

PARABOLIC INEQUALITIES WITH NONSTANDARD GROWTHS AND L^1 DATA

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We prove an existence result for solutions of nonlinear parabolic inequalities with L^1 data in Orlicz spaces.

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1. Introduction

Let Ω be an open bounded subset of \mathbb{R}^N , $N \geq 2$, let Q be the cylinder $\Omega \times (0, T)$ with some given $T > 0$. Consider the following nonlinear parabolic problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + A(u) &= \chi && \text{in } Q, \\ u(x, t) &= 0 && \text{on } \partial\Omega \times (0, T), \\ u(x, 0) &= u_0 && \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $A(u) = -\operatorname{div}(a(x, t, u, \nabla u))$ is a Leray-Lions operator defined on $D(A) \subset W_0^{1,x}L_M(\Omega)$, with M is an N -function, and χ is a given data.

In the variational case (i.e., where $\chi \in W^{-1,x}\bar{E}_M(\Omega)$), it is well known that the solvability of (1.1) is done by Donaldson [2] and Robert [11] when the operator A is monotone, $t^2 \ll M(t)$, and \bar{M} satisfies a Δ_2 condition, and by finally the recent work [3] for the general case.

In the L^1 case, an existence theorem is given in [4]. However, the techniques used in [4] do not allow us to adapt it for parabolic inequalities. It is our purpose in this paper to solve the obstacle problem associated to (1.1) in the case where $\chi \in L^1(Q) + W^{-1,x}\bar{E}_M(Q)$ and without assuming any growth restriction on M . The existence of solutions is proved via a sequence of penalized problems, with solutions u_n . A priori estimates of the truncation of u_n are obtained in some suitable Orlicz space. For the passage to the limit, the

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almost everywhere convergence of ∇u_n is proved via new techniques. As operators models, we can consider slow or fast growth:

$$A(u) = -\operatorname{div} \left((1 + |u|)^2 \nabla u \frac{\log(1 + |\nabla u|)}{|\nabla u|} \right), \quad (1.2)$$

$$A(u) = -\operatorname{div}(\nabla u \exp(|\nabla u|)).$$

For some classical and recent results in the setting of Orlicz spaces dealing with elliptic and parabolic equations, the reader can be referred to [8, 10, 12–14].

2. Preliminaries

2.1. Let $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an N -function, that is, M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \rightarrow 0$ as $t \rightarrow 0$, and $M(t)/t \rightarrow \infty$ as $t \rightarrow \infty$.

Equivalently, M admits the representation $M(t) = \int_0^t a(s)ds$, where $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is non-decreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$, and $a(t)$ tends to ∞ as $t \rightarrow \infty$.

The N -function \bar{M} conjugate to M is defined by $\bar{M}(t) = \int_0^t \bar{a}(s)ds$, where $\bar{a} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is given by $\bar{a}(t) = \sup\{s : a(s) \leq t\}$ (see [1]).

The N -function is said to satisfy the Δ_2 condition if, for some $k > 0$,

$$M(2t) \leq kM(t), \quad \forall t \geq 0, \quad (2.1)$$

when (2.1) holds only for $t \geq$ some $t_0 > 0$, then M is said to satisfy the Δ_2 condition near infinity.

We will extend these N -functions into even functions on all \mathbb{R} .

Let P and Q be two N -functions. $P \ll Q$ means that P grows essentially less rapidly than Q , that is, for each $\epsilon > 0$, $P(t)/Q(\epsilon t) \rightarrow 0$ as $t \rightarrow \infty$. This is the case if and only if $\lim_{t \rightarrow \infty} (Q^{-1}(t))/(P^{-1}(t)) = 0$.

2.2. Let Ω be an open subset of \mathbb{R}^N . The Orlicz class $K_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that

$$\int_{\Omega} M(u(x))dx < +\infty \quad \left(\text{resp., } \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0 \right). \quad (2.2)$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right)dx \leq 1 \right\} \quad (2.3)$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_{\Omega} uv dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.

