

# ON A PERIODIC BOUNDARY VALUE PROBLEM FOR SECOND-ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Unimprovable efficient sufficient conditions are established for the unique solvability of the periodic problem  $u''(t) = \ell(u)(t) + q(t)$  for  $0 \leq t \leq \omega$ ,  $u^{(i)}(0) = u^{(i)}(\omega)$  ( $i = 0, 1$ ), where  $\omega > 0$ ,  $\ell : C([0, \omega]) \rightarrow L([0, \omega])$  is a linear bounded operator, and  $q \in L([0, \omega])$ .

## 1. Introduction

Consider the equation

$$u''(t) = \ell(u)(t) + q(t) \quad \text{for } 0 \leq t \leq \omega \quad (1.1)$$

with the periodic boundary conditions

$$u^{(i)}(0) = u^{(i)}(\omega) \quad (i = 0, 1), \quad (1.2)$$

where  $\omega > 0$ ,  $\ell : C([0, \omega]) \rightarrow L([0, \omega])$  is a linear bounded operator and  $q \in L([0, \omega])$ .

By a solution of the problem (1.1), (1.2) we understand a function  $u \in \tilde{C}'([0, \omega])$ , which satisfies (1.1) almost everywhere on  $[0, \omega]$  and satisfies the conditions (1.2).

The periodic boundary value problem for functional differential equations has been studied by many authors (see, for instance, [1, 2, 3, 4, 5, 6, 8, 9] and the references therein). Results obtained in this paper on the one hand generalise the well-known results of Lasota and Opial (see [7, Theorem 6, page 88]) for linear ordinary differential equations, and on the other hand describe some properties which belong only to functional differential equations. In the paper [8], it was proved that the problem (1.1), (1.2) has a unique solution if the inequality

$$\int_0^\omega |\ell(1)(s)| ds \leq \frac{d}{\omega} \quad (1.3)$$

with  $d = 16$  is fulfilled. Moreover, there was also shown that the condition (1.3) is non-improvable. This paper attempts to find a specific subset of the set of linear monotone operators, in which the condition (1.3) guarantees the unique solvability of the problem

(1.1), (1.2) even for  $d \geq 16$  (see Corollary 2.3). It turned out that if  $A$  satisfies some conditions dependent only on the constants  $d$  and  $\omega$ , then  $K_{[0,\omega]}(A)$  (see Definition 1.2) is such a subset of the set of linear monotone operators.

The following notation is used throughout.

$N$  is the set of all natural numbers.

$R$  is the set of all real numbers,  $R_+ = [0, +\infty[$ .

$C([a,b])$  is the Banach space of continuous functions  $u : [a,b] \rightarrow R$  with the norm  $\|u\|_C = \max\{|u(t)| : a \leq t \leq b\}$ .

$\tilde{C}'([a,b])$  is the set of functions  $u : [a,b] \rightarrow R$  which are absolutely continuous together with their first derivatives.

$L([a,b])$  is the Banach space of Lebesgue integrable functions  $p : [a,b] \rightarrow R$  with the norm  $\|p\|_L = \int_a^b |p(s)| ds$ .

If  $x \in R$ , then  $[x]_+ = (|x| + x)/2$ ,  $[x]_- = (|x| - x)/2$ .

*Definition 1.1.* We will say that an operator  $\ell : C([a,b]) \rightarrow L([a,b])$  is *nonnegative (nonpositive)*, if for any nonnegative  $x \in C([a,b])$  the inequality

$$\ell(x)(t) \geq 0 \quad (\ell(x)(t) \leq 0) \quad \text{for } a \leq t \leq b \quad (1.4)$$

is satisfied.

We will say that an operator  $\ell$  is *monotone* if it is nonnegative or nonpositive.

*Definition 1.2.* Let  $A \subset [a,b]$  be a nonempty set. We will say that a linear operator  $\ell : C([a,b]) \rightarrow L([a,b])$  belongs to the set  $K_{[a,b]}(A)$  if for any  $x \in C([a,b])$ , satisfying

$$x(t) = 0 \quad \text{for } t \in A, \quad (1.5)$$

the equality

$$\ell(x)(t) = 0 \quad \text{for } a \leq t \leq b \quad (1.6)$$

holds.

We will say that  $K_{[a,b]}(A)$  is the set of operators *concentrated* on the set  $A \subset [a,b]$ .

## 2. Main results

Define, for any nonempty set  $A \subseteq R$ , the continuous (see Lemma 3.1) functions:

$$\rho_A(t) = \inf \{|t - s| : s \in A\}, \quad \sigma_A(t) = \rho_A(t) + \rho_A\left(t + \frac{\omega}{2}\right) \quad \text{for } t \in R. \quad (2.1)$$

**THEOREM 2.1.** Let  $A \subset [0,\omega]$ ,  $A \neq \emptyset$  and a linear monotone operator  $\ell \in K_{[0,\omega]}(A)$  be such that the conditions

$$\int_0^\omega \ell(1)(s) ds \neq 0, \quad (2.2)$$

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right) \int_0^\omega |\ell(1)(s)| ds \leq \frac{16}{\omega} \quad (2.3)$$

are satisfied, where

$$\delta = \min \left\{ \sigma_A(t) : 0 \leq t \leq \frac{\omega}{2} \right\}. \quad (2.4)$$

Then the problem (1.1), (1.2) has a unique solution.

