

# PARABOLIC INEQUALITIES WITH NONSTANDARD GROWTHS AND $L^1$ DATA

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*Received 25 July 2005; Revised 13 December 2005; Accepted 19 December 2005*

We prove an existence result for solutions of nonlinear parabolic inequalities with  $L^1$  data in Orlicz spaces.

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## 1. Introduction

Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^N$ ,  $N \geq 2$ , let  $Q$  be the cylinder  $\Omega \times (0, T)$  with some given  $T > 0$ . Consider the following nonlinear parabolic problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + A(u) &= \chi \quad \text{in } Q, \\ u(x, t) &= 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(x, 0) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where  $A(u) = -\operatorname{div}(a(x, t, u, \nabla u))$  is a Leray-Lions operator defined on  $D(A) \subset W_0^{1,x}L_M(\Omega)$ , with  $M$  is an  $N$ -function, and  $\chi$  is a given data.

In the variational case (i.e., where  $\chi \in W^{-1,x}E_{\overline{M}}(\Omega)$ ), it is well known that the solvability of (1.1) is done by Donaldson [2] and Robert [11] when the operator  $A$  is monotone,  $t^2 \ll M(t)$ , and  $\overline{M}$  satisfies a  $\Delta_2$  condition, and by finally the recent work [3] for the general case.

In the  $L^1$  case, an existence theorem is given in [4]. However, the techniques used in [4] do not allow us to adapt it for parabolic inequalities. It is our purpose in this paper to solve the obstacle problem associated to (1.1) in the case where  $\chi \in L^1(Q) + W^{-1,x}E_{\overline{M}}(Q)$  and without assuming any growth restriction on  $M$ . The existence of solutions is proved via a sequence of penalized problems, with solutions  $u_n$ . A priori estimates of the truncation of  $u_n$  are obtained in some suitable Orlicz space. For the passage to the limit, the

## 2 Parabolic inequalities in $L^1$

almost everywhere convergence of  $\nabla u_n$  is proved via new techniques. As operators models, we can consider slow or fast growth:

$$\begin{aligned} A(u) &= -\operatorname{div} \left( (1 + |u|)^2 \nabla u \frac{\log(1 + |\nabla u|)}{|\nabla u|} \right), \\ A(u) &= -\operatorname{div}(\nabla u \exp(|\nabla u|)). \end{aligned} \quad (1.2)$$

For some classical and recent results in the setting of Orlicz spaces dealing with elliptic and parabolic equations, the reader can be referred to [8, 10, 12–14].

### 2. Preliminaries

**2.1.** Let  $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be an  $N$ -function, that is,  $M$  is continuous, convex, with  $M(t) > 0$  for  $t > 0$ ,  $M(t)/t \rightarrow 0$  as  $t \rightarrow 0$ , and  $M(t)/t \rightarrow \infty$  as  $t \rightarrow \infty$ .

Equivalently,  $M$  admits the representation  $M(t) = \int_0^t a(s) ds$ , where  $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is non-decreasing, right continuous, with  $a(0) = 0$ ,  $a(t) > 0$  for  $t > 0$ , and  $a(t)$  tends to  $\infty$  as  $t \rightarrow \infty$ .

The  $N$ -function  $\overline{M}$  conjugate to  $M$  is defined by  $\overline{M}(t) = \int_0^t \bar{a}(s) ds$ , where  $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is given by  $\bar{a}(t) = \sup\{s : a(s) \leq t\}$  (see [1]).

The  $N$ -function is said to satisfy the  $\Delta_2$  condition if, for some  $k > 0$ ,

$$M(2t) \leq kM(t), \quad \forall t \geq 0, \quad (2.1)$$

when (2.1) holds only for  $t \geq$  some  $t_0 > 0$ , then  $M$  is said to satisfy the  $\Delta_2$  condition near infinity.

We will extend these  $N$ -functions into even functions on all  $\mathbb{R}$ .

Let  $P$  and  $Q$  be two  $N$ -functions.  $P \ll Q$  means that  $P$  grows essentially less rapidly than  $Q$ , that is, for each  $\epsilon > 0$ ,  $P(t)/Q(\epsilon t) \rightarrow 0$  as  $t \rightarrow \infty$ . This is the case if and only if  $\lim_{t \rightarrow \infty} (Q^{-1}(t))/(P^{-1}(t)) = 0$ .

**2.2.** Let  $\Omega$  be an open subset of  $\mathbb{R}^N$ . The Orlicz class  $K_M(\Omega)$  (resp., the Orlicz space  $L_M(\Omega)$ ) is defined as the set of (equivalence classes of) real-valued measurable functions  $u$  on  $\Omega$  such that

$$\int_{\Omega} M(u(x)) dx < +\infty \quad \left( \text{resp., } \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0 \right). \quad (2.2)$$

$L_M(\Omega)$  is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) dx \leq 1 \right\} \quad (2.3)$$

and  $K_M(\Omega)$  is a convex subset of  $L_M(\Omega)$ .

The closure in  $L_M(\Omega)$  of the set of bounded measurable functions with compact support in  $\overline{\Omega}$  is denoted by  $E_M(\Omega)$ .

The equality  $E_M(\Omega) = L_M(\Omega)$  holds if and only if  $M$  satisfies the  $\Delta_2$  condition, for all  $t$  or for  $t$  large, according to whether  $\Omega$  has infinite measure or not.

The dual of  $E_M(\Omega)$  can be identified with  $L_{\overline{M}}(\Omega)$  by means of the pairing  $\int_{\Omega} uv dx$ , and the dual norm of  $L_{\overline{M}}(\Omega)$  is equivalent to  $\|\cdot\|_{\overline{M},\Omega}$ .

The space  $L_M(\Omega)$  is reflexive if and only if  $M$  and  $\overline{M}$  satisfy the  $\Delta_2$  condition, for all  $t$  or for  $t$  large, according to whether  $\Omega$  has infinite measure or not.

**2.3.** We now turn to the Orlicz-Sobolev space,  $W^1 L_M(\Omega)$  (resp.,  $W^1 E_M(\Omega)$ ) is the space of all functions  $u$  such that  $u$  and its distributional derivatives up to order 1 lie in  $L_M(\Omega)$  (resp.,  $E_M(\Omega)$ ). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^{\alpha} u\|_M. \quad (2.4)$$

Thus,  $W^1 L_M(\Omega)$  and  $W^1 E_M(\Omega)$  can be identified with subspaces of product of  $N+1$  copies of  $L_M(\Omega)$ . Denoting this product by  $\prod L_M$ , we will use the weak topologies  $\sigma(\prod L_M, \prod L_{\overline{M}})$  and  $\sigma(\prod L_M, \prod E_{\overline{M}})$ .