The space $L_M(\Omega)$ is reflexive if and only if M and \overline{M} satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

2.3. We now turn to the Orlicz-Sobolev space, $W^1 L_M(\Omega)$ (resp., $W^1 E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^{\alpha} u\|_M. \quad (2.4)$$

Thus, $W^1 L_M(\Omega)$ and $W^1 E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1 E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1 E_M(\Omega)$ and the space $W_0^1 L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1 L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1 L_M(\Omega)$ if for some $\lambda > 0$,

$$\int_{\Omega} M\left(\frac{D^{\alpha} u_n - D^{\alpha} u}{\lambda}\right) dx \rightarrow 0, \quad \forall |\alpha| \leq 1. \quad (2.5)$$

This implies convergence for $\sigma(\prod L_M, \prod L_{\overline{M}})$. If M satisfies the Δ_2 condition on \mathbb{R}^+ , then modular convergence coincides with norm convergence.

2.4. Let $W^{-1} L_{\overline{M}}(\Omega)$ (resp., $W^{-1} E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}$ (resp., $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $D(\Omega)$ is dense in $W_0^1 L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\prod L_M, \prod L_{\overline{M}})$ (cf. [6, 7]). Consequently, the action of a distribution in $W^{-1} L_{\overline{M}}(\Omega)$ on an element of $W_0^1 L_M(\Omega)$ is well defined.

2.5. Let Ω be a bounded open subset of \mathbb{R}^N , $T > 0$, and set $Q = \Omega \times (0, T)$. Let M be an N -function. For each $\alpha \in \mathbb{N}^N$, denote by D_x^{α} the distributional derivatives on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Orlicz-Sobolev spaces of order 1 are defined as follows:

$$\begin{aligned} W^{1,x} L_M(Q) &= \{u \in L_M(Q) : D_x^{\alpha} u \in L_M(Q), \forall |\alpha| \leq 1\}, \\ W^{1,x} E_M(Q) &= \{u \in E_M(Q) : D_x^{\alpha} u \in E_M(Q), \forall |\alpha| \leq 1\}. \end{aligned} \quad (2.6)$$

The latest space is a subset of the first one. They are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|=1} \|D_x^{\alpha} u\|_{M,Q}. \quad (2.7)$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product spaces $\prod L_M(Q)$

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which has $N + 1$ copies. We will also consider the weak topologies $\sigma(\prod L_M, \prod E_{\bar{M}})$ and $\sigma(\prod L_M, \prod L_{\bar{M}})$. If $u \in W^{1,x}L_M(Q)$, then the function $t \rightarrow u(t) = u(\cdot, t)$ is defined on $(0, T)$ with values in $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q)$, then $u(t)$ is $W^1E_M(\Omega)$ -valued and is strongly measurable. Furthermore, the following continuous imbedding holds: $W^{1,x}E_M(Q) \subset L^1(0, T; W^1E_M(\Omega))$. The space $W^{1,x}L_M(Q)$ is not in general separable, if $u \in W^{1,x}L_M(Q)$, we cannot conclude that the function $u(t)$ is measurable from $(0, T)$ into $W^1L_M(\Omega)$. However, the scalar function $t \rightarrow \|D_x^\alpha u(t)\|_{M,\Omega}$ is in $L^1(0, T)$ for all $|\alpha| \leq 1$.

2.6. The space $W_0^{1,x}E_M(Q)$ is defined as the (norm) closure in $W^{1,x}E_M(Q)$ of $D(Q)$. We can easily show as in [7] that when Ω has the segment property, then for all $u \in \overline{D(Q)}^{\sigma(\prod L_M, \prod E_{\bar{M}})}$ there exist some $\lambda > 0$ and a sequence $(u_n) \subset D(Q)$ such that for all $|\alpha| \leq 1$, $\int_{\Omega} M((D_x^\alpha u_n - D_x^\alpha u)/\lambda) dx \rightarrow 0$ when $n \rightarrow \infty$. Consequently, $\overline{D(Q)}^{\sigma(\prod L_M, \prod E_{\bar{M}})} = \overline{D(Q)}^{\sigma(\prod L_M, \prod L_{\bar{M}})}$, this space will be denoted by $W_0^{1,x}L_M(Q)$. Furthermore, $W_0^{1,x}E_M(Q) = W_0^{1,x}L_M(Q) \cap \prod E_{\bar{M}}$. Poincaré's inequality also holds in $W_0^{1,x}L_M(Q)$ and then there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$, one has

$$\sum_{|\alpha| \leq 1} \|D_x^\alpha u\|_{M,Q} \leq C \sum_{|\alpha| = 1} \|D_x^\alpha u\|_{M,Q}, \quad (2.8)$$

thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q)$. We have then the following complementary system:

$$\left(\frac{W_0^{1,x}L_M(Q)}{W_0^{1,x}E_M(Q)} \middle| \frac{F}{F_0} \right), \quad (2.9)$$

