 0$$

and from this we conclude

$$D^2 q - \frac{(Dq)^2}{q} \geq 2(K_u'(u)Du \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}) . \tag{2.3}$$

If $\ell(t) = \ln\, q(t)$, then

$$\frac{Dq}{q} = D\ell(t) \quad \text{and} \quad D^2 \ell(t) = \frac{D^2 q}{q} - \frac{(Dq)^2}{q^2}$$

and for $\ell(t)$ we have

$$D^2 \ell(t) \geq \frac{2(K_u'(u)Du \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x})}{q(t)} \tag{2.4}$$

and

$$D\ell(t) = \frac{2(K(u)Du \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x})}{q(t)} . \tag{2.5}$$

LEMMA 2.1[4]. Let $\ell(t)$ be a twice differentiable function in the interval $T = (0,1]$, satisfying the following second-order differential inequality

$$D^2 \ell(t) + t\alpha(t)|D\ell(t)| + t\beta(t) \geq 0 \tag{2.6}$$

where $\alpha(t)$, $\beta(t)$ are non-negative continuous functions in the interval [0,1]. Then for each solution of (2.6) there exist non-negative constants ν, μ, such that

$$\ell(t) \geq \ell(t_0) + \nu \ln\left(\frac{t}{t_0}\right) + \mu \ln\left(\frac{t}{t_0}\right)$$

or

$$\exp \ell(t) \geq [\exp \ell(t_0)] \left(\frac{t}{t_0}\right)^{\nu} \cdot \left(\frac{t}{t_0}\right)^{\mu} \tag{2.7}$$

where ν depends on $\alpha(t)$ and $\beta(t)$ only, and μ depends on $\alpha(t)$, $\beta(t)$ and the solution $\ell(t)$.

PROOF. From (2.6), it follows that

$$D^2\ell(t) + C\, t|D\ell(t)| + C\, t \geq 0 .$$

By the change of variable

$$t = e^{-\tau}$$

we have for $\ell(t)$

$$\ell(\tau) + C\, e^{-\tau}|\dot{\ell}(\tau)| + C\, e^{-\tau} > 0.$$

From Lemma 1.2 of [2] we get

$$\ell(\tau) > \ell(\tau_0) + \min\{0, \ell(\tau_0)\}\, e^{Ce^{\tau_0}} (\tau - \tau_0) - Ce^{Ce^{\tau_0}} e^{\tau_0} (\tau - \tau_0)$$

or,

$$\ell(\tau) \geq \ell(t_0) + \min\{0, t_0\dot{\ell}(t_0)\} \exp\left(\frac{C}{t_0}\right) \ln\left(\frac{t_0}{t}\right)$$

$$- C \exp\left(\frac{C}{t_0}\right) t_o^{-1} \ln\frac{t_0}{t} = \ell(t_0) + \max\{0, -t_0\dot{\ell}(t_o)\} \exp\left(\frac{C}{t_o}\right) \ln\frac{t}{t_o}$$

$$+ C \exp\left(\frac{C}{t_0}\right) t_o^{-1} \ln\frac{t}{t_0} = \ell(t_0) + \mu(t_0) \ln\frac{t}{t_0} + \nu(t_0) \ln\frac{t}{t_0}$$

where

$$\mu(t_0) = \max\{0, -t_0\dot{\ell}(t_0)\} \exp\left(\frac{C}{t_0}\right)$$

and

$$\nu(t_0) = C \exp\left(\frac{C}{t_0}\right) t_0^{-1} .$$

REMARK. In the similar way from the second-order inequality of the type

$$D^2\ell(t) + ta(t)|D\ell(t)| + tb(t) > 0$$

we obtain for $\ell(t)$ the following estimate

$$\exp \ell(t) > \exp \ell(t_0) \left(\frac{t}{t_0}\right)^{\nu} \left(\frac{t}{t_0}\right)^{\mu} \quad \text{for} \quad t < t_0 .$$