*Example 2.2.* The example below shows that condition (2.3) in Theorem 2.1 is optimal and it cannot be replaced by the condition

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right) \int_0^\omega |\ell(1)(s)| ds \leq \frac{16}{\omega} + \varepsilon, \quad (2.3_\varepsilon)$$

no matter how small  $\varepsilon \in ]0, 1]$  would be. Let  $\omega = 1$ ,  $\varepsilon_0 \in ]0, 1/16[$ ,  $\delta_1 \in ]0, 1/4 - 2\varepsilon_0[$  and  $\mu_i, \nu_i$  ( $i = 1, 2$ ) be the numbers given by the equalities

$$\mu_i = \frac{1 - 2\delta_1}{4} + (-1)^i \varepsilon_0, \quad \nu_i = \frac{3 + 2\delta_1}{4} + (-1)^i \varepsilon_0 \quad (i = 1, 2). \quad (2.5)$$

Let, moreover, the functions  $x \in \tilde{C}'([\mu_1, \mu_2])$ ,  $y \in \tilde{C}'([\nu_1, \nu_2])$  be such that

$$x(\mu_1) = x(\mu_2) = 1, \quad x'(\mu_1) = \frac{1}{\mu_1}, \quad x'(\mu_2) = -\frac{1}{\mu_1 + \delta_1}, \quad (2.5_1)$$

$$x''(t) \leq 0 \quad \text{for } \mu_1 \leq t \leq \mu_2,$$

$$y(\nu_1) = y(\nu_2) = -1, \quad y'(\nu_1) = -\frac{1}{\mu_1 + \delta_1}, \quad y'(\nu_2) = \frac{1}{\mu_1}, \quad (2.5_2)$$

$$y''(t) \geq 0 \quad \text{for } \nu_1 \leq t \leq \nu_2.$$

Define a function

$$u_0(t) = \begin{cases} \frac{t}{\mu_1} & \text{for } 0 \leq t \leq \mu_1 \\ x(t) & \text{for } \mu_1 < t < \mu_2 \\ \frac{1 - 2t}{\nu_1 - \mu_2} & \text{for } \mu_2 \leq t \leq \nu_1 \\ y(t) & \text{for } \nu_1 < t < \nu_2 \\ \frac{t - 1}{\mu_1} & \text{for } \nu_2 \leq t \leq 1. \end{cases} \quad (2.6)$$

Obviously,  $u_0 \in \tilde{C}'([0, \omega])$ . Now let  $A = \{\mu_1, \nu_2\}$ , the function  $\tau : [0, \omega] \rightarrow A$  and the operator  $\ell : C([0, \omega]) \rightarrow L([0, \omega])$  be given by the equalities:

$$\tau(t) = \begin{cases} \mu_1 & \text{if } u_0''(t) \geq 0 \\ \nu_2 & \text{if } u_0''(t) < 0, \end{cases} \quad \ell(z)(t) = |u_0''(t)|z(\tau(t)). \quad (2.7)$$

It is clear from the definition of the functions  $\tau$  and  $\sigma_A$  that the nonnegative operator  $\ell$  is concentrated on the set  $A$  and the condition (2.4) is satisfied with  $\delta = \delta_1 + 2\varepsilon_0$ . In view

of (2.5<sub>1</sub>), (2.5<sub>2</sub>), and (2.7) we obtain

$$\int_0^\omega \ell(1)(s)ds = \int_{\nu_1}^{\nu_2} y''(s)ds - \int_{\mu_1}^{\mu_2} x''(s)ds = 2 \frac{2\mu_1 + \delta_1}{\mu_1(\mu_1 + \delta_1)} = 16 \frac{1 - 4\varepsilon_0}{(1 - 4\varepsilon_0)^2 - 4\delta_1^2}. \quad (2.8)$$

When  $\varepsilon$  is small enough, the last equality it implies the existence of  $\varepsilon_0$  such that

$$0 < \int_0^\omega \ell(1)(s)ds = \frac{16 + \varepsilon}{1 - 4\delta_1^2}. \quad (2.9)$$

Thus, because  $\delta_1 < \delta$ , all the assumptions of Theorem 2.1 are satisfied except (2.3), and instead of (2.3) the condition (2.3<sub>ε</sub>) is fulfilled with  $\omega = 1$ . On the other hand, from the definition of the function  $u_0$  and from (2.7), it follows that  $\ell(u_0)(t) = |u_0''(t)|u_0(\tau(t)) = |u_0''(t)|\operatorname{sign} u_0''(t)$ , that is,  $u_0$  is a nontrivial solution of the homogeneous problem  $u''(t) = \ell(u)(t)$ ,  $u^{(i)}(0) = u^{(i)}(1)$  ( $i = 1, 2$ ) which contradicts the conclusion of Theorem 2.1.