F being the dual space of $W_0^{1,x}E_M(Q)$. It is also, up to an isomorphism, the quotient of $\prod L_{\bar{M}}$ by the polar set $W_0^{1,x}E_M(Q)^\perp$, and will be denoted by $F = W^{-1,x}L_{\bar{M}}(Q)$ and it is shown that

$$W^{-1,x}L_{\bar{M}}(Q) = \left\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\bar{M}}(Q) \right\}. \quad (2.10)$$

This space will be equipped with the usual quotient norm:

$$\|f\| = \inf \sum_{|\alpha| \leq 1} \|f_\alpha\|_{\bar{M},Q}, \quad (2.11)$$

where the inf is taken on all possible decompositions $f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha$, $f_\alpha \in L_{\bar{M}}(Q)$. The space F_0 is then given by $F_0 = \{f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\bar{M}}(Q)\}$ and is denoted by $F_0 = W^{-1,x}E_{\bar{M}}(Q)$.

Defintion 2.1. We say that $u_n \rightarrow u$ in $W^{-1,x}L_{\bar{M}}(Q) + L^1(Q)$ for the modular convergence if we can write

$$u_n = \sum_{|\alpha| \leq 1} D_x^\alpha u_n^\alpha + u_n^0, \quad u = \sum_{|\alpha| \leq 1} D_x^\alpha u^\alpha + u^0 \quad (2.12)$$

with $u_n^\alpha \rightarrow u^\alpha$ in $L_{\bar{M}}(Q)$ for the modular convergence for all $|\alpha| \leq 1$ and $u_n^0 \rightarrow u^0$ strongly in $L^1(Q)$.

We will give the following approximation theorem which plays a crucial role when proving the existence result of solutions for parabolic inequalities.

THEOREM 2.2. *Let $\phi \in W_0^{1,x}E_M(Q) \cap L^\infty(Q)$ and consider the convex set $\mathcal{K}_\phi = \{v \in W_0^{1,x}L_M(Q) : v \geq \phi \text{ a.e. in } Q\}$. Then for every $u \in \mathcal{K}_\phi \cap L^\infty(Q)$ such that $\partial u/\partial t \in W^{-1,x}L_{\bar{M}}(Q) + L^1(Q)$, there exists $v_j \in \mathcal{K}_\phi \cap D(\bar{Q})$ such that*

$$\begin{aligned} v_j &\longrightarrow u \quad \text{in } W^{1,x}L_M(Q), \\ \frac{\partial v_j}{\partial t} &\longrightarrow \frac{\partial u}{\partial t} \quad \text{in } W^{-1,x}L_{\bar{M}}(Q) + L^1(Q) \end{aligned} \tag{2.13}$$

for the modular convergence.

Proof. It is easily adapted from that given in [4, Theorem 3] and the approximation techniques of [9]. \square

Remark 2.3. The result is still true for $\phi \in W^{1,x}E_M(Q) \cap L^\infty(Q)$, when Ω is more regular (see [9]).

In order to deal with the time derivative, we introduce a time mollification of a function $v \in L_M(Q)$. Thus, we define, for all $\mu > 0$ and all $(x, t) \in Q$,

$$v_\mu(x, t) = \mu \int_{-\infty}^t \tilde{v}(x, s) \exp(\mu(s - t)) ds, \tag{2.14}$$

where $\tilde{v}(x, s) = v(x, s)\chi_{(0, T)}(s)$ is the zero extension of v . The following proposition is fundamental in the sequel.

PROPOSITION 2.4 [5]. *If $v \in L_M(Q)$, then v_μ is measurable in Q , $\partial v_\mu/\partial t = \mu(v - v_\mu)$ and*

$$\int_Q M(v_\mu) dx dt \leq \int_Q M(v) dx dt. \tag{2.15}$$

Recall now the following compactness result which is proved in [5].