From this lemma we see that if a solution $u(t)$ of equation (2.1) satisfies the following condition

$$(K'(u)Du\,\frac{\partial u}{\partial x},\frac{\partial u}{\partial x}) \geq -\,t_\alpha(t)|(K(u)\,\frac{\partial u}{\partial x},\frac{\partial u}{\partial x})| - t_\beta(t)q(t) \tag{2.8}$$

then

$$q(t) \geq q(t_0)\,(\frac{t}{t_0})^\nu\,(\frac{t}{t_0})^\mu \,. \tag{2.9}$$

Hence if $u(t)$ is a flat-solution of equation (2.1) satisfying condition (2.8), then $u(t) \equiv 0$ in T. It is possible to restrict condition (2.8) and to rewrite it in the following form

CONDITION C:

For each $v(t,x)$ there exists a constant C depending on v such that

$$K'(v)v_t' + CK(v) \geq 0 \tag{2.10}$$

for each (t,x), $x \in \Omega$, $t \in T_\varepsilon = (0,\epsilon)$ and $\varepsilon > 0$.

THEOREM 2.1. Let $u(t)$ be a C^2 non-trivial solution of equation (2.1) and $K(u)$ be a real-valued function satisfying condition C. Then for $u(t)$ we have the following estimate

$$\|u(t)\| \geq \|u(t_0)\|\, t^{\nu+\mu} \quad \text{for} \quad t \leq t_0 \tag{2.11}$$

with ν depending on K and t_0, μ depending on K, t_0 and the solution $u(t)$.

PROOF. This follows immediately from the previous discussion and Lemma 2.1.

Let us now consider the following differential inequality in the interval (0,1],

$$f'(t) + cf(t) \geq 0 \,. \tag{2.12}$$

Define $\phi(t)$ by

$$f'(t) + cf(t) = \phi(t) \geq 0$$

and so

$$[e^{ct}f(t)]' = e^{ct}\phi(t) \,.$$

After integrating we have

$$e^{ct}f(t) - e^{ct_0}f(t_0) = \int_{t_0}^{t} e^{c\tau}\phi(\tau)d\tau, \; t < t_0$$

where

$$\int_{t_0}^{t} e^{c\tau}\phi(\tau)d\tau \leq 0$$

since $t < t_0$ and $\phi(\tau) \geq 0$. From this we have

$$e^{ct}f(t) - e^{ct_0}f(t_0) \leq 0 \quad t < t_0 \tag{2.13}$$

for each pair of points t, t_0 in the interval $T = (0,1)$ with $t < t_0$. In other words, the function $e^{ct}f(t)$ will be monotonic and does not decrease in the interval $(0,1)$.

From (2.13) we have

$$f(t) \le e^{-c(t-t_0)} f(t_0) = e^{c(t_0-t)} f(t_0) \ . \tag{2.14}$$

Now we can rewrite Condition C in the following form:

For each $u(t,x)$ there exists a constant c such that $K(u(t))$ as a function of t will satisfy condition (2.14) for (t,x), $x \in \Omega$, $t \in T$.

In other words, $K(u)$ as a function of t increases as $t \to 0$ no more than e^{-ct}.

For example, if $K(u)$ is a monotonic decreasing function with $K(0) > 0$, then $K(u)$ satisfies our condition with $c = 0$. From this discussion we have the following theorem.

THEOREM 2.2. Let $u(t)$ be a non-trivial solution of equation (2.1) under condition C. If $\|u(t)\|$ is flat then $u(t) \equiv 0$ in the interval $T = (0,1)$.

REMARK 2.1. We can obtain in the same way results for an inequality of the type

$$\| t \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} [K(u) \frac{\partial u}{\partial x}] \| \le Ct \| u \| \ . \tag{2.15}$$

COROLLARY 2.1. Let $K(u)$ be continuous and differentiable at the point $u = 0$. If $K(0) \ne 0$, then quasiuniqueness takes place for the equation

$$t \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} [-K(u) \frac{\partial u}{\partial x}] \ .$$

PROOF. If $K(0) \ne 0$ we have the two following cases.

1) If $K(0) < 0$ quasiuniqueness follows from Example 2 of part 4 of Section 1.
2) If $K(0) > 0$ and $K'(0)$ is bounded, then there exists for each $u(t)$ with flat norm $\| u(t) \|$, an $\varepsilon > 0$ such that for $t \in [0,\varepsilon]$

$$K'u' + C\,K(u) \ge 0$$

and in this case the quasiuniqueness takes place also. This follows from Theorem 2.2.