The space  $W_0^1 E_M(\Omega)$  is defined as the (norm) closure of the Schwartz space  $D(\Omega)$  in  $W^1 E_M(\Omega)$  and the space  $W_0^1 L_M(\Omega)$  as the  $\sigma(\prod L_M, \prod E_{\overline{M}})$  closure of  $D(\Omega)$  in  $W^1 L_M(\Omega)$ . We say that  $u_n$  converges to  $u$  for the modular convergence in  $W^1 L_M(\Omega)$  if for some  $\lambda > 0$ ,

$$\int_{\Omega} M\left(\frac{D^{\alpha} u_n - D^{\alpha} u}{\lambda}\right) dx \rightarrow 0, \quad \forall |\alpha| \leq 1. \quad (2.5)$$

This implies convergence for  $\sigma(\prod L_M, \prod L_{\overline{M}})$ . If  $M$  satisfies the  $\Delta_2$  condition on  $\mathbb{R}^+$ , then modular convergence coincides with norm convergence.

**2.4.** Let  $W^{-1} L_{\overline{M}}(\Omega)$  (resp.,  $W^{-1} E_{\overline{M}}(\Omega)$ ) denote the space of distributions on  $\Omega$  which can be written as sums of derivatives of order  $\leq 1$  of functions in  $L_{\overline{M}}$  (resp.,  $E_{\overline{M}}(\Omega)$ ). It is a Banach space under the usual quotient norm.

If the open set  $\Omega$  has the segment property, then the space  $D(\Omega)$  is dense in  $W_0^1 L_M(\Omega)$  for the modular convergence and thus for the topology  $\sigma(\prod L_M, \prod L_{\overline{M}})$  (cf. [6, 7]). Consequently, the action of a distribution in  $W^{-1} L_{\overline{M}}(\Omega)$  on an element of  $W_0^1 L_M(\Omega)$  is well defined.

**2.5.** Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^N$ ,  $T > 0$ , and set  $Q = \Omega \times (0, T)$ . Let  $M$  be an  $N$ -function. For each  $\alpha \in \mathbb{N}^N$ , denote by  $D_x^{\alpha}$  the distributional derivatives on  $Q$  of order  $\alpha$  with respect to the variable  $x \in \mathbb{R}^N$ . The inhomogeneous Orlicz-Sobolev spaces of order 1 are defined as follows:

$$\begin{aligned} W^{1,x} L_M(Q) &= \{u \in L_M(Q) : D_x^{\alpha} u \in L_M(Q), \forall |\alpha| \leq 1\}, \\ W^{1,x} E_M(Q) &= \{u \in E_M(Q) : D_x^{\alpha} u \in E_M(Q), \forall |\alpha| \leq 1\}. \end{aligned} \quad (2.6)$$

The latest space is a subset of the first one. They are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|=1} \|D_x^{\alpha} u\|_{M,Q}. \quad (2.7)$$

We can easily show that they form a complementary system when  $\Omega$  satisfies the segment property. These spaces are considered as subspaces of the product spaces  $\prod L_M(Q)$

#### 4 Parabolic inequalities in $L^1$

which has  $N + 1$  copies. We will also consider the weak topologies  $\sigma(\prod L_M, \prod E_{\overline{M}})$  and  $\sigma(\prod L_M, \prod L_{\overline{M}})$ . If  $u \in W^{1,x}L_M(Q)$ , then the function  $t \rightarrow u(t) = u(\cdot, t)$  is defined on  $(0, T)$  with values in  $W^1L_M(\Omega)$ . If, further,  $u \in W^{1,x}E_M(Q)$ , then  $u(t)$  is  $W^1E_M(\Omega)$ -valued and is strongly measurable. Furthermore, the following continuous imbedding holds:  $W^{1,x}E_M(Q) \subset L^1(0, T; W^1E_M(\Omega))$ . The space  $W^{1,x}L_M(Q)$  is not in general separable, if  $u \in W^{1,x}L_M(Q)$ , we cannot conclude that the function  $u(t)$  is measurable from  $(0, T)$  into  $W^1L_M(\Omega)$ . However, the scalar function  $t \rightarrow \|D_x^\alpha u(t)\|_{M,\Omega}$  is in  $L^1(0, T)$  for all  $|\alpha| \leq 1$ .

**2.6.** The space  $W_0^{1,x}E_M(Q)$  is defined as the (norm) closure in  $W^{1,x}E_M(Q)$  of  $D(Q)$ . We can easily show as in [7] that when  $\Omega$  has the segment property, then for all  $u \in \overline{D(Q)}^{\sigma(\prod L_M, \prod E_{\overline{M}})}$  there exist some  $\lambda > 0$  and a sequence  $(u_n) \subset D(Q)$  such that for all  $|\alpha| \leq 1$ ,  $\int_\Omega M((D_x^\alpha u_n - D_x^\alpha u)/\lambda) dx \rightarrow 0$  when  $n \rightarrow \infty$ . Consequently,  $\overline{D(Q)}^{\sigma(\prod L_M, \prod E_{\overline{M}})} = \overline{D(Q)}^{\sigma(\prod L_M, \prod L_{\overline{M}})}$ , this space will be denoted by  $W_0^{1,x}L_M(Q)$ . Furthermore,  $W_0^{1,x}E_M(Q) = W_0^{1,x}L_M(Q) \cap \prod E_{\overline{M}}$ . Poincaré's inequality also holds in  $W_0^{1,x}L_M(Q)$  and then there is a constant  $C > 0$  such that for all  $u \in W_0^{1,x}L_M(Q)$ , one has

$$\sum_{|\alpha| \leq 1} \|D_x^\alpha u\|_{M,Q} \leq C \sum_{|\alpha| = 1} \|D_x^\alpha u\|_{M,Q}, \quad (2.8)$$

thus both sides of the last inequality are equivalent norms on  $W_0^{1,x}L_M(Q)$ . We have then the following complementary system:

$$\left( \frac{W_0^{1,x}L_M(Q)}{W_0^{1,x}E_M(Q)} \middle| \frac{F}{F_0} \right), \quad (2.9)$$

$F$  being the dual space of  $W_0^{1,x}E_M(Q)$ . It is also, up to an isomorphism, the quotient of  $\prod L_{\overline{M}}$  by the polar set  $W_0^{1,x}E_M(Q)^\perp$ , and will be denoted by  $F = W^{-1,x}L_{\overline{M}}(Q)$  and it is shown that

$$W^{-1,x}L_{\overline{M}}(Q) = \left\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{M}}(Q) \right\}. \quad (2.10)$$

This space will be equipped with the usual quotient norm:

$$\|f\| = \inf \sum_{|\alpha| \leq 1} \|f_\alpha\|_{\overline{M},Q}, \quad (2.11)$$

where the inf is taken on all possible decompositions  $f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha$ ,  $f_\alpha \in L_{\overline{M}}(Q)$ . The space  $F_0$  is then given by  $F_0 = \{f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q)\}$  and is denoted by  $F_0 = W^{-1,x}E_{\overline{M}}(Q)$ .