PROPOSITION 2.5. *Assume that $(u_n)_n$ is a bounded sequence in $W_0^1L_M(Q)$ such that $\partial u_n/\partial t$ is bounded in $W^{-1,x}L_{\bar{M}}(Q) + L^1(Q)$, then u_n is relatively compact in $L^1(Q)$.*

3. The main result

Let Ω be an open bounded subset of \mathbb{R}^N , $N \geq 2$, with the segment property. Let P and M be two N -functions such that $P \ll M$. Consider now the operator $A : D(A) \subset W_0^{1,x}L_M(Q) \rightarrow W^{-1}L_{\bar{M}}(Q)$ in divergence form $A(u) = -\operatorname{div}(a(x, t, u, \nabla u))$, where $a : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$ and for all $\zeta, \zeta' \in \mathbb{R}^N$,

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$(\zeta \neq \zeta')$ and all $s, t \in \mathbb{R}$:

$$\begin{aligned} |a(x, t, s, \zeta)| &\leq h(x, t) + k_1 \bar{P}^{-1} M(k_2 |s|) + k_3 \bar{M}^{-1} M(k_4 |\zeta|), \\ (a(x, t, s, \zeta) - a(x, t, s, \zeta'))(\zeta - \zeta') &> 0, \\ a(x, t, s, \zeta)\zeta &\geq \alpha M(|\zeta|) - d(x, t), \end{aligned} \quad (3.1)$$

with $d \in L^1(Q)$, $\alpha, k_1, k_2, k_3, k_4 > 0$, and $h \in E_{\bar{M}}(Q)$. Let

$$\psi \in W_0^1 E_M(\Omega) \cap L^\infty(\Omega). \quad (3.2)$$

Finally, consider

$$f \in L^1(Q). \quad (3.3)$$

We define for all $t \in \mathbb{R}$, $k \geq 0$, $T_k(t) = \max(-k, \min(k, t))$, and $S_k(t) = \int_0^t T_k(\eta) d\eta$.

We will prove the following existence theorem.

THEOREM 3.1. *Let $u_0 \in L^1(\Omega)$ such that $u_0 \geq 0$. Assume that (3.1)–(3.3) hold true. Then there exists at least one solution $u \in C([0, T]; L^1(\Omega))$ such that $u(x, 0) = u_0$ a.e. and for all $\tau \in]0, T]$,*

$$\begin{aligned} u &\geq \psi \quad \text{a.e. in } Q, \\ T_k(u) &\in W_0^{1,x} L_M(Q), \\ \int_{\Omega} S_k(u(\tau) - v(\tau)) dx + \left\langle \frac{\partial v}{\partial t}, T_k(u - v) \right\rangle_{Q_\tau} &+ \int_{Q_\tau} a(x, t, u, \nabla u) \nabla T_k(u - v) dx dt \\ &\leq \int_{Q_\tau} f T_k(u - v) dx dt + \int_{\Omega} S_k(u_0 - v(x, 0)) dx, \\ \forall k > 0 \text{ and } \forall v \in \mathcal{K}_\psi \cap L^\infty(Q) \text{ such that } \frac{\partial v}{\partial t} &\in W^{-1,x} L_{\bar{M}}(Q) + L^1(Q), \end{aligned} \quad (p_\psi)$$

where $Q_\tau = \Omega \times]0, \tau[$.

Remark 3.2. Since $\{v \in \mathcal{K}_\psi \cap L^\infty(Q) : \partial v / \partial t \in W^{-1,x} L_{\bar{M}}(Q) + L^1(Q)\} \subset C([0, T], L^1(\Omega))$, (see [4]), the first and the latest terms of problem (p_ψ) are well defined.

Proof

Step 1. A priori estimates.

For the sake of simplicity, we assume that $d(x, t) = 0$.