THE NON-GENERATE EQUATION.

Consider the following equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} [K(u) \frac{\partial u}{\partial x}] \tag{2.16}$$

where $t \in T = [1,+\infty]$. After the change of variables

$$s = e^{-t}$$

we obtain from equation (2.16) the following equation

$$s \frac{\partial u}{\partial s} - \frac{\partial}{\partial x} [K(u) \frac{\partial u}{\partial x}] \tag{2.17}$$

where

$$s \in T = (0,1] \ .$$

The equation (2.17) is the same as equation (2.1), and hence it is possible to rewrite all of our theorems for equation (2.16)

For equation (2.16) we have a condition analogous to condition C.

Condition C'. For each $v(t)$ there exists a constant C depending on v such that

$$K'(v)v_t' + Ce^{-t} K(v) \geq 0$$

for each (t,x), $x \in \Omega$, $t > N$ for some N.

The following theorems follow immediately.

THEOREM 2.3. Let $u(t)$ be a non-trivial solution of equation (2.16) and $K(u)$ a real-valued function satisfying condition C'. Then for $u(t)$ we have the following estimate

$$\|u(t)\| > M \|u(t_0)\| e^{-\nu t} e^{-\mu t} \tag{2.18}$$

where ν depends on K and t_0 and μ depend on K, t_0 and u.

THEOREM 2.4. Let $K(u)$ be a real-valued function satisfying condition C'. Let $u(t)$ be a solution of equation (2.16) satisfying the following condition

$$\text{for each positive } C, e^{Ct} \|u(t)\| \to 0 \text{ as } t \to +\infty .$$

Then $u(t) \equiv 0$ in the interval $\bar{T} = [1,+\infty)$.

COROLLARY 2.2. Let $K(u)$ be continuous and differentiable at the point $u = 0$. Suppose $K(0) \neq 0$, and $K'(0)$ is bounded. Let $u(t)$ be a solution of equation

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left[K(u)\frac{\partial u}{\partial x}\right]$$

satisfying the condition

$$\|u(t)\| e^{-ct} \to 0 \text{ as } t \to +\infty \text{ for each positive } c.$$

Then $u(t) \equiv 0$ in the interval $\bar{T} = [1,+\infty)$.

This Corollary follows from Corollary 6 of §2.

CASE $K(0) = 0$.

In this section we obtain results about quasiuniqueness for equation (2.1) in the case $K(0) = 0$.

Let us consider the equation

$$t\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left[K(u)\frac{\partial u}{\partial x}\right] \tag{2.19}$$

where $t \in T = (0,1)$ and $K(u)$ is differentiable with respect to u, $K(0) = 0$, $x \in \Omega \subset R^1$, $u_{|\partial\Omega} = 0$, and Ω is compact.

In this case $K(u) = u K_1(u)$ where $K_1(u)$ is continuous. From this we obtain

$$\left(t\frac{\partial u}{\partial t}, u\right) = -\left(K(u)\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right) .$$

If

$$q(t) = (u(t), u(t))$$

and $T_1 = (t_1,t_0)$ is a subinterval in T such that $q(t) \neq 0$ in T_1, we may introduce a new function $v(t)$ by the formula

$$v(t) = \frac{u(t)}{q^{\frac{1}{2}}(t)}$$

or

$$u(t) = q^{\frac{1}{2}}(t)\, v(t) .$$

After taking the scalar product of (2.19) with $u(t)$ we obtain the following equation for $q(t)$

$$\tfrac{1}{2}\, t\, \dot{q}(t) = -\left(K(u)\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right) = q^{3/2}\left(-vK_1(vq^{\frac{1}{2}})\frac{\partial v}{\partial x}, \frac{\partial v}{\partial x}\right) .$$

Let $u(t)$ be a C^1 solution of (2.19) with flat norm or, in other words, with flat $q(t)$. Then the following function

$$u_\delta(t) = q^\delta(t)v(t)$$

is flat for each $\delta > 0$ and

$$\frac{\partial u_\delta(t)}{\partial x} = q^\delta(t)\,\frac{\partial v(t)}{\partial x} .$$

