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ON FIRST-ORDER DIFFERENTIAL OPERATORS WITH BOHR-NEUGEBAUER TYPE PROPERTY

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ABSTRACT. We consider a differential equation $\frac{d}{dt}u(t)-Bu(t) = f(t)$, where the functions u and f map the real line into a Banach space X and B: X → X is a bounded linear operator. Assuming that any Stepanov-bounded solution u is Stepanov almost-periodic when f is Bochner almost-periodic, we establish that any Stepanov-bounded solution u is Bochner almost-periodic when f is Stepanov almost-periodic. Some examples are given in which the operator $\frac{d}{dt} - B$ is shown to satisfy our assumption.

KEY WORDS AND PHRASES. Bounded linear operator, differential operator, Bohr-Neugebauer property, Bochner (Stepanov) almost-periodic function. 1980 AMS SUBJECT CLASSIFICATION CODE. 34Gxx, 34G10, 34C27.

1. INTRODUCTION.

Suppose X is a Banach space and J is the interval $-\infty < t < \infty$. A function $f \in L^p_{loc}(J;X)$ with $1 \leqq p < \infty$ is said to be Stepanov-bounded or S^p-bounded on J if

$$\| f \|_{S^p} = \sup_{t \in J} \left[\int_t^{t+1} \| f(s) \|^p ds \right]^{1/p} < \infty . \tag{1.1}$$

Our first result is as follows.

THEOREM 1. Suppose $f : J \to X$ is a continuously differentiable S^1-bounded function, and f' is an S^p-bounded function with $1 \leqq p < \infty$. Then, (a) if p = 1, f is bounded on J, and (b) if p > 1, f is bounded and uniformly continuous on J.

2. PROOF OF THEOREM 1.

(a) p = 1. For an arbitrary but fixed $t \in J$, there exists at least one point $\tau_t \in [t - 1, t]$ such that

$$\| f(\tau_t) \| = \inf_{t-1 \leqq s \leqq t} \| f(s) \| . \tag{2.1}$$

Consequently, we have

$$\| f(\tau_t) \| \leqq \int_{t-1}^{t} \| f(s) \| ds \leqq \| f \|_{S^1}, \text{ by (1.1)}. \tag{2.2}$$

Hence, from the S^1-boundedness of f', we obtain

$$\|f(t)\| = \|f(\tau_t) + \int_{\tau_t}^{t} f'(s)\,ds\|$$

$$\leq \|f(\tau_t)\| + \int_{\tau_t}^{t} \|f'(s)\|\,ds$$

$$\leq \|f\|_{S^1} + \|f'\|_{S^1}. \tag{2.3}$$

(b) $p > 1$. By Hölder's inequality, the S^p-boundedness of f' implies the S^1-boundedness of f'. Hence, as shown above, f is bounded on J.

Moreover, for $0 < t_2 - t_1 < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have, again by Hölder's inequality,

$$\|f(t_2) - f(t_1)\| = \|\int_{t_1}^{t_2} f'(s)\,ds\|$$

$$\leq (t_2-t_1)^{1/q}\left[\int_{t_1}^{t_2} \|f'(s)\|^p ds\right]^{1/p}$$

$$\leq (t_2-t_1)^{1/q}\left[\int_{t_1}^{t_1+1} \|f'(s)\|^p ds\right]^{1/p}$$

$$\leq (t_2-t_1)^{1/q}\,\|f'\|_{S^p}. \tag{2.4}$$

Therefore f is uniformly continuous on J, completing the proof of the theorem.

REMARK. If $f : J \to X$ is a continuously differentiable S^1-almost periodic function, with f' being S^p-bounded on J $(1 < p < \infty)$, then f is (uniformly) almost-periodic from J to X (see pp. 3 and 77, Amerio-Prouse [1] for the definitions of (uniform) almost-periodicity and S^p-almost periodicity).

PROOF. By Theorem 1, f is uniformly continuous on J. Hence, by Theorem 7, p. 78, Amerio-Prouse [1], f is (uniformly) almost-periodic from J to X.

3. MAIN RESULT.

Let B be a bounded linear operator on a Banach space X into itself. Then the differential operator $\frac{d}{dt} - B$ is said to have Bohr-Neugebauer property if, for any (uniformly) almost-periodic X-valued function f, any bounded (on J) solution of the equation

$$\frac{d}{dt} u(t) - Bu(t) = f(t) \text{ on } J \tag{3.1}$$

is (uniformly) almost-periodic.

Our result is as follows.