Consider the approximate equations

$$\begin{aligned} \frac{\partial u_n}{\partial t} - \operatorname{div}(a(x, t, u_n, \nabla u_n)) - n T_n(u_n - \psi)^- &= f_n, \\ u_n &\in W_0^{1,x} L_M(Q), \quad u_n(x, 0) = u_0^n, \end{aligned} \quad (P_n)$$

where $f_n \rightarrow f$ strongly in $L^1(Q)$ and $u_0^n \rightarrow u_0$ strongly in $L^1(\Omega)$. Thanks to [3, Theorem 3.1], there exists at least one solution u_n of problem (P_n) . By choosing $T_k(u_n - T_h(u_n)), h \geq \|\psi\|_\infty$ as test function in (P_n) , we get

$$\begin{aligned} & \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - T_h(u_n)) \right\rangle + \int_{h \leq |u_n| \leq h+k} a(u_n, \nabla u_n) \nabla u_n dx dt \\ & - \int_Q n T_n(u_n - \psi)^- T_k(u_n - T_h(u_n)) dx dt = \int_Q f_n T_k(u_n - T_h(u_n)) dx dt. \end{aligned} \quad (3.4)$$

On the one hand, we have

$$\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - T_h(u_n)) \right\rangle = \int_\Omega S_k^h(u_n(T)) dx - \int_\Omega S_k^h(u_n^0) dx, \quad (3.5)$$

where $S_k^h(s) = \int_0^s T_k(q - T_h(q)) dq$, and by using the fact that $\int_\Omega S_k^h(u_n(T)) dx \geq 0$ and $|\int_\Omega S_k^h(u_n^0)| \leq k \|u_n^0\|_1$, we get

$$\alpha \int_{h \leq |u_n| \leq h+k} M(|\nabla u_n|) dx dt - \int_Q n T_n(u_n - \psi)^- T_k(u_n - T_h(u_n)) dx dt \leq Ck, \quad \forall n \in \mathbb{N}, \quad (3.6)$$

so that

$$- \int_Q n T_n(u_n - \psi)^- \frac{T_k(u_n - T_h(u_n))}{k} dx dt \leq C. \quad (3.7)$$

Since $-n T_n(u_n - \psi)^- T_k(u_n - T_h(u_n)) \geq 0$, for every $h \geq \|\psi\|_\infty$, we deduce by Fatou's lemma as $k \rightarrow 0$ that

$$\int_Q n T_n(u_n - \psi)^- \leq C. \quad (3.8)$$

Using in (P_n) the test function $T_k(u_n)\chi(0, \tau)$, we get for every $\tau \in (0, T)$,

$$\begin{aligned} & \int_\Omega S_k(u_n(\tau)) dx + \int_{Q_\tau} a(x, t, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx dt \\ & + \int_{Q_\tau} n T_n((u_n - \psi)^-) T_k(u_n) dx dt \leq Ck \end{aligned} \quad (3.9)$$

which gives thanks to (3.8)

$$\int_\Omega S_k(u_n(\tau)) dx + \int_{Q_\tau} a(x, t, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx dt \leq Ck, \quad (3.10)$$

$$\int_Q M(|\nabla T_k(u_n)|) dx dt \leq Ck. \quad (3.11)$$

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On the other hand, by using [6, Lemma 5.7], there exist two positive constants μ_1 and μ_2 such that

$$\int_Q M\left(\frac{T_k(u_n)}{\mu_1}\right) dx dt \leq \mu_2 \int_Q M(|\nabla T_k(u_n)|) dx dt \quad (3.12)$$

which implies, by using (3.11), that

$$\text{meas}\{ |u_n| > k\} \leq \frac{\mu_2 C k}{M(k/\mu_1)} \quad (3.13)$$

so that

$$\lim_{k \rightarrow \infty} \text{meas}\{ |u_n| > k\} = 0 \quad \text{uniformly with respect to } n. \quad (3.14)$$

Take now a nondecreasing function $\theta_k \in C^2(\mathbb{R})$ such that $\theta_k(s) = s$ for $|s| \leq k/2$ and $\theta_k(s) = k \text{sign}(s)$ for $|s| > k$. By multiplying the approximate equation by $\theta'_k(u_n)$, we get

$$\begin{aligned} \frac{\partial \theta_k(u_n)}{\partial t} - \text{div}(a(x, t, u_n, \nabla u_n) \theta'(u_n)) + a(x, t, u_n, \nabla u_n) \nabla u_n \theta''(u_n) \\ - n T_n(u_n - \psi)^- \theta'_k(u_n) = f_n \theta'_k(u_n), \end{aligned} \quad (3.15)$$

which implies that $\partial \theta_k(u_n)/\partial t$ is bounded in $W^{-1,x} L_{\bar{M}}(Q) + L^1(Q)$. Since $\theta_k(u_n)$ is bounded in $W_0^{1,x} L_M(Q)$, we have by Proposition 2.5 that $\theta_k(u_n)$ is relatively compact in $L^1(Q)$ and so that $u_n \rightarrow u$ a.e. in Q , and from (3.8) by using Fatou's lemma, we get $u \geq \psi$ a.e. in Q . Consequently,

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,x} L_M(Q) \quad (3.16)$$

for the topology $\sigma(\prod L_M, \prod E_{\bar{M}})$.