**COROLLARY 2.3.** *Let the set  $A \subset [0, \omega]$ , number  $d \geq 16$ , and a linear monotone operator  $\ell \in K_{[0, \omega]}(A)$  be such that the conditions (2.2)*

$$\int_0^\omega |\ell(1)(s)|ds \leq \frac{d}{\omega}, \quad (2.10)$$

*are satisfied and*

$$\sigma_A(t) \geq \frac{\omega}{2} \sqrt{1 - \frac{16}{d}} \quad \text{for } 0 \leq t \leq \frac{\omega}{2}. \quad (2.11)$$

*Then the problem (1.1), (1.2) has a unique solution.*

**COROLLARY 2.4.** *Let  $\alpha \in [0, \omega]$ ,  $\beta \in [\alpha, \omega]$ , and a linear monotone operator  $\ell \in K_{[0, \omega]}(A)$  be such that the conditions (2.2) and (2.3) are satisfied, where*

$$A = [\alpha, \beta], \quad \delta = \left[ \frac{\omega}{2} - (\beta - \alpha) \right]_+ \quad (2.11_1)$$

*or*

$$A = [0, \alpha] \cup [\beta, \omega], \quad \delta = \left[ \frac{\omega}{2} - (\beta - \alpha) \right]_- \quad (2.11_2)$$

*Then the problem (1.1), (1.2) has a unique solution.*

Consider the equation with deviating arguments

$$u''(t) = p(t)u(\tau(t)) + q(t) \quad \text{for } 0 \leq t \leq \omega, \quad (2.12)$$

where  $p \in L([0, \omega])$  and  $\tau : [0, \omega] \rightarrow [0, \omega]$  is a measurable function.

**COROLLARY 2.5.** *Let there exist  $\sigma \in \{-1, 1\}$  such that*

$$\sigma p(t) \geq 0 \quad \text{for } 0 \leq t \leq \omega, \quad (2.13)$$

$$\int_0^\omega p(s)ds \neq 0. \quad (2.14)$$

Moreover, let  $\delta \in [0, \omega/2]$  and the function  $p$  be such that

$$\left(1 - 4\left(\frac{\delta}{\omega}\right)^2\right) \int_0^\omega |p(s)| ds \leq \frac{16}{\omega}, \quad (2.15)$$

and let at least one of the following items be fulfilled:

- (a) the set  $A \subset [0, \omega]$  is such that the condition (2.4) holds and

$$p(t) = 0 \quad \text{if } \tau(t) \notin A \quad (2.16)$$

on  $[0, \omega]$ ;

- (b) the constants  $\alpha \in [0, \omega]$ ,  $\beta \in [\alpha, \omega]$  are such that

$$\tau(t) \in [\alpha, \beta] \quad \text{for } 0 \leq t \leq \omega, \quad (2.17)$$

$$\delta = \left[ \frac{\omega}{2} - (\beta - \alpha) \right]_+. \quad (2.18)$$

Then the problem (2.12), (1.2) has a unique solution.

Now consider the ordinary differential equation

$$u''(t) = p(t)u(t) + q(t) \quad \text{for } 0 \leq t \leq \omega, \quad (2.19)$$

where  $p, q \in L([0, \omega])$ .

**COROLLARY 2.6.** Let

$$p(t) \leq 0 \quad \text{for } 0 \leq t \leq \omega. \quad (2.20)$$

Moreover, let  $\delta \in [0, \omega/2]$  and the function  $p$  be such that the conditions (2.14), (2.15) hold, and let at least one of the following items be fulfilled:

- (a) the set  $A \subset [0, \omega]$  is such that  $\text{mes } A \neq 0$ , the condition (2.4) holds and

$$p(t) = 0 \quad \text{for } t \notin A; \quad (2.21)$$

- (b) the constants  $\alpha \in [0, \omega]$ ,  $\beta \in [\alpha, \omega]$  are such that

$$p(t) = 0 \quad \text{for } t \in [0, \alpha \cup \beta, \omega], \quad (2.22)$$

and  $\delta \in [0, \omega/2]$  satisfies (2.18). Then the problem (2.19), (1.2) has a unique solution.

**Remark 2.7.** As for the case where  $p(t) \geq 0$  for  $0 \leq t \leq \omega$ , the necessary and sufficient condition for the unique solvability of (2.19), (1.2) is  $p(t) \neq 0$  (see [2, Proposition 1.1, page 72]).

### 3. Auxiliary propositions

LEMMA 3.1. *The function  $\rho_A : R \rightarrow R$  defined by the equalities (2.1), is continuous and*

$$\rho_{\bar{A}}(t) = \rho_A(t) \quad \text{for } t \in R, \quad (3.1)$$

where  $\bar{A}$  is the closure of the set  $A$ .

*Proof.* Since  $A \subseteq \bar{A}$ , it is clear that

$$\rho_{\bar{A}}(t) \leq \rho_A(t) \quad \text{for } t \in R. \quad (3.2)$$

Let  $t_0 \in R$  be an arbitrary point,  $s_0 \in \bar{A}$ , and the sequence  $s_n \in A$  ( $n \in N$ ) be such that  $\lim_{n \rightarrow \infty} s_n = s_0$ . Then  $\rho_A(t_0) \leq \lim_{n \rightarrow \infty} |t_0 - s_n| = |t_0 - s_0|$ , that is,

$$\rho_{\bar{A}}(t) \geq \rho_A(t) \quad \text{for } t \in R. \quad (3.3)$$

From the last relation and (3.2) we get the equality (3.1).