Thus it is possible to rewrite the equation for $q(t)$ in the following form

$$\tfrac{1}{2}\, t\, \dot{q}(t) = q^{3/2-2\delta}\left(-v\, K_1(vq^{\frac{1}{2}})\,\frac{\partial u_\delta}{\partial u}, \frac{\partial u_\delta}{\partial u}\right) .$$

Since

$$\left|\left(-v\, K_1(vq^{\frac{1}{2}})\,\frac{\partial u_\delta}{\partial x}, \frac{\partial u_\delta}{\partial x}\right)\right| \le |K_1(vq^{\frac{1}{2}})|\, \left\| \frac{\partial u_\delta}{\partial x} \right\|^2 ,$$

and $K_1(vq^{\frac{1}{2}})$ is continuous in a neighborhood of the point $q = 0$ there exists a constant $C > 0$ such that

$$|K_1(vq^{\frac{1}{2}})| < C \quad \text{for } q < \varepsilon_0$$

and the following inequality for $q(t)$ holds,

$$t\,\dot{q} \ge 2C\, q^{3/2-2\delta}\, \left\| \frac{\partial u_\delta}{\partial x} \right\|^2 .$$

From the smoothness and flatness of $u(t)$ we see that $\left\| \frac{\partial u_\delta}{\partial x} \right\|$ is flat and bounded. From this we obtain the following inequality

$$t\,\dot{q} \le 2C\, q^{3/2-2\delta} \quad \text{for } q < \varepsilon_0 ,$$

with the constant C' depending on $u(t)$ itself. Since $q(t) \to 0$ as $t \to 0$, for each $\delta_1 > 0$ there exists $\varepsilon > 0$ such that

$$2C'q^{3/2-2\delta} < 2\delta_1\, q \quad \text{if } t \in (0,\varepsilon), \quad \delta < \tfrac{1}{4}.$$

Here the constants ε, δ_1 depend on $q(t)$. If

$$\ell(t) = \ln q(t)$$

we have for $t < \varepsilon$,

$$t\,\dot{\ell}(t) < 2\delta_1$$

or

$$t\,\frac{\partial}{\partial t}\left[\ell(t) - 2\delta_1 \ln t\right] < 0 .$$

From this we conclude that

$$\ell(t) - 2\delta_1 \ln t$$

is monotonic in the interval $(0,\varepsilon)$ and it is not increasing. Thus for $t < t_0$ we have

$$\ell(t) - 2\delta_1 \ln t > \ell(t_0) - 2\delta_1 \ln t_0,$$

or

$$\ell(t) - \ell(t_0) > 2\delta_1 \ln \frac{t}{t_0}$$

or

$$\ln \frac{q(t)}{q(t_0)} > 2\delta_1 \ln \frac{t}{t_0} ,$$

and hence

$$q(t) > q(t_0) \left(\frac{t}{t_0}\right)^{2\delta_1} \text{ for } t < t_0 < \varepsilon .$$

For $\|u(t)\|$ we have the following estimate

$$\|u(t)\| > \|u(t_0)\| \left(\frac{t}{t_0}\right)^{\delta_1} . \tag{2.20}$$

From estimate (2.20) we obtain the quasiuniqueness for solutions of equation (2.19) in the case $K(0) = 0$.

From this we obtain the following statements.

THEOREM 2.5. For a real-valued function $K(u) \in C^1$ quasiuniqueness takes place at the point $t = 0$ for equation (2.19).

THEOREM 2.6. Quasiuniqueness takes place at the point $t = +\infty$ for the equation (2.16) for every real-valued function $K(u) \in C^1$.

Remark 2.2. Theorem 2.5 completes the Corollary 2.1 of §1 and Theorem 2.6 completes the Corollary 2.2 of §2.

CONVEXITY OF THE NORM OF A SOLUTION. THE INFLUENCE OF A BOUNDED OPERATOR.

First we study the convexity of the norm of a solution of equation (2.1). It is possible to obtain the same type of convexity as in Hadamard's three circles theorem. In the linear case, there are complete results on this type of convexity [1,2]. After this, we study the influence of a bounded operator on the quasiuniqueness at the point $t = 0$.