Step 2. Almost everywhere convergence of the gradients.

Since $T_k(u) \in W_0^{1,x} L_M(Q)$, then there exists a sequence $(\alpha_j^k) \subset D(Q)$ such that $\alpha_j^k \rightarrow T_k(u)$ for the modular convergence in $W_0^{1,x} L_M(Q)$. In the sequel and throughout the paper, $\chi_{j,s}$ and χ_s will denote, respectively, the characteristic functions of the sets $Q^{j,s} = \{(x, t) \in \Omega : |\nabla T_k(\alpha_j^k)| \leq s\}$ and $Q^s = \{(x, t) \in \Omega : |\nabla T_k(u)| \leq s\}$. For the sake of simplicity, we will write only $\epsilon(n, j, \mu, s)$ to mean all quantities (possibly different) such that $\lim_{s \rightarrow \infty} \lim_{\mu \rightarrow \infty} \lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \epsilon(n, j, \mu, s) = 0$.

Taking now $T_\eta(u_n - T_k(\alpha_j^k)_\mu)$, $\eta > 0$ as test function in (P_n) , we get

$$\begin{aligned} \left\langle \frac{\partial u_n}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle + \int_Q a(x, u_n, \nabla u_n) \nabla T_\eta(u_n - T_k(\alpha_j^k)_\mu) \\ - \int_Q n T_n((u_n - \psi)^-) T_\eta(u_n - T_k(\alpha_j^k)_\mu) dx dt \leq C\eta, \end{aligned} \quad (3.17)$$

and by using (3.8), we get

$$\left\langle \frac{\partial u_n}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle + \int_Q a(u_n, \nabla u_n) \nabla T_\eta(u_n - T_k(\alpha_j^k)_\mu) \leq C\eta. \quad (3.18)$$

The first term of the left-hand side of the last equality reads as

$$\begin{aligned} \left\langle \frac{\partial u_n}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle &= \left\langle \frac{\partial u_n}{\partial t} - \frac{\partial T_k(\alpha_j^k)_\mu}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle \\ &\quad + \left\langle \frac{\partial T_k(\alpha_j^k)_\mu}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle. \end{aligned} \quad (3.19)$$

The second term of the last equality can be written as

$$\begin{aligned} &\left\langle \frac{\partial u_n}{\partial t} - \frac{\partial T_k(\alpha_j^k)_\mu}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle \\ &= \int_{\Omega} S_\eta(u_n(T) - T_k(\alpha_j^k)_\mu(T)) dx - \int_{\Omega} S_\eta(u_0^n) dx \geq -\eta \int_{\Omega} |u_0^n| dx \geq -\eta C, \end{aligned} \quad (3.20)$$

the third term can be written as

$$\left\langle \frac{\partial T_k(\alpha_j^k)_\mu}{\partial t}, T_\eta(u_n - T_k(\alpha_j^k)_\mu) \right\rangle = \mu \int_Q (T_k(\alpha_j^k) - T_k(\alpha_j^k)_\mu) (T_\eta(u_n - T_k(\alpha_j^k)_\mu)) dx dt, \quad (3.21)$$

thus by letting $n, j \rightarrow \infty$ and since $\alpha_j^k \rightarrow T_k(u)$ a.e. in Q and by using Lebesgue theorem,

$$\begin{aligned} &\int_Q (T_k(\alpha_j^k) - T_k(\alpha_j^k)_\mu) (T_\eta(u_n - T_k(\alpha_j^k)_\mu)) dx dt \\ &= \int_Q (T_k(u) - T_k(u)_\mu) (T_\eta(u - T_k(u)_\mu)) dx dt + \epsilon(n, j). \end{aligned} \quad (3.22)$$