THEOREM 2. In a Banach space X, let the differential operator $\frac{d}{dt} - B$ be such that, for any (uniformly) almost-periodic X-valued function f, any S^1-bounded solution of the equation (3.1) is S^1-almost periodic. Then, for any S^1-almost periodic continuous X-valued function g, any S^1-bounded solution $u : J \to X$ of the equation

$$\frac{d}{dt} u(t) - Bu(t) = g(t) \text{ on } J \tag{3.2}$$

is (uniformly) almost-periodic.

PROOF. Since g is S^1-almost periodic from J to X, it is S^1-bounded on J. Consequently, $u' = Bu + g$ is S^1-bounded on J. Hence, by Theorem 1, u is bounded on J.

Now consider a sequence $\{\phi_n(t)\}_{n=1}^{\infty}$ of non-negative continuous functions on J such that

$$\phi_n(t) = 0 \quad \text{for} \quad |t| \geq n^{-1}, \quad \int_{-n^{-1}}^{n^{-1}} \phi_n(t)\, dt = 1. \tag{3.3}$$

The convolution of u and ϕ_n is defined by

$$(u*\phi_n)(t) = \int_J u(t-s)\, \phi_n(s)\, ds = \int_J u(s)\, \phi_n(t-s)\, ds. \tag{3.4}$$

Then, by (3.2), we have

$$\frac{d}{dt}(u*\phi_n)(t) - B(u*\phi_n)(t) = (g*\phi_n)(t) \text{ on } J. \tag{3.5}$$

We note that

$$\sup_{t\in J} \| (u*\phi_n)(t) \| \leq \sup_{t\in J} \| u(t) \| . \tag{3.6}$$

Further, we can show that $g*\phi_n$ is (uniformly) almost-periodic from J to X (see the proof of Theorem 7, p. 78, Amerio-Prouse [1]).

Therefore, by our assumption on the operator $\frac{d}{dt} - B$, $(u*\phi_n)(t)$ is S^1-almost periodic for all n = 1, 2,....

By (3.2), we have the representation

$$u(t) = u(0) + \int_0^t Bu(s)\, ds + \int_0^t g(s)\, ds \text{ on } J. \tag{3.7}$$

If $t_2 > t_1$, then

$$\left\| \int_{t_1}^{t_2} Bu(s)\, ds \right\| \leq \| B \| \cdot \sup_{t\in J} \| u(t) \| \cdot (t_2 - t_1). \tag{3.8}$$

Hence $\int_0^t Bu(s)\, ds$ is uniformly continuous on J. Also, by Theorem 8, p. 79, Amerio-Prouse [1], $\int_0^t g(s)\, ds$ is uniformly continuous on J. Consequently, u is uniformly continuous on J.

Similarly, from (3.5), it follows that $u*\phi_n$ is uniformly continuous on J. So, by Theorem 7, p. 78, Amerio-Prouse [1], $u*\phi_n$ is (uniformly) almost-periodic for all n = 1, 2,....

Now, by the uniform continuity of u on J, the sequence of convolutions $(u*\phi_n)(t)$ converges to u (t) uniformly on J. Hence u is (uniformly) almost-periodic from J to X, which completes the proof of the theorem.

4. NOTES.

(i) Suppose X is a Hilbert space and B is a self-adjoint bounded linear operator on X into itself. Then we know that the operator $\frac{d}{dt} - B$ has Bohr-Neugebauer property (see Zaidman [4]). Given an (uniformly) almost-periodic X-valued function f, suppose that u is an S^1-bounded solution of the equation (3.1). If we replace g by f in the proof of our Theorem 2, then, by the Bohr-Neugebauer property of the operator $\frac{d}{dt} - B$, it follows that u is (uniformly) almost-periodic from J to X. Thus the operator $\frac{d}{dt} - B$ satisfies the hypothesis of Theorem 2.

(ii) Now suppose X is a separable Hilbert space and B is a completely continuous normal operator on X into itself. Then, by Theorem 1 of Cooke [3], the operator $\frac{d}{dt} - B$ has Bohr-Neugebauer property. Consequently, the operator $\frac{d}{dt} - B$ satisfies the assumption of Theorem 2.

(iii) Finally, suppose X is a reflexive space and B = 0. Then the operator $\frac{d}{dt}$ has Bohr-Neugebauer property (see Amerio-Prouse [1], p. 55 and Authors' Remark on p. 82). Hence the operator $\frac{d}{dt}$ satisfies the assumption of Theorem 2.

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