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NONLINEAR INTEGRAL INEQUALITY IN TWO INDEPENDENT VARIABLES

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ABSTRACT. In this note, the authors obtain a generalization of the integral inequality of Bihari [1] to a nonlinear inequality in two independent variables. With the aid of this inequality a bound for the solution of a nonlinear partial differential equation is established.

KEYS WORDS AND PHRASES. *Nonlinear integral inequality, submultiplicative function, nonlinear partial differential equation.*
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1. INTRODUCTION.

In the qualitative analysis of differential equations integral inequalities play a vital role [2]. An inequality due to Gronwall continues to draw the attention of mathematicians because of its usefulness. The nonlinear generalization of this inequality due to Bihari [1] is as follows:

LEMMA 1. Let $Y(x)$, $F(x)$ be positive continuous functions in $a \leqq x \leqq b$ and $K \geqq o$, $M \geqq o$, further $W(u)$ a non-negative non-decreasing continuous function for $u \geqq o$. Then the inequality

$$Y(x) \leqq K + M \int_a^x F(t) \; W(Y(t))dt \qquad (a \leqq x \leqq b)$$

implies the inequality

$$Y(x) \leqq G^{-1}(G(K) + M \int_a^x F(t) \, dt) \qquad (a \leqq x \leqq b' \leqq b)$$

where

$$G(u) = \int_{u_o}^{u} \frac{dt}{W(t)} \qquad (u_o > o, \; u \geqq o).$$

This inequality has been further generalized in several directions by Beesack [3].

It has been recently established that the inequalities of this type in two and more independent variables can be profitably employed in the analysis of partial differential equations [4, and references listed therein]. An interesting inequality by Wendroff given without proof in [5] is as follows:

LEMMA 2. Let $J = \{x \mid o \leqq x \leqq X < \infty\}$ and $K = \{y \mid o \leqq y \leqq Y < \infty\}$. Further let $U(x,y)$ be a scalar, non-negative continuous function defined on $J \times K$, then for any arbitrary non-negative constants a, b and c, the integral inequality

$$U(x,y) \leqq c + a \int_o^x U(s,y)ds + b \int_o^y U(x,t)dt$$

implies that $U(x,y) \leqq c \exp(ax + by + abxy)$.

The proof of this inequality can be constructed by observing that $U \leqq P$, where $P(x,y)$ satisfies the equation

$$P(x,y) = c + a \int_o^x P(s,y)ds + b \int_o^y P(x,t)dt.$$

Observe that $P(x,o) = c \exp(ax)$ and $P(o,y) = c \exp(by)$. Now assuming $P(x,y) = \exp(ax + by)q(x,y)$, one can determine the inequality satisfied by $q(x,y)$ which results into the given conclusion.

Lemma 2, in view of Lemma 1, suggests that it is possible to consider a nonlinear generalization of the Wendroff's inequality. We do this in the present paper and further show by an example that the generalization of this kind is truely beneficial in the study of some nonlinear partial differential equations.

2. NOTATION.

Let $\mathbb{R}$ denote the real line. For any rectangle $J \times K$ we define the following classes of functions:

(i) $C^+(J\times K)$ = the space of continuous functions $u : J \times K \to \mathbb{R}^+$

(ii) $C_1(\mathbb{R}^+)$ = the space of non-decreasing, non-negative, submultiplicative, continuous functions on $\mathbb{R}^+$.

(iii) $C_2(\mathbb{R}^+)$ = the space of non-decreasing, non-negative, continuous functions on $\mathbb{R}^+$ and such that for $g_2 \in C_2(\mathbb{R}^+)$ and for any real-valued function $h(x,y)$, $(x,y) \in J \times K$,

$$\frac{g_2(u(x,y))}{h(x,y)} \leqq g_2\left(\frac{u(x,y)}{h(x,y)}\right), \quad h(x,y) \geqq 1. \tag{2.1}$$

Further, we define

$$G_i(u) = \int_{u_o}^{u} \frac{dt}{g_i(t)}, \quad (u \geqq o,\ u_o > o). \tag{2.2}$$

Let G_i^{-1} be the inverse of G_i, $i = 1,2$.

3. MAIN RESULT.

THEOREM 3.1. Assume that

(a) $u(x,y) \in C^+(J \times K)$,

(b) $g_1(u) \in C_1(\mathbb{R}^+)$,

(c) $g_2(u) \in C_2(\mathbb{R}^+)$,

(d) there exists a $u_o > o$ such that $g_1(u) > o$ and $g_2(u) > o$ for $u \geqq u_o$. Then for any arbitrary non-negative constants a, b and c, with $c \geqq 1$, the inequality

$$u(x,y) \leq c + a \int_o^x g_1(u(s,y)\,ds + b \int_o^y g_2(u(x,t))dt, \quad (o \leq x \leq X, o \leq y \leq Y) \tag{3.1}$$

implies, on a nonempty rectangle, the inequality

$$u(x,y) \leq G_1^{-1}\left\{G_1(c) + axg_1\,[G_2^{-1}(G_2(1) + by)]\right\}g_2^{-1}(G_2(1)+by), \tag{3.2}$$

where $G_i(u)$, $i = 1,2$, are as defined in (2.2), $(o \leq x \leq X' \leq X$ and $o \leq y \leq Y' \leq Y)$.

PROOF. We define

$$h(x,y) = c + a \int_o^x g_1(u(s,y))ds, \quad (x,y) \in J \times K. \tag{3.3}$$

It is clear that $h(x,y)$ is non-decreasing and $h(x,y) \geq 1$ on $J \times K$.