*Defintion 2.1.* We say that  $u_n \rightarrow u$  in  $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$  for the modular convergence if we can write

$$u_n = \sum_{|\alpha| \leq 1} D_x^\alpha u_n^\alpha + u_n^0, \quad u = \sum_{|\alpha| \leq 1} D_x^\alpha u^\alpha + u^0 \quad (2.12)$$

with  $u_n^\alpha \rightarrow u^\alpha$  in  $L_{\overline{M}}(Q)$  for the modular convergence for all  $|\alpha| \leq 1$  and  $u_n^0 \rightarrow u^0$  strongly in  $L^1(Q)$ .

We will give the following approximation theorem which plays a crucial role when proving the existence result of solutions for parabolic inequalities.

**THEOREM 2.2.** *Let  $\phi \in W_0^{1,x}E_M(Q) \cap L^\infty(Q)$  and consider the convex set  $\mathcal{H}_\phi = \{v \in W_0^{1,x}L_M(Q) : v \geq \phi \text{ a.e. in } Q\}$ . Then for every  $u \in \mathcal{H}_\phi \cap L^\infty(Q)$  such that  $\partial u/\partial t \in W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$ , there exists  $v_j \in \mathcal{H}_\phi \cap D(\overline{Q})$  such that*

$$\begin{aligned} v_j &\longrightarrow u \quad \text{in } W^{1,x}L_M(Q), \\ \frac{\partial v_j}{\partial t} &\longrightarrow \frac{\partial u}{\partial t} \quad \text{in } W^{-1,x}L_{\overline{M}}(Q) + L^1(Q) \end{aligned} \quad (2.13)$$

for the modular convergence.

*Proof.* It is easily adapted from that given in [4, Theorem 3] and the approximation techniques of [9].  $\square$

**Remark 2.3.** The result is still true for  $\phi \in W^{1,x}E_M(Q) \cap L^\infty(Q)$ , when  $\Omega$  is more regular (see [9]).

In order to deal with the time derivative, we introduce a time mollification of a function  $v \in L_M(Q)$ . Thus, we define, for all  $\mu > 0$  and all  $(x, t) \in Q$ ,

$$v_\mu(x, t) = \mu \int_{-\infty}^t \tilde{v}(x, s) \exp(\mu(s - t)) ds, \quad (2.14)$$

where  $\tilde{v}(x, s) = v(x, s)\chi_{(0, T)}(s)$  is the zero extension of  $v$ . The following proposition is fundamental in the sequel.

**PROPOSITION 2.4** [5]. *If  $v \in L_M(Q)$ , then  $v_\mu$  is measurable in  $Q$ ,  $\partial v_\mu/\partial t = \mu(v - v_\mu)$  and*

$$\int_Q M(v_\mu) dx dt \leq \int_Q M(v) dx dt. \quad (2.15)$$

Recall now the following compactness result which is proved in [5].

**PROPOSITION 2.5.** *Assume that  $(u_n)_n$  is a bounded sequence in  $W_0^1L_M(Q)$  such that  $\partial u_n/\partial t$  is bounded in  $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$ , then  $u_n$  is relatively compact in  $L^1(Q)$ .*

### 3. The main result

Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^N$ ,  $N \geq 2$ , with the segment property. Let  $P$  and  $M$  be two  $N$ -functions such that  $P \ll M$ . Consider now the operator  $A : D(A) \subset W_0^{1,x}L_M(Q) \rightarrow W^{-1,x}L_{\overline{M}}(Q)$  in divergence form  $A(u) = -\operatorname{div}(a(x, t, u, \nabla u))$ , where  $a : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a Carathéodory function satisfying for a.e.  $x \in \Omega$  and for all  $\zeta, \zeta' \in \mathbb{R}^N$ ,

## 6 Parabolic inequalities in $L^1$

$(\zeta \neq \zeta')$  and all  $s, t \in \mathbb{R}$ :

$$\begin{aligned} |a(x, t, s, \zeta)| &\leq h(x, t) + k_1 \bar{P}^{-1} M(k_2 |s|) + k_3 \bar{M}^{-1} M(k_4 |\zeta|), \\ (a(x, t, s, \zeta) - a(x, t, s, \zeta'))(\zeta - \zeta') &> 0, \\ a(x, t, s, \zeta)\zeta &\geq \alpha M(|\zeta|) - d(x, t), \end{aligned} \quad (3.1)$$

with  $d \in L^1(Q)$ ,  $\alpha, k_1, k_2, k_3, k_4 > 0$ , and  $h \in E_{\bar{M}}(Q)$ . Let

$$\psi \in W_0^1 E_M(\Omega) \cap L^\infty(\Omega). \quad (3.2)$$

Finally, consider

$$f \in L^1(Q). \quad (3.3)$$

We define for all  $t \in \mathbb{R}$ ,  $k \geq 0$ ,  $T_k(t) = \max(-k, \min(k, t))$ , and  $S_k(t) = \int_0^t T_k(\eta) d\eta$ .

We will prove the following existence theorem.