For arbitrary  $s \in A$ ,  $t_1, t_2 \in R$ , we have

$$\rho_A(t_i) \leq |t_i - s| \leq |t_2 - t_1| + |t_{3-i} - s| \quad (i = 1, 2). \quad (3.4)$$

Consequently  $\rho_A(t_i) - |t_2 - t_1| \leq \rho_A(t_{3-i})$  ( $i = 1, 2$ ). Thus the function  $\rho_A$  is continuous.  $\square$

LEMMA 3.2. *Let  $A \subseteq [0, \omega]$  be a nonempty set,  $A_1 = \{t + \omega : t \in A\}$ ,  $B = A \cup A_1$ , and*

$$\min \left\{ \sigma_A(t) : 0 \leq t \leq \frac{\omega}{2} \right\} = \delta. \quad (3.5)$$

*Then*

$$\min \left\{ \sigma_B(t) : 0 \leq t \leq \frac{3\omega}{2} \right\} = \delta. \quad (3.6)$$

*Proof.* Let  $\alpha = \inf A$ ,  $\beta = \sup A$ , and let  $t_0 \in [0, 3\omega/2]$  be such that

$$\sigma_B(t_0) = \min \left\{ \sigma_B(t) : 0 \leq t \leq \frac{3\omega}{2} \right\}. \quad (3.7)$$

Assume that  $t_1 \in [0, 3\omega/2]$  is such that  $t_1 \notin \bar{B}$ ,  $t_1 + \omega/2 \notin \bar{B}$ . Then

$$\varepsilon = \min \{ \rho_B(t_1), \rho_B(t_1 + \omega/2) \} > 0, \quad (3.8)$$

and either

$$\sigma_B(t_1 - \varepsilon) \leq \sigma_B(t_1) \quad \text{and} \quad \rho_B(t_1 - \varepsilon) = 0 \quad \text{or} \quad \rho_B\left(t_1 + \frac{\omega}{2} - \varepsilon\right) = 0 \quad (3.9)$$

or

$$\sigma_B(t_1 + \varepsilon) \leq \sigma_B(t_1) \quad \text{and} \quad \rho_B(t_1 + \varepsilon) = 0 \quad \text{or} \quad \rho_B\left(t_1 + \frac{\omega}{2} + \varepsilon\right) = 0. \quad (3.10)$$

In view of this fact, without loss of generality we can assume that

$$t_0 \in \bar{B} \quad \text{or} \quad t_0 + \frac{\omega}{2} \in \bar{B}. \quad (3.11)$$

From (3.5) and the condition  $A \subseteq [0, \omega]$ , we have

$$\min \left\{ \sigma_A(t) : 0 \leq t \leq \frac{3\omega}{2} \right\} = \delta. \quad (3.12)$$

First suppose that  $0 \leq t_0 \leq \beta - \omega/2$ . From this inequality by the inclusion  $\beta \in \bar{A}$ , we get

$$\inf \left\{ \left| t_0 + \frac{\omega i}{2} - s \right| : s \in B \right\} = \inf \left\{ \left| t_0 + \frac{\omega i}{2} - s \right| : s \in A \right\} \quad (3.12_i)$$

for  $i = 0, 1$ . Then  $\sigma_B(t_0) = \sigma_A(t_0)$  and in view of (3.12)

$$\sigma_B(t_0) \geq \delta. \quad (3.13)$$

Let now

$$\beta - \frac{\omega}{2} < t_0 \leq \beta. \quad (3.14)$$

Obviously, either

$$\left( t_0 + \frac{\omega}{2} \right) - \beta \leq \alpha + \omega - \left( t_0 + \frac{\omega}{2} \right), \quad (3.14_1)$$

or

$$\left( t_0 + \frac{\omega}{2} \right) - \beta > \alpha + \omega - \left( t_0 + \frac{\omega}{2} \right). \quad (3.14_2)$$

If (3.14<sub>1</sub>) is satisfied, then, in view of (3.14) and  $\beta \in \bar{A}$ , the equalities (3.12<sub>i</sub>) ( $i = 0, 1$ ) hold. Therefore  $\sigma_B(t_0) = \sigma_A(t_0)$  and, in view of (3.12), the inequality (3.13) is fulfilled. Let now (3.14<sub>2</sub>) be satisfied. If  $\alpha + \omega > t_0 + \omega/2$ , then, in view of (3.14), we have  $t_0 + \omega/2 \notin \bar{B}$ . Consequently, from (3.12) and (3.14<sub>2</sub>) by virtue of (3.11) and the inclusions  $\alpha, \beta \in \bar{A}$ , we get

$$\sigma_B(t_0) = \rho_B \left( t_0 + \frac{\omega}{2} \right) = \alpha + \frac{\omega}{2} - t_0 \geq \rho_A \left( \alpha + \frac{\omega}{2} \right) \geq \delta. \quad (3.15)$$

If  $\alpha + \omega \leq t_0 + \omega/2$ , then  $t_0 + \omega/2 \in \bar{A}_1$  and

$$\inf \left\{ \left| t_0 + \frac{\omega}{2} - s \right| : s \in B \right\} = \inf \left\{ \left| t_0 - \frac{\omega}{2} - s \right| : s \in A \right\}, \quad (3.16)$$

that is,  $\rho_B(t_0 + \omega/2) = \rho_A(t_0 - \omega/2)$  and in view of (3.12), (3.14) we get

$$\sigma_B(t_0) = \rho_A(t_0) + \rho_A \left( t_0 - \frac{\omega}{2} \right) = \sigma_A \left( t_0 - \frac{\omega}{2} \right) \geq \delta. \quad (3.17)$$

Consequently the inequality (3.13) is fulfilled as well.