Let $u(t)$ be a C^2-solution of equation (2.1) under the same assumptions as above, and $K(u) \geq \varepsilon > 0$ for any real u. If $K'(u)$ is bounded, then there exists a constant C such that

$$D^2\ell(t) + tCD\ell(t) + tC \geq 0 \tag{2.21}$$

where

$$\ell(t) = \ln q(t),\ q(t) = (u(t),u(t)) .$$

(This follows from the discussion in §2.1.) After the change of variables

$$t = e^{-s}$$

we obtain for $\ell(t)$ the following inequality

$$\frac{d^2\ell(s)}{ds^2} + Ce^{-s} \frac{d\ell}{ds} + Ce^{-s} \geq 0, \quad s \in [1,+\infty) . \tag{2.22}$$

It follows [2, Lemma 1.4] $\ell(s)$ will satisfy the following condition:

If $1 < s_1 < s < s_2 < \infty$, then

$$\ell(s) \le \ell(s_1)\, \frac{\int_s^{s_2} e^{\mp C \int_{s_1}^{\tau} e^{-\lambda} d\lambda} d\tau}{\int_{s_1}^{s_2} e^{\mp C \int_{s_1}^{\tau} e^{-\lambda} d\lambda} d\tau} +$$

$$\ell(s_2)\, \frac{\int_{s_1}^{s} e^{\mp C \int_{s_1}^{\tau} e^{-\lambda} d\lambda} d\tau}{\int_{s_1}^{s_2} e^{\mp C \int_{s_1}^{\tau} e^{-\lambda} d\lambda} d\tau} + e^{2C \int_{s_2}^{s_1} e^{-\lambda} d\lambda} C \int_{s_1}^{s_2} (s-\tau) e^{-\lambda} d\lambda$$

Since

$$\int_{s_1}^{s} e^{-\lambda} d\lambda = -e^{-\lambda} \Big|_{s_1}^{s} = e^{-s_1} - e^{-s} ,$$

we see that

$$\ell(s) \le \ell(s_1)\, \frac{\int_s^{s_2} e^{\mp C(e^{-s_1} - e^{-\tau})} d\tau}{\int_{s_1}^{s_2} e^{\mp C(e^{-s_1} - e^{-\tau})} d\tau} +$$

$$\ell(s_2)\, \frac{\int_s^{s_2} e^{\mp C(e^{-s_1} - e^{-\tau})} d\tau}{\int_{s_1}^{s_2} e^{\mp C(e^{-s_1} - e^{-\tau})} d\tau} + K$$

$(\pm)$ is taken as $-$ if $\ell(s_1) \le \ell(s_2)$ and
$+$, if $\ell(s_1) > \ell(s_2)$.

From this, we see that there exist two non-negative functions $\alpha(t)$, $\beta(t)$ and a constant k such that

$$\ell(s) \le \alpha(s)\ell(s_1) + \beta(s)\ell(s_2) + k \tag{2.23}$$

where

$$\alpha(s) + \beta(s) = 1 \quad \text{for} \quad s_1 \le s \le s_2$$

and k depends on C, s_1, s_2.

For $u(t)$ we have Hadamard's three circles theorem. Namely, the following statement is true.

THEOREM 2.7. Let $|K(u)| \ge \varepsilon > 0$ and $|K'(u)| \le L$ for every u. If $u(t)$ is a C^2-solution of (2.1) and $0 < t_2 < t < t_1 < 1$ then there exist two nonnegative continuous functions $\alpha(t)$, $\beta(t)$ such that the following holds:

i) $\alpha(t) + \beta(t) = 1$, $t_2 \le t \le t_1$.

ii) $\ell(t) \le \alpha(t)\ell(t_1) + \beta(t)\ell(t_2) + k$.

iii) k is a constant depending on C from (2.21) (or on $K(u)$) and t_1, t_2

Recall that

$$\ell(t) = \ln \left(\frac{1}{t^0} \int_{t=\text{const}} | u(t,x) |^2 dx\right) . \tag{2.24}$$

This is essentially Hadamard's three circles theorem in this case. Since the proof for $K(u)$ positive has been discussed above, we only consider the case where

$K(u) \le -\varepsilon < 0$. Set $\ell_1(t) = -\ln q(t)$. Ther

$$D\ell_1(t) = -\frac{Dq}{q} : D^2\ell_1(t) = D\,[D\ell_1(t)] = D(\frac{Dq}{q})$$
$$= \frac{(Dq)^2}{q^2} - \frac{D^2q}{q}. \tag{2.25}$$