**THEOREM 3.1.** *Let  $u_0 \in L^1(\Omega)$  such that  $u_0 \geq 0$ . Assume that (3.1)–(3.3) hold true. Then there exists at least one solution  $u \in C([0, T]; L^1(\Omega))$  such that  $u(x, 0) = u_0$  a.e. and for all  $\tau \in ]0, T]$ ,*

$$\begin{aligned} u &\geq \psi \quad \text{a.e. in } Q, \\ T_k(u) &\in W_0^{1,x} L_M(Q), \\ \int_\Omega S_k(u(\tau) - v(\tau)) dx &+ \left\langle \frac{\partial v}{\partial t}, T_k(u - v) \right\rangle_{Q_\tau} + \int_{Q_\tau} a(x, t, u, \nabla u) \nabla T_k(u - v) dx dt \\ &\leq \int_{Q_\tau} f T_k(u - v) dx dt + \int_\Omega S_k(u_0 - v(x, 0)) dx, \\ \forall k > 0 \text{ and } \forall v &\in \mathcal{H}_\psi \cap L^\infty(Q) \text{ such that } \frac{\partial v}{\partial t} \in W^{-1,x} L_{\bar{M}}(Q) + L^1(Q), \end{aligned} \quad (p_\psi)$$

where  $Q_\tau = \Omega \times ]0, \tau[$ .

**Remark 3.2.** Since  $\{v \in \mathcal{H}_\psi \cap L^\infty(Q) : \partial v / \partial t \in W^{-1,x} L_{\bar{M}}(Q) + L^1(Q)\} \subset C([0, T], L^1(\Omega))$ , (see [4]), the first and the latest terms of problem  $(p_\psi)$  are well defined.

*Proof*

**Step 1.** A priori estimates.

For the sake of simplicity, we assume that  $d(x, t) = 0$ .

Consider the approximate equations

$$\begin{aligned} \frac{\partial u_n}{\partial t} - \operatorname{div}(a(x, t, u_n, \nabla u_n)) - n T_n(u_n - \psi)^- &= f_n, \\ u_n &\in W_0^{1,x} L_M(Q), \quad u_n(x, 0) = u_0^n, \end{aligned} \quad (P_n)$$

where  $f_n \rightarrow f$  strongly in  $L^1(Q)$  and  $u_0^n \rightarrow u_0$  strongly in  $L^1(\Omega)$ . Thanks to [3, Theorem 3.1], there exists at least one solution  $u_n$  of problem  $(P_n)$ . By choosing  $T_k(u_n - T_h(u_n))$ ,  $h \geq \|\psi\|_\infty$  as test function in  $(P_n)$ , we get

$$\begin{aligned} & \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - T_h(u_n)) \right\rangle + \int_{h \leq |u_n| \leq h+k} a(u_n, \nabla u_n) \nabla u_n dx dt \\ & - \int_Q n T_n(u_n - \psi)^- T_k(u_n - T_h(u_n)) dx dt = \int_Q f_n T_k(u_n - T_h(u_n)) dx dt. \end{aligned} \quad (3.4)$$

On the one hand, we have

$$\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - T_h(u_n)) \right\rangle = \int_\Omega S_k^h(u_n(T)) dx - \int_\Omega S_k^h(u_n^0) dx, \quad (3.5)$$

where  $S_k^h(s) = \int_0^s T_k(q - T_h(q)) dq$ , and by using the fact that  $\int_\Omega S_k^h(u_n(T)) dx \geq 0$  and  $|\int_\Omega S_k^h(u_n^0)| \leq k \|u_n^0\|_1$ , we get

$$\alpha \int_{h \leq |u_n| \leq h+k} M(|\nabla u_n|) dx dt - \int_Q n T_n(u_n - \psi)^- T_k(u_n - T_h(u_n)) dx dt \leq Ck, \quad \forall n \in \mathbb{N}, \quad (3.6)$$

so that

$$- \int_Q n T_n(u_n - \psi)^- \frac{T_k(u_n - T_h(u_n))}{k} dx dt \leq C. \quad (3.7)$$

Since  $-n T_n(u_n - \psi)^- T_k(u_n - T_h(u_n)) \geq 0$ , for every  $h \geq \|\psi\|_\infty$ , we deduce by Fatou's lemma as  $k \rightarrow 0$  that

$$\int_Q n T_n(u_n - \psi)^- \leq C. \quad (3.8)$$

Using in  $(P_n)$  the test function  $T_k(u_n) \chi(0, \tau)$ , we get for every  $\tau \in (0, T)$ ,

$$\begin{aligned} & \int_\Omega S_k(u_n(\tau)) dx + \int_{Q_\tau} a(x, t, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx dt \\ & + \int_{Q_\tau} n T_n((u_n - \psi)^-) T_k(u_n) dx dt \leq Ck \end{aligned} \quad (3.9)$$

which gives thanks to (3.8)

$$\int_\Omega S_k(u_n(\tau)) dx + \int_{Q_\tau} a(x, t, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx dt \leq Ck, \quad (3.10)$$

$$\int_Q M(|\nabla T_k(u_n)|) dx dt \leq Ck. \quad (3.11)$$

## 8 Parabolic inequalities in $L^1$

On the other hand, by using [6, Lemma 5.7], there exist two positive constants  $\mu_1$  and  $\mu_2$  such that

$$\int_Q M\left(\frac{T_k(u_n)}{\mu_1}\right) dx dt \leq \mu_2 \int_Q M(|\nabla T_k(u_n)|) dx dt \quad (3.12)$$

which implies, by using (3.11), that

$$\text{meas}\{|u_n| > k\} \leq \frac{\mu_2 Ck}{M(k/\mu_1)} \quad (3.13)$$

so that

$$\lim_{k \rightarrow \infty} \text{meas}\{|u_n| > k\} = 0 \quad \text{uniformly with respect to } n. \quad (3.14)$$

Take now a nondecreasing function  $\theta_k \in C^2(\mathbb{R})$  such that  $\theta_k(s) = s$  for  $|s| \leq k/2$  and  $\theta_k(s) = k \text{sign}(s)$  for  $|s| > k$ . By multiplying the approximate equation by  $\theta'_k(u_n)$ , we get

$$\begin{aligned} \frac{\partial \theta_k(u_n)}{\partial t} - \text{div}(a(x, t, u_n, \nabla u_n) \theta'_k(u_n)) + a(x, t, u_n, \nabla u_n) \nabla u_n \theta''_k(u_n) \\ - n T_n(u_n - \psi)^- \theta'_k(u_n) = f_n \theta'_k(u_n), \end{aligned} \quad (3.15)$$

which implies that  $\partial \theta_k(u_n)/\partial t$  is bounded in  $W^{-1,x}L_{\overline{M}}(Q) + L^1(Q)$ . Since  $\theta_k(u_n)$  is bounded in  $W_0^{1,x}L_M(Q)$ , we have by Proposition 2.5 that  $\theta_k(u_n)$  is relatively compact in  $L^1(Q)$  and so that  $u_n \rightarrow u$  a.e. in  $Q$ , and from (3.8) by using Fatou's lemma, we get  $u \geq \psi$  a.e. in  $Q$ . Consequently,

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,x}L_M(Q) \quad (3.16)$$

for the topology  $\sigma(\prod L_M, \prod E_{\overline{M}})$ .