Further, let  $\beta \leq t_0 \leq t_0 + \omega/2 \leq \alpha + \omega$ . Then  $t_0 - \alpha \leq \alpha + \omega - t_0$ , and also  $t_0 - \beta \leq \alpha + \omega - t_0$ . On account of (3.12) and  $\beta \in \bar{A}$  we have

$$\sigma_B(t_0) = \alpha + \frac{\omega}{2} - \beta \geq \rho_A\left(\alpha + \frac{\omega}{2}\right) = \sigma_A(\alpha) \geq \delta. \quad (3.18)$$

Thus the inequality (3.13) is fulfilled.

Let now

$$\beta \leq t_0 \leq \alpha + \omega \leq t_0 + \frac{\omega}{2}. \quad (3.19)$$

From (3.19) it follows that

$$\begin{aligned} \inf \left\{ \left| t_0 + \frac{\omega}{2} - s \right| : s \in B \right\} &= \inf \left\{ \left| t_0 + \frac{\omega}{2} - s \right| : s \in A_1 \right\} \\ &= \inf \left\{ \left| t_0 - \frac{\omega}{2} - s \right| : s \in A \right\} \geq \inf \left\{ \left| t_0 - \frac{\omega}{2} - s \right| : s \in B \right\}, \end{aligned} \quad (3.20)$$

and therefore,

$$\sigma_B(t_0) \geq \rho_B\left(t_0 - \frac{\omega}{2}\right) + \rho_B(t_0) = \sigma_B\left(t_0 - \frac{\omega}{2}\right). \quad (3.21)$$

The inequalities (3.19) imply  $t_0 - \omega/2 \leq \alpha + \omega$  and, according to the case considered above, we have  $\sigma_B(t_0 - \omega/2) \geq \delta$ . Consequently, (3.21) results in (3.13).

Finally, if  $\alpha + \omega \leq t_0$ , the validity of (3.13) can be proved analogously to the previous cases. Then we have

$$\sigma_B(t) \geq \delta \quad \text{for } 0 \leq t \leq \frac{3\omega}{2}. \quad (3.22)$$

On the other hand, since  $A \subset B$ , it is clear that

$$\sigma_B(t) \leq \sigma_A(t) \quad \text{for } 0 \leq t \leq \frac{3\omega}{2}. \quad (3.23)$$

The last two relations and (3.5) yields the equality (3.6).  $\square$

LEMMA 3.3. *Let  $\sigma \in \{-1, 1\}$ ,  $D \subset [a, b]$ ,  $D \neq \emptyset$ ,  $\ell_1 \in K_{[a,b]}(D)$ , and let  $\sigma\ell_1$  be nonnegative. Then, for an arbitrary  $v \in C([a, b])$ ,*

$$\begin{aligned} \min \{v(s) : s \in \bar{D}\} |\ell_1(1)(t)| \\ \leq \sigma\ell_1(v)(t) \leq \max \{v(s) : s \in \bar{D}\} |\ell_1(1)(t)| \quad \text{for } a \leq t \leq b. \end{aligned} \quad (3.24)$$

*Proof.* Let  $\alpha = \inf D, \beta = \sup D$ ,

$$v_0(t) = \begin{cases} v(\alpha) & \text{for } t \in [a, \alpha[ \\ v(t) & \text{for } t \in \bar{D} \\ \frac{v(\mu(t)) - v(\nu(t))}{\mu(t) - \nu(t)} (t - \nu(t)) + v(\nu(t)) & \text{for } t \in [\alpha, \beta] \setminus \bar{D} \\ v(\beta) & \text{for } t \in ]\beta, b], \end{cases} \quad (3.25)$$

where

$$\mu(t) = \min\{s \in \bar{D} : t \leq s\}, \quad \nu(t) = \max\{s \in \bar{D} : t \geq s\} \quad \text{for } \alpha \leq t \leq \beta. \quad (3.26)$$

It is clear that  $v_0 \in C([a, b])$  and

$$\begin{aligned} \min\{v(s) : s \in \bar{D}\} \leq v_0(t) \leq \max\{v(s) : s \in \bar{D}\} & \quad \text{for } a \leq t \leq b, \\ v_0(t) = v(t) & \quad \text{for } t \in D. \end{aligned} \quad (3.27)$$

Since  $\ell_1 \in K_{[a, b]}(D)$  and the operator  $\sigma\ell_1$  is nonnegative, it follows from (3.27) that (3.24) is true.  $\square$

LEMMA 3.4. *Let  $a \in [0, \omega]$ ,  $D \subset [a, a + \omega]$ ,  $c \in [a, a + \omega]$ , and  $\delta \in [0, \omega/2]$  be such that*

$$\sigma_D(t) \geq \delta \quad \text{for } a \leq t \leq a + \frac{\omega}{2}, \quad (3.28)$$

$$A_c = \bar{D} \cap [a, c] \neq \emptyset, \quad B_c = \bar{D} \cap [c, a + \omega] \neq \emptyset. \quad (3.29)$$

*Then the estimate*

$$\left( \frac{(c - t_1)(t_1 - a)(a + \omega - t_2)(t_2 - c)}{(c - a)(a + \omega - c)} \right)^{1/2} \leq \frac{\omega^2 - 4\delta^2}{8\omega} \quad (3.30)$$