Inequality (3.1) may be written as

$$u(x,y) \leq h(x,y) + b \int_o^y g_2(u(x,t))dt. \tag{3.4}$$

Dividing throughout by $h(x,y)$ and using (2.1) of assumption (c) we have

$$\frac{u(x,y)}{h(x,y)} \leq 1 + b \int_o^y g_2\left(\frac{u(x,t)}{h(x,t)}\right) dt. \tag{3.5}$$

For fixed $x \in J$, an application of Lemma 1 yields

$$u(x,y) \leq h(x,y).G_2^{-1}(G_2(1) + by), \quad (o \leq x \leq X, o \leq y \leq Y' \leq Y) \tag{3.6}$$

Substituting (3.6) in (3.3) and employing submultiplicative property of g_1 we obtain

$$h(x,y) \leq c + a \int_o^x g_1(h(s,y))g_1(G_2^{-1}(G_2(1) + by))\,ds. \tag{3.7}$$

An application of Lemma 1 again, to (3.7), yields a bound for $h(x,y)$ on a nonempty rectangle. The desired inequality now follows by substituting the bound for $h(x,y)$ in (3.6).

REMARK 3.1. In particular, if $b = o$ and $g_1(u) = u$, then the estimate in (3.2) reduces to

$$u(x,y) \leq G_1^{-1}\,[G_1(c) + ax].$$

In view of (2.2), it is clear that for fixed $y \in K$, our estimate further reduces to

$$u(x,y) \leq c \exp(ax).$$

Thus, Gronwall's estimate is included in (3.2).

REMARK 3.2. In the case $b = o$, the estimate in (3.2) reduces to

$$u(x,y) \leq G_1^{-1}\,[G_1(c) + ax\;g_1(1)]$$

for each $y \in K$, $o \leq x \leq X' \leq X$, which is a Bihari-like estimate.

Further, if in (3.1), $a = o$, then for fixed $x \in J$,

$$u(x,y) \leq c + b \int_o^y g_2(u(x,t))dt$$

implies

$$\frac{u(x,y)}{c} \le 1 + b \int_o^y g_2\left(\frac{u(x,t)}{c}\right)dt \ , \qquad y \in K.$$

An application of Lemma 1 yields

$$u(x,y) \le c\ G_2^{-1}\ [G_2(1) + by],$$
$$(x \in J \text{ is fixed, } o \le y \le Y' \le Y)$$

This estimate is the same as that obtained from (3.2) with $a = o$.

REMARK 3.3. Let $g_i(u) = u$, $i = 1,2$. Then

$$G_i(u) = \log \frac{u}{uo}, \ u \ge o, \ u_o > o, \ i = 1,2.$$

Clearly

$$G_i^{-1}(v) = u_o \exp(v), \qquad i = 1,2.$$

Hence the estimate (3.2) reduces to

$$u(x,y) \le u_o \exp\left[\log \frac{c}{u_o} + ax\, u_o \exp(\log \frac{1}{u_o} + by)\right] \cdot u_o \exp(\log \frac{1}{u_o} + by)$$

$$\le c \exp\left[ax \exp(by) + by\right]$$

$$= c \exp\left[ax + by + abxy + \text{higher order terms}\right] .$$

This estimate is obviously not as sharp as the one obtained in Lemma 2.

4. AN APPLICATION.

Consider the characteristic initial value problem for the nonlinear partial differential equation

$$u_{xy} = u^a u_y + be^u u_x, \qquad a,\ b \ge o,$$
$$(o \le x \le X, \quad o \le y \le Y) \tag{4.1}$$

satisfying the initial values

$$u(x,o) = u(o,y) = u(o,o) = c \ge 1. \tag{4.2}$$

Under the condition (4.2) equation (4.1) can be reformulated in terms of the integral equation

$$u(x,y) = c - \frac{1}{a+1} \int_o^x u^{a+1}(s,o)ds - b \int_o^x e^{u(o,t)}dt$$
$$+ \frac{1}{a+1} \int_o^y u^{a+1}(s,y)ds + b \int_o^y e^{u(x,t)}dt. \tag{4.3}$$

Therefore, using the initial - data, we obtain the inequality

$$|u(x,y)| \le k + \frac{1}{a+1} \int_o^x |u(s,y)|^{a+1} ds + b \int_o^y e^{|u(x,t)|}dt,$$

where $k = (|c| + \frac{|c|^{a+1}X}{a+1} + be^{|c|}Y)$.

The present inequality is equivalent to (3.1). A direct application of Theorem 3.1 yields

$$|u(x,y)| \le G_1^{-1}\,[G_1(k) + \frac{x}{a+1}\,[G_2^{-1}(G_2(1) + by)]^{a+1}\,] \cdot G_2^{-1}(G_2(1) + by)$$

where

$$G_1(u) = \int_o^u \frac{1}{p^{a+1}}\,dp = -\frac{1}{a}\,[\frac{1}{u^a} - \frac{1}{u_o^a}\,], \text{ and } G_2(u) = \int_o^u \frac{1}{e^p}\,dp = -[\,e^{-u} - e^{-u_o}\,].$$

Therefore,

$$G_1^{-1}(v) = [\,u_o^{-a} - av\,]^{-1/a} \text{ and } G_2^{-1}(v) = \log_e\,[\,e^{-u_o} - v\,]^{-1}.$$

Hence,

$$|u(x,y)| \le \{k^{-a} - \frac{ax}{a+1}\,[\,\log(e^{-1} - by)^{-1}\,]^{a+1}\,\}^{-1/a} \cdot \log(e^{-1} - by)^{-1}.$$

This provides a pointwise estimate for solutions of the given equation (4.1).

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