From §1 we have

$$D^2q = 2(K_u'(u)Du\,\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}) + 4\,\|\frac{\partial}{\partial x}[-K(u)\,\frac{\partial u}{\partial x}]\,\|^2,$$

$$\frac{(Dq)^2}{q} = \frac{4[-(\frac{\partial}{\partial x}[K(u)\,\frac{\partial u}{\partial x}]u)]^2}{q}$$

and

$$D^2\ell_1(t) = \frac{4(\frac{\partial}{\partial x}[-K(u)\,\frac{\partial u}{\partial x}\,u)^2}{q^2} - \frac{4\,\|\frac{\partial}{\partial x}[-K(u)\,\frac{\partial u}{\partial x}]\,\|^2}{q}$$
$$= \frac{2(K_u'(u)Du\,\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x})}{q} \le -\frac{2(K_u'(u)Du\,\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x})}{q(t)}.$$

Thus

$$D^2\ell(t) \ge \frac{2(K_u'(u)Du\,\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x})}{q(t)} \quad \text{where} \quad \ell(t) = \ln q(t)$$

and

$$D\ell(t) = \frac{2(K(u)\,\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x})}{q(t)},\ D\ell_1(t) = \frac{2(K(u)Du\,\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x})}{q(t)}.$$

Since $K(u) \le -\varepsilon < 0$, $D\ell(t) \le 0$, and $K'(u)$ is bounded, then there exists $C < 0$ such that

$$D^2\ell(t) - CtD\ell(t) \ge 0 \quad \text{or} \quad D^2\ell(t) + Ct|D\ell(t)| \ge 0. \tag{2.26}$$

Consider now the following equation

$$t\,\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[K(u)\,\frac{\partial u}{\partial x}] + T(u) \tag{2.27}$$

under the same assumptions as above and with $T(u)$ a bounded operator in the following sense,

$$\|T(u)\| \le C\,\|u\|,\quad u \in H. \tag{2.28}$$

Here we consider three cases:

i) $K(0) > 0$,

ii) $K(0) = 0$,

iii) $K(0) < 0$.

For the first two cases we use the following fact.

LEMMA 2.2 [4]. Let $\phi(t)$ be a positive function in the interval $I = (0,1]$ and $m_\phi(t)$ the following "almost" logarithmic derivative of $\phi(t)$,

$$m_\phi(t) = t\,\frac{\phi'(t)}{\phi(t)} = t\,\frac{\partial}{\partial t}[\ln \phi(t)]. \tag{2.29}$$

Then $\phi(t)$ is flat at the point $t = 0$, if and only if the following condition holds,

$$m_\phi(t) \to +\infty \quad \text{as} \quad t \to 0.$$

PROOF. If $\phi(t) \neq 0$ for $t \neq 0$, then $m_\phi(t)$ is a continuous function for $t \neq 0$. For all $t_0 \in (0,1]$

$$\phi(t) = \phi(t_0) \exp \int_{t_0}^{t} \frac{m_\phi(\tau)}{\tau} d\tau$$

and $\phi(t) \to 0$ for $t \to 0$, is equivalent to

$$\int_0^{t_0} \frac{m_\phi(\tau)}{\tau} d\tau = +\infty \tag{2.30}$$

Suppose $m_\phi(t) \to +\infty$ as $t \to 0$. To prove that $\phi(t)$ is flat we need to show that $\forall\, n > 0$, $t^{-n}\phi(t) \to 0$ as $t \to 0$. Introducing a new function,

$$\phi_n(t) = t^{-n}\phi(t)$$

we have

$$m_{\phi_n}(t) = m_\phi(t) - n$$

and since $m_{\phi_n}(t) \to +\infty$ as $t \to 0$ we have

$$\int_0^{t_0} \frac{m_{\phi_n}(\tau)}{\tau} d\tau = \int_0^{t_0} \frac{m_\phi(\tau)-n}{\tau} d\tau = +\infty \tag{2.31}$$

for all $t_0 \in I_n$ and so $t^{-n}\phi(t) \to 0$ as $t \to 0$.