*for all  $t_1 \in A_c$ ,  $t_2 \in B_c$  is satisfied.*

*Proof.* Put  $b = a + \omega$  and

$$\sigma_1 = \rho_D\left(\frac{a + c}{2}\right), \quad \sigma_2 = \rho_D\left(\frac{c + b}{2}\right). \quad (3.31)$$

Then, from the condition (3.28) it is clear

$$\sigma_1 + \sigma_2 \geq \delta. \quad (3.32)$$

Obviously, either

$$\max(\sigma_1, \sigma_2) \geq \delta \quad (3.32_1)$$

or

$$\max(\sigma_1, \sigma_2) < \delta. \quad (3.32_2)$$

First note that from (3.29) and (3.31) the equalities

$$\begin{aligned}\max \{(c-t_1)(t_1-a) : t_1 \in A_c\} &= (c-t'_1)(t'_1-a), \\ \max \{(b-t_2)(t_2-c) : t_2 \in B_c\} &= (b-t'_2)(t'_2-c),\end{aligned}\tag{3.33}$$

follow, where  $t'_1 = (a+c)/2 - \sigma_1$ ,  $t'_2 = (c+b)/2 - \sigma_2$ . Hence, on account of well-known inequality

$$d_1 d_2 \leq \frac{(d_1 + d_2)^2}{4},\tag{3.34}$$

we have

$$\begin{aligned}\left( \frac{(c-t_1)(t_1-a)(b-t_2)(t_2-c)}{(c-a)(b-c)} \right)^{1/2} \\ \leq \left( \frac{c-a}{4} - \frac{\sigma_1^2}{c-a} \right)^{1/2} \left( \frac{b-c}{4} - \frac{\sigma_2^2}{b-c} \right)^{1/2} \leq \frac{1}{2} \left( \frac{\omega}{4} - \frac{\sigma_1^2}{c-a} - \frac{\sigma_2^2}{b-c} \right)\end{aligned}\tag{3.35}$$

for all  $t_1 \in A_c$ ,  $t_2 \in B_c$ . In the case, where inequality (3.32<sub>1</sub>) is fulfilled, we have

$$\frac{\omega}{4} - \frac{\sigma_1^2}{c-a} - \frac{\sigma_2^2}{b-c} \leq \frac{\omega}{4} - \frac{(\max(\sigma_1, \sigma_2))^2}{\omega} \leq \frac{\omega^2 - 4\delta^2}{4\omega}.\tag{3.36}$$

This, together with (3.35), yields the estimate (3.30). Suppose now that the condition (3.32<sub>2</sub>) is fulfilled. Then in view of Lemma 3.1, we can choose  $\alpha, \beta \in \bar{D}$  such that

$$\rho_D\left(\frac{a+c}{2}\right) = \left| \frac{a+c}{2} - \alpha \right|, \quad \rho_D\left(\frac{c+b}{2}\right) = \left| \frac{c+b}{2} - \beta \right|,\tag{3.37}$$

which together with (3.31) yields

$$\frac{\omega}{4} - \frac{\sigma_1^2}{c-a} - \frac{\sigma_2^2}{b-c} = \omega - (\beta - \alpha) - \eta(c),\tag{3.38}$$

where  $\eta(t) = (\alpha - a)^2/(t - a) + (b - \beta)^2/(b - t)$ . It is not difficult to verify that the function  $\eta$  achieves its minimum at the point  $t_0 = ((\alpha - a)b + (b - \beta)a)/(\omega - (\beta - \alpha))$ . Thus,

$$\omega - (\beta - \alpha) - \eta(c) \leq (\omega - (\beta - \alpha)) \frac{\beta - \alpha}{\omega}.\tag{3.39}$$

Put

$$\sigma = \min(\sigma_1, \sigma_2).\tag{3.40}$$

Then it follows from (3.37) that either

$$\alpha \leq \frac{a+c}{2} - \sigma \quad (3.40_1)$$

or

$$\alpha \geq \frac{a+c}{2} + \sigma, \quad (3.40_2)$$

and either

$$\beta \geq \frac{c+b}{2} + \sigma \quad (3.40_3)$$

or

$$\beta \leq \frac{c+b}{2} - \sigma. \quad (3.40_4)$$

Consider now the case where  $\alpha$  satisfies the inequality (3.40<sub>1</sub>) and assume that  $\beta$  satisfies the inequality (3.40<sub>4</sub>). Then from (3.37), (3.40<sub>1</sub>), and (3.40<sub>4</sub>) we get

$$\rho_D\left(\frac{a+c}{2} - \sigma\right) = \rho_D\left(\frac{a+c}{2}\right) - \sigma, \quad \rho_D\left(\frac{c+b}{2} - \sigma\right) = \rho_D\left(\frac{c+b}{2}\right) - \sigma. \quad (3.41)$$