*Step 2.* Almost everywhere convergence of the gradients.

Since  $T_k(u) \in W_0^{1,x}L_M(Q)$ , then there exists a sequence  $(\alpha_j^k) \subset D(Q)$  such that  $\alpha_j^k \rightarrow T_k(u)$  for the modular convergence in  $W_0^{1,x}L_M(Q)$ . In the sequel and throughout the paper,  $\chi_{j,s}$  and  $\chi_s$  will denote, respectively, the characteristic functions of the sets  $Q^{j,s} = \{(x, t) \in \Omega : |\nabla T_k(\alpha_j^k)| \leq s\}$  and  $Q^s = \{(x, t) \in \Omega : |\nabla T_k(u)| \leq s\}$ . For the sake of simplicity, we will write only  $\epsilon(n, j, \mu, s)$  to mean all quantities (possibly different) such that  $\lim_{s \rightarrow \infty} \lim_{\mu \rightarrow \infty} \lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \epsilon(n, j, \mu, s) = 0$ .

Taking now  $T_\eta(u_n - T_k(\alpha_j^k)_\mu)$ ,  $\eta > 0$  as test function in  $(P_n)$ , we get

$$\begin{aligned} \left\langle \frac{\partial u_n}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle + \int_Q a(x, u_n, \nabla u_n) \nabla T_\eta(u_n - T_k(\alpha_j^k)_\mu) \\ - \int_Q n T_n((u_n - \psi)^-) T_\eta(u_n - T_k(\alpha_j^k)_\mu) dx dt \leq C\eta, \end{aligned} \quad (3.17)$$

and by using (3.8), we get

$$\left\langle \frac{\partial u_n}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle + \int_Q a(u_n, \nabla u_n) \nabla T_\eta(u_n - T_k(\alpha_j^k)_\mu) \leq C\eta. \quad (3.18)$$



The first term of the left-hand side of the last equality reads as

$$\begin{aligned} \left\langle \frac{\partial u_n}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle &= \left\langle \frac{\partial u_n}{\partial t} - \frac{\partial T_k(\alpha_j^k)_\mu}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle \\ &+ \left\langle \frac{\partial T_k(\alpha_j^k)_\mu}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle. \end{aligned} \quad (3.19)$$

The second term of the last equality can be written as

$$\begin{aligned} &\left\langle \frac{\partial u_n}{\partial t} - \frac{\partial T_k(\alpha_j^k)_\mu}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle \\ &= \int_\Omega S_\eta(u_n(T) - T_k(\alpha_j^k)_\mu(T)) dx - \int_\Omega S_\eta(u_0^n) dx \geq -\eta \int_\Omega |u_0^n| dx \geq -\eta C, \end{aligned} \quad (3.20)$$

the third term can be written as

$$\left\langle \frac{\partial T_k(\alpha_j^k)_\mu}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle = \mu \int_Q (T_k(\alpha_j^k) - T_k(\alpha_j^k)_\mu) (T_\eta(u_n - T_k(\alpha_j^k)_\mu)), \quad (3.21)$$

thus by letting  $n, j \rightarrow \infty$  and since  $\alpha_j^k \rightarrow T_k(u)$  a.e. in  $Q$  and by using Lebesgue theorem,

$$\begin{aligned} &\int_Q (T_k(\alpha_j^k) - T_k(\alpha_j^k)_\mu) (T_\eta(u_n - T_k(\alpha_j^k)_\mu)) dx dt \\ &= \int_Q (T_k(u) - T_k(u)_\mu) (T_\eta(u - T_k(u)_\mu)) dx dt + \epsilon(n, j). \end{aligned} \quad (3.22)$$

Consequently,

$$\left\langle \frac{\partial u_n}{\partial t}, T_\eta(T_k(u_n - T_k(\alpha_j^k)_\mu)) \right\rangle \geq \epsilon(n, j) - \eta C. \quad (3.23)$$

On the other hand,

$$\begin{aligned} &\int_Q a(u_n, \nabla u_n) \nabla T_\eta(u_n - T_k(\alpha_j^k)_\mu) dx dt \\ &= \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} dx dt \\ &+ \int_{\{|k| < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)_\mu| < \eta\}} a(u_n, \nabla u_n) \nabla u_n dx dt \\ &- \int_{\{|k| < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)_\mu| < \eta\}} a(u_n, \nabla u_n) \nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)_\mu| > s\}} dx dt \end{aligned} \quad (3.24)$$

which implies, by using the fact that  $\int_{\{k < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)|_\mu < \eta\}} a(u_n, \nabla u_n) \nabla u_n dx dt \geq 0$ , that

$$\begin{aligned} & \int_{\{|T_k(u_n) - T_k(\alpha_j^k)|_\mu < \eta\}} a(u_n, \nabla u_n) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s} dx dt \\ & \leq C\eta + \int_{\{k < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)|_\mu < \eta\}} a(u_n, \nabla u_n) \nabla T_k(\alpha_j^k) \chi_{\{| \nabla T_k(\alpha_j^k) | > s\}} dx dt. \end{aligned} \quad (3.25)$$

Since  $a(T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n))$  is bounded in  $(L_{\overline{M}}(Q))^N$ , there exists some  $h_{k+\eta} \in (L_{\overline{M}}(Q))^N$  such that

$$a(T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n)) \rightharpoonup h_{k+\eta} \quad \text{weakly in } (L_{\overline{M}}(Q))^N \text{ for } \sigma\left(\prod L_{\overline{M}}, \prod E_M\right). \quad (3.26)$$