Conversely, suppose $\phi(t)$ is flat. Then the integral

$$\int_0^{t_0} \frac{m_\phi(\tau)-n}{\tau} d\tau = +\infty \quad \text{for all } n.$$

From here we have that $m_\phi(t)$ is unbounded. Let us prove that if $\phi(t)$ is flat then $\delta > 0$ such that $\phi(t)$ is monotonic in the interval $(0,\delta)$. Assume that $\phi(t)$ is not monotonic and $\phi(t) > 0$ for $t > 0$. It follows then that $\{t_k\} \in I$ such that $t_{k+1} < t_k$ and $\phi(t_k) < \phi(t_{k+1})$ and $t_k \to 0$. But this contradicts the fact that $\phi(t) \to 0$ as $t \to 0$ since $\phi(t_k) > \phi(t_0) > 0$ $\forall_k$.

It follows from here that $m_\phi(t)$ is non-negative in the interval $I_\delta = (0,\delta)$. The same considerations carried out for $\phi_n(t) = t\ \phi(t)$ show that $m_{\phi_n}(t)$ is also non-negative. Since $m_\phi(t) - n$ is non-negative in some neighborhood $(0,\delta_n)$, $\lim_{t\to 0} m_\phi(t)$ exists and this limit is $+\infty$ due to the unboundedness of $m_\phi(t)$. The Lemma is proved.

We have the following theorem.

THEOREM 2.8. Let $K(u)$ be C^1 real-valued function and $T(u)$ satisfying the condition (2.28). If $u(t)$ is C^2 flat solution of (2.27), then $u(t)$ is identically zero in the interval I.

In other words, quasiuniqueness takes place for solutions of equation (2.27) at the point $t = 0$. That is, the operator $T(u)$ has no influence on quasiuniqueness.

PROOF.

i) If $K(u) > 0$ near the origin the proof is trivial and follows immediately from Lemma 2.2.

ii) If $K(0) = 0$ by the same reasoning as in §2.3 and using the above Lemma we obtain that quasiuniqueness takes place in this case.

iii) If $K(u) < 0$ near the origin and $u(t)$ is a C^2 flat solution of (2.27), then $u(t)$ can be written in the form

$$u(t) = t^\lambda v(t), \ \lambda \in \mathbb{R}$$

with $v(t)$ a C^2 flat function. For $v(t)$ we have the following equation from (2.27):

$$t \frac{\partial v}{\partial t} - \frac{\partial}{\partial x} [K(t^\lambda v(t) \frac{\partial v}{\partial x}] + t^{-\lambda}T(t^\lambda v(t)) - \lambda v(t) . \tag{2.32}$$

From (2.28) it follows that the operator $t^{-\lambda}T(t^\lambda v(t))$ is bounded. If $-\lambda > C$, the operator

$$B(t,v) \equiv t^{-\lambda}T(t^\lambda v(t)) - \lambda v(t)$$

will be non-negative in the sense that

$$(B(t,v),v) \geq 0$$

for any $v \in H$. As above, let

$$q(t) = (v(t),v(t))$$

and

$$\ell(t) = \ln q(t) .$$

Then using a similar argument as in §1 we obtain that

$$D\ell(t) = - (K(t^\lambda u(t)) \frac{\partial v}{\partial x}, \frac{\partial v}{\partial x}) + (B(t,v),v)$$

and

$$D\ell(t) \geq - (K(t^\lambda v(t)) \frac{\partial v}{\partial x}, \frac{\partial v}{\partial x}) . \tag{2.33}$$

Now in a similar way as in §1 we obtain for $\ell(t)$ an inequality of the following type

$$D^2\ell(t) + Ct|D\ell(t)| \geq 0 .$$

From here as in §1 it follows that

$$\|v(t)\| \geq M \|v(t_0)\| t^\mu \quad \text{for} \quad t < t_0 \tag{2.34}$$

with μ depending on $v(t)$ itself. From (2.34) we obtain that $u(t)$ satisfies the following estimate

$$\|u(t)\| \geq M^1 \|u(t_0)\| t^{\mu+\lambda} . \tag{2.35}$$

This contradicts the flatness of $u(t)$ and so Theorem 2.8 is proved.

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