These equalities in view of (3.31) and (3.40) yield

$$\sigma_D\left(\frac{a+c}{2} - \sigma\right) = (\sigma_1 - \sigma) + (\sigma_2 - \sigma) = \max(\sigma_1, \sigma_2) - \sigma, \quad (3.42)$$

but in view of (3.32<sub>2</sub>) this contradicts the condition (3.28). Consequently,  $\beta$  satisfies the inequality (3.40<sub>3</sub>). Then from (3.31), (3.37), by (3.40<sub>1</sub>) and (3.40<sub>3</sub>), we get  $\sigma_1 = (a+c)/2 - \alpha$ ,  $\sigma_2 = \beta - (c+b)/2$ , that is,

$$\beta - \alpha = \sigma_1 + \sigma_2 + \frac{\omega}{2}. \quad (3.42_1)$$

Now suppose that (3.40<sub>2</sub>) holds. It can be proved in a similar manner as above that, in this case, the inequality (3.40<sub>4</sub>) is satisfied. Therefore, from (3.31), (3.37), (3.40<sub>2</sub>), and (3.40<sub>4</sub>) we obtain

$$\beta - \alpha = \frac{\omega}{2} - (\sigma_1 + \sigma_2). \quad (3.42_2)$$

Then, on account of (3.32), in both (3.42<sub>1</sub>) and (3.42<sub>2</sub>) cases we have

$$(\omega - (\beta - \alpha)) \frac{\beta - \alpha}{\omega} = \frac{\omega^2 - 4(\sigma_1 + \sigma_2)^2}{4\omega} \leq \frac{\omega^2 - 4\delta^2}{4\omega}. \quad (3.43)$$

Consequently from (3.35), (3.38), (3.39), and (3.43) we obtain the estimate (3.30), also in case where the inequality (3.32<sub>2</sub>) holds.  $\square$

#### 4. Proof of the main results

*Proof of Theorem 2.1.* Consider the homogeneous problem

$$v''(t) = \ell(v)(t) \quad \text{for } 0 \leq t \leq \omega, \quad (4.1)$$

$$v^{(i)}(0) = v^{(i)}(\omega) \quad (i = 0, 1). \quad (4.2)$$

It is known from the general theory of boundary value problems for functional differential equations that if  $\ell$  is a monotone operator, then problem (1.1), (1.2) has the Fredholm property (see [3, Theorem 1.1, page 345]). Thus, the problem (1.1), (1.2) is uniquely solvable iff the homogeneous problem (4.1), (4.2) has only the trivial solution.

Assume that, on the contrary, the problem (4.1), (4.2) has a nontrivial solution  $v$ . If  $v \equiv \text{const}$ , then, in view of (4.1) we obtain a contradiction with the condition (2.2). Consequently,  $v \not\equiv \text{const}$ . Then, in view of the conditions (4.2), there exist subsets  $I_1$  and  $I_2$  from  $[0, \omega]$  which have positive measure and

$$v''(t) > 0 \quad \text{for } t \in I_1, \quad v''(t) < 0 \quad \text{for } t \in I_2. \quad (4.3)$$

Assume that  $v$  is either nonnegative or nonpositive on the entire set  $A$ . Without loss of generality we can suppose  $v(t) \geq 0$  for  $t \in A$ . Then, from Lemma 3.3 with  $a = 0$ ,  $b = \omega$ ,  $D = A$ , and  $\ell_1 \equiv \ell$  we obtain

$$\sigma\ell(v)(t) \geq 0 \quad \text{for } 0 \leq t \leq \omega. \quad (4.4)$$

In view of (4.1), the inequality (4.4) contradicts one of the inequalities in (4.3). Therefore, the function  $v$  changes its sign on the set  $A$ , that is, there exist  $t'_1, t_1 \in \bar{A}$  such that

$$v(t'_1) = \min \{v(t) : t \in \bar{A}\}, \quad v(t_1) = \max \{v(t) : t \in \bar{A}\}, \quad (4.5)$$

and  $v(t'_1) < 0$ ,  $v(t_1) > 0$ . Without loss of generality we can assume that  $t'_1 < t_1$ . Then, in view of the last inequalities, there exists  $a \in ]t'_1, t_1[$  such that  $v(a) = 0$ .

Let us set  $C_\omega([a, a + \omega]) = \{x \in C([a, a + \omega]) : x(a) = x(a + \omega)\}$ , and let the continuous operators  $\gamma : L([0, \omega]) \rightarrow L([a, a + \omega])$ ,  $\ell_1 : C_\omega([a, a + \omega]) \rightarrow L([a, a + \omega])$  and the function  $v_0 \in C([a, a + \omega])$  be given by the equalities

$$\gamma(x)(t) = \begin{cases} x(t) & \text{for } a \leq t \leq \omega \\ x(t - \omega) & \text{for } \omega < t \leq a + \omega, \end{cases} \quad (4.6)$$

$$v_0(t) = \gamma(v(t)), \quad \ell_1(x)(t) = \gamma(\ell(\gamma^{-1}(x)))(t) \quad \text{for } a \leq t \leq a + \omega.$$