Consequently,

$$\begin{aligned} & \int_{\{k < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)|_\mu < \eta\}} a(u_n, \nabla u_n) \nabla T_k(\alpha_j^k) \chi_{\{| \nabla T_k(\alpha_j^k) | > s\}} dx dt \\ & = \int_{\{k < |u|\} \cap \{|u - T_k(\alpha_j^k)|_\mu < \eta\}} h_{k+\eta} \nabla T_k(\alpha_j^k) \chi_{\{| \nabla T_k(\alpha_j^k) | > s\}} dx dt + \epsilon(n), \end{aligned} \quad (3.27)$$

where we have used the fact that  $\nabla T_k(\alpha_j^k) \chi_{\{k < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)|_\mu < \eta\}}$  tends strongly to  $\nabla T_k(\alpha_j^k) \chi_{\{k < |u|\} \cap \{|u - T_k(\alpha_j^k)|_\mu < \eta\}}$  in  $(E_M(Q))^N$ . Letting  $j \rightarrow \infty$ , we obtain

$$\begin{aligned} & \int_{\{k < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)|_\mu < \eta\}} a(u_n, \nabla u_n) \nabla T_k(\alpha_j^k) \chi_{\{| \nabla T_k(\alpha_j^k) | > s\}} dx dt \\ & = \int_{\{k < |u|\} \cap \{|u - T_k(u)|_\mu < \eta\}} h_{k+\eta} \nabla T_k(u) \chi_{\{| \nabla T_k(u) | > s\}} dx dt + \epsilon(n, j). \end{aligned} \quad (3.28)$$

Thanks to Proposition 2.4, one easily has

$$\begin{aligned} & \int_{\{k < |u|\} \cap \{|u - T_k(u)|_\mu < \eta\}} h_{k+\eta} \nabla T_k(u) \chi_{\{| \nabla T_k(u) | > s\}} dx dt \\ & = \int_{\{k < |u|\} \cap \{|u - T_k(u)|_\mu < \eta\}} h_{k+\eta} \nabla T_k(u) \chi_{\{| \nabla T_k(u) | > s\}} dx dt + \epsilon(\mu) = \epsilon(\mu, s). \end{aligned} \quad (3.29)$$

Hence

$$\int_{\{|T_k(u_n) - T_k(\alpha_j^k)|_\mu < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s} dx dt \leq C\eta + \epsilon(n, j, \mu, s). \quad (3.30)$$

On the other hand, note that

$$\begin{aligned}
& \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} dx dt \\
&= \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} dx dt \\
&+ \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(\alpha_j^k)_\mu \chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}] dx dt.
\end{aligned} \tag{3.31}$$

The latest integral tends to 0 as  $n$  and  $j$  go to  $\infty$ . Indeed, we have that

$$\int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(\alpha_j^k)_\mu \chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}] dx dt \tag{3.32}$$

tends to

$$\int_{\{|T_k(u) - T_k(\alpha_j^k)_\mu| < \eta\}} h_k [\nabla T_k(\alpha_j^k)_\mu \chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}] dx dt \tag{3.33}$$

as  $n \rightarrow \infty$ , since

$$a(T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \quad \text{weakly in } (L_{\overline{M}}(Q))^N \text{ for } \sigma\left(\prod L_{\overline{M}}, \prod E_M\right) \tag{3.34}$$

while  $\nabla T_k(\alpha_j^k)_\mu \chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} \in (E_{\overline{M}}(Q))^N$ . It is obvious that

$$\int_{\{|T_k(u) - T_k(\alpha_j^k)_\mu| < \eta\}} h_k [\nabla T_k(\alpha_j^k)_\mu \chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}] dx dt \tag{3.35}$$

goes to 0 as  $j \rightarrow \infty$  by using Lebesgue theorem. We deduce then that

$$\int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} dx dt \leq C\eta + \epsilon(n, j, \mu, s). \tag{3.36}$$

Let now  $0 < \delta < 1$ . We have

$$\begin{aligned}
& \int_{Q^r} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)]^\delta dx dt \\
& \leq C \text{meas} \left\{ \left| T_k(u_n) - T_k(\alpha_j^k)_\mu \right| > \eta \right\}^\delta \\
& + C \left[ \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\} \cap Q^r} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] \right. \\
& \quad \left. \times [\nabla T_k(u_n) - \nabla T_k(u)] dx dt \right]^\delta.
\end{aligned} \tag{3.37}$$

## 12 Parabolic inequalities in $L^1$

On the other hand, we have for every  $s \geq r$ ,  $r > 0$ ,

$$\begin{aligned}
& \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta \cap Q^r\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] dx dt \\
& \leq \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)\chi_s)] \\
& \quad \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] dx dt \\
& \leq \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(\alpha_j^k)\chi_{j,s})] \\
& \quad \times [\nabla T_k(u_n) - \nabla T_k(\alpha_j^k)\chi_{j,s}] dx dt \\
& + \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(\alpha_j^k)\chi_{j,s} - \nabla T_k(u)\chi_s] dx dt \\
& + \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} [a(T_k(u_n), \nabla T_k(\alpha_j^k)\chi_{j,s}) - a(T_k(u_n), \nabla T_k(u)\chi_s)] \nabla T_k(u_n) dx dt \\
& - \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(\alpha_j^k)\chi_{j,s}) \nabla T_k(\alpha_j^k)\chi_{j,s} dx dt \\
& + \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u)\chi_s) \nabla T_k(u)\chi_s dx dt \\
& \leq I_1(n, j, \mu, s) + I_2(n, j, \mu, s) + I_3(n, j, \mu, s) + I_4(n, j, \mu, s) + I_5(n, j, \mu, s).
\end{aligned} \tag{3.38}$$

We will go to the limit as  $n, j, \mu$ , and  $s \rightarrow \infty$  in the last fifth integrals of the last side. Starting with  $I_1$ , we have

$$\begin{aligned}
I_1(n, j, \mu, s) & \leq C\eta + \epsilon(n, j, \mu, s) \\
& - \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(\alpha_j^k)\chi_{j,s}) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k)\chi_{j,s} dx dt
\end{aligned} \tag{3.39}$$

since

$$\begin{aligned}
& a(T_k(u_n), \nabla T_k(\alpha_j^k)\chi_{j,s}) \chi_{\{|T_k(u) - T_k(\alpha_j^k)_\mu| < \eta\}} \\
& \longrightarrow a(T_k(u), \nabla T_k(\alpha_j^k)\chi_{j,s}) \chi_{\{|T_k(u) - T_k(\alpha_j^k)_\mu| < \eta\}} \quad \text{in } (E_{\overline{M}}(Q))^N,
\end{aligned} \tag{3.40}$$

while

$$\nabla T_k(u_n) \longrightarrow \nabla T_k(u) \quad \text{weakly in } (L_{\overline{M}}(\Omega))^N. \tag{3.41}$$