Let, moreover,  $t_2 = t'_1 + \omega$  and  $D = A \cup \{t + \omega : t \in A\} \cap [a, a + \omega]$ . Then (4.1), (4.2) with regard for (4.6) and the definitions of  $a$ ,  $t'_1$ ,  $t_1$ , imply that  $v_0 \in \tilde{C}'([a, a + \omega])$ ,  $t_1, t_2 \in D$ ,

$$v_0''(t) = \ell_1(v_0)(t) \quad \text{for } a \leq t \leq a + \omega, \quad (4.7)$$

$$v_0(a) = 0, \quad v_0(a + \omega) = 0, \quad (4.8)$$

$$v_0(t_1) = \max \{v_0(t) : t \in \bar{D}\}, \quad v_0(t_2) = \min \{v_0(t) : t \in \bar{D}\}, \quad (4.9)$$

$$v_0(t_1) > 0, \quad v_0(t_2) < 0, \quad (4.10)$$

and there exists  $c \in ]t_1, t_2[$  such that

$$\nu_0(c) = 0. \quad (4.11)$$

It is not difficult to verify that the condition  $\ell \in K_{[0,\omega]}(A)$  implies

$$\ell_1 \in K_{[a,a+\omega]}(D). \quad (4.12)$$

Since  $D \subset A \cup \{t + \omega : t \in A\}$ , it follows from condition (2.4) and Lemma 3.2 that

$$\sigma_D(t) \geq \delta \quad \text{for } a \leq t \leq a + \frac{\omega}{2}. \quad (4.13)$$

Thus, from the general theory of ordinary differential equations (see [6, Theorem 1.1, page 2348]), in view of (4.7), (4.8), (4.9), and (4.11), we obtain the representations

$$\nu_0(t_1) = - \int_a^c |G_1(t_1, s)| \ell_1(\nu_0)(s) ds, \quad (4.13_1)$$

$$|\nu_0(t_2)| = \int_c^{a+\omega} |G_2(t_2, s)| \ell_1(\nu_0)(s) ds, \quad (4.13_2)$$

where  $G_1(G_2)$  is Green's function of the problem

$$\begin{aligned} z''(t) &= 0 \quad \text{for } a \leq t \leq c \ (c \leq t \leq a + \omega), \\ z(a) &= 0, \quad z(c) = 0 \quad (z(c) = 0, z(a + \omega) = 0). \end{aligned} \quad (4.14)$$

If  $\ell$  is a nonnegative operator, then from (4.6) it is clear that  $\ell_1$  is also nonnegative. Then, from (4.13<sub>1</sub>) and (4.13<sub>2</sub>), by Lemma 3.3 and relations (4.9), (4.10), and (4.12), we get the strict estimates

$$\begin{aligned} 0 &< \frac{\nu_0(t_1)}{|\nu_0(t_2)|} < \frac{(t_1 - a)(c - t_1)}{c - a} \int_a^c \ell_1(1)(s) ds, \\ 0 &< \frac{|\nu_0(t_2)|}{\nu_0(t_1)} < \frac{(t_2 - c)(a + \omega - t_2)}{a + \omega - c} \int_c^{a+\omega} \ell_1(1)(s) ds, \end{aligned} \quad (4.15)$$

respectively. By multiplying these estimates and applying the numerical inequality (3.34), we obtain

$$1 < \frac{1}{2} \left( \frac{(t_1 - a)(c - t_1)(t_2 - c)(a + \omega - t_2)}{(c - a)(a + \omega - c)} \right)^{1/2} \int_a^{a+\omega} |\ell_1(1)(s)| ds. \quad (4.16)$$

Reasoning analogously, we can show that the estimate (4.16) is valid also in case where the operator  $\ell$  is nonpositive.

From the definitions of  $t_1, t_2, c$ , and (4.13), it follows that all the conditions of Lemma 3.4 are satisfied. In view of the estimate (3.30) and the definition of the operator  $\ell_1$ , the inequality (4.16) contradicts the condition (2.3).  $\square$

*Proof of Corollary 2.3.* Let  $\delta = \omega/2(1 - 16/d)^{1/2}$ . Then, on account of (2.10) and (2.11), we obtain that the conditions (2.3) and (2.4) of Theorem 2.1 are fulfilled. Consequently, all the assumptions of Theorem 2.1 are satisfied.  $\square$

*Proof of Corollary 2.4.* It is not difficult to verify that if  $A = [\alpha, \beta]$  ( $A = [0, \alpha] \cup [\beta, \omega]$ ), then

$$\sigma_A(t) \geq \left[ \frac{\omega}{2} - \beta + \alpha \right]_+ \quad \left( \sigma_A(t) \geq \left[ \frac{\omega}{2} - \beta + \alpha \right]_- \right) \quad \text{for } 0 \leq t \leq \frac{\omega}{2}. \quad (4.17)$$

Consequently, in view of the condition (2.11<sub>1</sub>), (2.11<sub>2</sub>), all the assumptions of Theorem 2.1 are satisfied.  $\square$

*Proof of Corollary 2.5.* Let  $\ell(u)(t) \equiv p(t)u(\tau(t))$ . On account of (2.13), (2.14), and (2.15) we see that the operator  $\ell$  is monotone and the conditions (2.2) and (2.3) are satisfied.

(a) It is not difficult to verify that from the condition (2.16) it follows that  $\ell \in K_{[0, \omega]}(A)$ . Consequently, all the assumptions of Theorem 2.1 are satisfied.

(b) Let  $A = [\alpha, \beta]$ . Then in view of the condition (2.17) the inclusion  $\ell \in K_{[0, \omega]}(A)$  is satisfied. The inequality (4.17) obtained in the proof of Corollary 2.4, by virtue of (2.18), implies the inequality (2.4). Consequently, all the assumptions of Theorem 2.1 are satisfied.  $\square$

*Proof of Corollary 2.6.* The validity of this assertion follows immediately from Corollary 2.5(a).  $\square$

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