We deduce then that

$$\begin{aligned} & \int_{\{|T_k(u_n) - T_k(\alpha_j^k)|_\mu < \eta\}} a(T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s}) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s} dx dt \\ &= \int_{\{|T_k(u) - T_k(\alpha_j^k)|_\mu < \eta\}} a(T_k(u), \nabla T_k(\alpha_j^k) \chi_{j,s}) \nabla T_k(u) - \nabla T_k(\alpha_j^k) \chi_{j,s} dx dt + \epsilon(n) \end{aligned} \quad (3.42)$$

which gives by letting  $j \rightarrow \infty$  and using the modular convergence of  $\nabla T_k(\alpha_j^k)$ , that

$$\begin{aligned} & \int_{\{|T_k(u) - T_k(\alpha_j^k)|_\mu < \eta\}} a(T_k(u), \nabla T_k(\alpha_j^k) \chi_{j,s}) \nabla T_k(u) - \nabla T_k(\alpha_j^k) \chi_{j,s} dx dt \\ &= \int_Q a(T_k(u), \nabla T_k(u) \chi_s) \nabla T_k(u) - \nabla T_k(u) \chi_s dx dt + \epsilon(j) = \epsilon(j). \end{aligned} \quad (3.43)$$

Finally,

$$I_1(n, j, \mu, s) \leq C\eta + \epsilon(n, j, \mu, s) + \epsilon(n, j) = \epsilon(n, j, \mu, s, \eta). \quad (3.44)$$

For what concerns  $I_2$ , by letting  $n \rightarrow \infty$ , we have

$$I_2(n, j, \mu, s) = \int_{\{|T_k(u) - T_k(\alpha_j^k)|_\mu < \eta\}} h_k[\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(u) \chi_s] dx dt + \epsilon(n) \quad (3.45)$$

since

$$a(T_k(u_n), \nabla T_k(u_n)) \chi_{\{|T_k(u_n) - T_k(\alpha_j^k)|_\mu < \eta\}} \longrightarrow h_k \quad \text{weakly in } (L_{\overline{M}}(Q))^N \text{ for } \sigma\left(\prod L_{\overline{M}}, \prod E_{\overline{M}}\right), \quad (3.46)$$

while

$$\chi_{\{|T_k(u_n) - T_k(\alpha_j^k)|_\mu < \eta\}} [\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(u) \chi_s] \longrightarrow \chi_{\{|T_k(u) - T_k(\alpha_j^k)|_\mu < \eta\}} \nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(u) \chi_s \quad (3.47)$$

strongly in  $(E_M(Q))^N$ . By letting now  $j \rightarrow \infty$ , and using Lebesgue theorem, we deduce then that

$$I_2(n, j, \mu, s) = \epsilon(n, j). \quad (3.48)$$

Similar tools as above give

$$\begin{aligned} I_3(n, j, \mu, s) &= \epsilon(n, j), \\ I_4(n, j, \mu, s) &= \int_Q a(T_k(u), \nabla T_k(u)) \nabla T_k(u) + \epsilon(n, j, \mu, s), \\ I_5(n, j, \mu, s) &= \int_Q a(T_k(u), \nabla T_k(u)) \nabla T_k(u) + \epsilon(n, j, \mu, s). \end{aligned} \quad (3.49)$$

Combining (3.37)–(3.48) and (3.49), we get

$$\begin{aligned} & \int_{Q_r} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)]^\delta dx dt \\ & \leq C \text{meas} \left\{ \left| T_k(u_n) - T_k(\alpha_j^k) \right|_\mu < \eta \right\}^\delta + C(\epsilon(n, j, s, \mu, \eta))^{1-\delta}, \end{aligned} \quad (3.50)$$

and by passing to the limit sup over  $n, j, \mu, s$ , and,  $\eta$

$$\lim_{n \rightarrow \infty} \int_{Q_r} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)]^\delta dx dt = 0, \quad (3.51)$$

and thus there exists a subsequence also denoted by  $(u_n)$  such that

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } Q^r, \quad (3.52)$$

and since  $r$  is arbitrary, we obtain

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } Q. \quad (3.53)$$

*Step 3. Passage to the limit.*

Let  $\phi \in \mathcal{H}_\psi \cap D(\overline{Q})$ . Choosing now  $T_k(u_n - \phi)\chi_{(0, \tau)}$  as test function in  $(P_n)$ , we get

$$\begin{aligned} & \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \phi) \right\rangle_{Q_\tau} + \int_{Q_\tau} a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - \phi) dx dt \\ & - \int_{Q_\tau} n T_n(u_n - \psi)^- T_k(u_n - \phi) dx dt = \int_{Q_\tau} f_n T_k(u_n - \phi) dx dt \end{aligned} \quad (3.54)$$

which gives, by  $-\int_{Q_\tau} n T_n(u_n - \psi)^- T_k(u_n - \phi) dx dt \geq 0$ ,

$$\begin{aligned} & \int_\Omega S_k(u_n(\tau) - \phi(\tau)) dx + \left\langle \frac{\partial \phi}{\partial t}, T_k(u_n - \phi) \right\rangle_{Q_\tau} \\ & + \int_{Q_\tau} a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - \phi) dx dt \\ & \leq \int_{Q_\tau} f_n T_k(u_n - \phi) dx dt + \int_\Omega S_k(u_n(0) - \phi(0)) dx. \end{aligned} \quad (3.55)$$

We will show that

$$u_n \longrightarrow u \quad \text{in } C([0, T], L^1(\Omega)). \quad (3.56)$$

Since  $T_k(u) \in \mathcal{H}_\psi$ , for every  $k \geq \|\psi\|_\infty$ , there exists a sequence  $(w_j)$  in  $D(\overline{Q}) \cap \mathcal{H}_\phi$  such that

$$w_j \longrightarrow T_k(u) \quad \text{in } W_0^{1,x} L_M(Q) \quad (3.57)$$

for the modular convergence. Choosing now  $\Phi_{j,\mu}^{i,l} = T_l(w_j)_\mu + e^{-\mu t} T_l(\eta_i)$ , with  $\eta_i \geq 0$  converges to  $u_0$  in  $L^1(\Omega)$ , as test function in (3.55),

$$\begin{aligned} & \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Phi_{j,\mu}^{i,l}) \right\rangle_{Q_\tau} + \int_{Q_\tau} a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt \\ & - \int_{Q_\tau} n T_n(u_n - \psi)^- T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt = \int_{Q_\tau} f_n T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt. \end{aligned} \quad (3.58)$$

On the one hand, we have

$$\left\langle (\Phi_{j,\mu}^{i,l})', T_k(u_n - \Phi_{j,\mu}^{i,l}) \right\rangle_{Q_\tau} = \mu \int_{Q_\tau} (T_l(w_j) - \Phi_{j,\mu}^{i,l}) T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt \geq \epsilon(n, j, \mu, l); \quad (3.59)$$

on the other hand, by using the monotonicity of  $a$  and the fact that  $-\int_{Q_\tau} n T_n(u_n - \psi)^- T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt \geq 0$ , we deduce that

$$\begin{aligned} & \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Phi_{j,\mu}^{i,l}) \right\rangle_{Q_\tau} + \int_{Q_\tau} a(x, t, u_n, \nabla \Phi_{j,\mu}^{i,l}) \nabla T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt \\ & \leq \int_{Q_\tau} f_n T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt. \end{aligned} \quad (3.60)$$

Since, for every  $\epsilon > 0$ ,

$$\begin{aligned} & |\chi_{Q_\tau} a(x, t, u_n, \nabla \Phi_{j,\mu}^{i,l}) \nabla T_k(u_n - \Phi_{j,\mu}^{i,l})| \\ & \leq \epsilon \bar{M}(a(x, t, T_{k+\|l\|_\infty}(u_n), \nabla \Phi_{j,\mu}^{i,l})) + M \left( \frac{|\nabla T_k(u_n - \Phi_{j,\mu}^{i,l})|}{\epsilon} \right), \end{aligned} \quad (3.61)$$

we have by using Vitali's theorem

$$\limsup_{l \rightarrow \infty} \limsup_{i \rightarrow \infty} \limsup_{\mu \rightarrow \infty} \limsup_{j \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Phi_{j,\mu}^{i,l}) \right\rangle_{Q_\tau} \leq 0 \quad (3.62)$$

uniformly on  $\tau$ . Therefore, by writing

$$\begin{aligned} \int_{\Omega} S_k(u_n(\tau) - \Phi_{j,\mu}^{i,l}) dx &= \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Phi_{j,\mu}^{i,l}) \right\rangle_{Q_\tau} - \left\langle (\Phi_{j,\mu}^{i,l})', T_k(u_n - \Phi_{j,\mu}^{i,l}) \right\rangle_{Q_\tau} \\ &+ \int_{\Omega} S_k(u_0 - T_l(\eta_i)) dx \end{aligned} \quad (3.63)$$

and using (3.55) and (3.59), we see that

$$\int_{\Omega} S_k(u_n(\tau) - \Phi_{j,\mu}^{i,l}) dx \leq \epsilon(n, j, \mu, i, l) \quad (3.64)$$

which implies, by writing

$$\int_{\Omega} S_k \left( \frac{u_n(\tau) - u_m(\tau)}{2} \right) dx \leq \frac{1}{2} \left( \int_{\Omega} S_k(u_n(\tau) - \Phi_{j,\mu}^{i,l}) dx + \int_{\Omega} S_k(u_m(\tau) - \Phi_{j,\mu}^{i,l}) dx \right), \quad (3.65)$$

that

$$\int_{\Omega} S_k \left( \frac{u_n(\tau) - u_m(\tau)}{2} \right) dx \leq \epsilon_1(n, m), \quad (3.66)$$

we deduce then that

$$\int_{\Omega} |u_n(\tau) - u_m(\tau)| dx \leq \epsilon_2(n, m), \quad \text{not depending on } \tau, \quad (3.67)$$

and thus  $(u_n)$  is a Cauchy sequence in  $C([0, T], L^1(\Omega))$ , and since  $u_n \rightarrow u$ , a.e. in  $Q$ , we deduce that

$$u_n \longrightarrow u \quad \text{in } C([0, T], L^1(\Omega)). \quad (3.68)$$

Go back now to (3.48) and pass to the limit to obtain

$$\begin{aligned} & \int_{\Omega} S_k(u(\tau) - \phi(\tau)) dx + \left\langle \frac{\partial \phi}{\partial t}, T_k(u - \phi) \right\rangle_{Q_{\tau}} + \int_{Q_{\tau}} a(x, t, u, \nabla u) \nabla T_k(u - \phi) dx dt \\ & \leq \int_{Q_{\tau}} f T_k(u - \phi) dx dt + \int_{\Omega} S_k(u(0) - \phi(0)) dx \end{aligned} \quad (3.69)$$

since for every  $v \in \mathcal{H}_{\psi} \cap L^{\infty}(Q)$ , there exists  $v_j \in \mathcal{H}_{\psi} \cap D(\overline{Q})$  such that

$$\begin{aligned} v_j & \longrightarrow v \quad \text{for the modular convergence in } W_0^{1,x} L_M(Q), \\ \frac{\partial v_j}{\partial t} & \longrightarrow \frac{\partial v}{\partial t} \quad \text{for the modular in } W^{-1,x} L_{\overline{M}}(Q) + L^1(Q), \end{aligned} \quad (3.70)$$

we deduce then that

$$\begin{aligned} & \int_{\Omega} S_k(u(\tau) - v(\tau)) dx + \left\langle \frac{\partial v}{\partial t}, T_k(u - v) \right\rangle_{Q_{\tau}} + \int_{Q_{\tau}} a(x, t, u, \nabla u) \nabla T_k(u - v) dx dt \\ & \leq \int_{Q_{\tau}} f T_k(u - v) dx dt + \int_{\Omega} S_k(u(0) - v(0)) dx \end{aligned} \quad (3.71)$$

which completes the proof.  $\square$



*Remark 3.3.* A similar result can be proved when dealing with the right-hand side in  $L^1(Q) + W^{-1,x}E_{\overline{M}}(Q)$  or replacing the assumption (3.1) by the general one:

$$|a(x, t, s, \zeta)| \leq b(|s|)(h(x, t) + \overline{M}^{-1}M(k_4|\zeta|)), \quad (3.72)$$

where  $b: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an increasing continuous function. Indeed, we consider the following approximate problems:

$$\begin{aligned} \frac{\partial u_n}{\partial t} - \operatorname{div}(a(x, t, T_n(u_n), \nabla u_n)) - nT_n(u_n - \psi)^- &= f_n, \\ u_n &\in W_0^{1,x}L_M(Q), \quad u_n(x, 0) = u_0^n, \end{aligned} \quad (P_n)$$

and we conclude by adapting the same steps.

## Acknowledgments

The authors would like to thank the anonymous referees for interesting remarks. This work was supported in part by the Volkswagen Foundation, Grant number I/79315 and in the other part by the Moroccan-Tunisian Project 04/TM/19.

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