Consequently,

$$\left\langle \frac{\partial u_n}{\partial t}, T_\eta(T_k(u_n - T_k(\alpha_j^k)_\mu)) \right\rangle \geq \epsilon(n, j) - \eta C. \quad (3.23)$$

On the other hand,

$$\begin{aligned} &\int_Q a(u_n, \nabla u_n) \nabla T_\eta(u_n - T_k(\alpha_j^k)_\mu) dx dt \\ &= \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} dx dt \\ &\quad + \int_{\{k < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)_\mu| < \eta\}} a(u_n, \nabla u_n) \nabla u_n dx dt \\ &\quad - \int_{\{k < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)_\mu| < \eta\}} a(u_n, \nabla u_n) \nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)| > s\}} dx dt \end{aligned} \quad (3.24)$$

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which implies, by using the fact that $\int_{\{k < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)_\mu| < \eta\}} a(u_n, \nabla u_n) \nabla u_n dx dt \geq 0$, that

$$\begin{aligned} & \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(u_n, \nabla u_n) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} dx dt \\ & \leq C\eta + \int_{\{k < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)_\mu| < \eta\}} a(u_n, \nabla u_n) \nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)| > s\}} dx dt. \end{aligned} \quad (3.25)$$

Since $a(T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\bar{M}}(Q))^N$, there exists some $h_{k+\eta} \in (L_{\bar{M}}(Q))^N$ such that

$$a(T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n)) \rightharpoonup h_{k+\eta} \quad \text{weakly in } (L_{\bar{M}}(Q))^N \text{ for } \sigma\left(\prod L_{\bar{M}}, \prod E_M\right). \quad (3.26)$$

Consequently,

$$\begin{aligned} & \int_{\{k < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)_\mu| < \eta\}} a(u_n, \nabla u_n) \nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)| > s\}} dx dt \\ & = \int_{\{k < |u|\} \cap \{|u - T_k(\alpha_j^k)_\mu| < \eta\}} h_{k+\eta} \nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)| > s\}} dx dt + \epsilon(n), \end{aligned} \quad (3.27)$$

where we have used the fact that $\nabla T_k(\alpha_j^k)_\mu \chi_{\{k < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)_\mu| < \eta\}}$ tends strongly to $\nabla T_k(\alpha_j^k)_\mu \chi_{\{k < |u|\} \cap \{|u - T_k(\alpha_j^k)_\mu| < \eta\}}$ in $(E_M(Q))^N$. Letting $j \rightarrow \infty$, we obtain

$$\begin{aligned} & \int_{\{k < |u_n|\} \cap \{|u_n - T_k(\alpha_j^k)_\mu| < \eta\}} a(u_n, \nabla u_n) \nabla T_k(\alpha_j^k)_\mu \chi_{\{|\nabla T_k(\alpha_j^k)| > s\}} dx dt \\ & = \int_{\{k < |u|\} \cap \{|u - T_k(u)_\mu| < \eta\}} h_{k+\eta} \nabla T_k(u)_\mu \chi_{\{|\nabla T_k(u)| > s\}} dx dt + \epsilon(n, j). \end{aligned} \quad (3.28)$$

Thanks to Proposition 2.4, one easily has

$$\begin{aligned} & \int_{\{k < |u|\} \cap \{|u - T_k(u)_\mu| < \eta\}} h_{k+\eta} \nabla T_k(u)_\mu \chi_{\{|\nabla T_k(u)| > s\}} dx dt \\ & = \int_{\{k < |u|\} \cap \{|u - T_k(u)| < \eta\}} h_{k+\eta} \nabla T_k(u) \chi_{\{|\nabla T_k(u)| > s\}} dx dt + \epsilon(\mu, s) = \epsilon(\mu, s). \end{aligned} \quad (3.29)$$

Hence

$$\int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} dx dt \leq C\eta + \epsilon(n, j, \mu, s). \quad (3.30)$$

On the other hand, note that

$$\begin{aligned}
& \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} dx dt \\
&= \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s} dx dt \\
&+ \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}] dx dt.
\end{aligned} \tag{3.31}$$

The latest integral tends to 0 as n and j go to ∞ . Indeed, we have that

$$\int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}] dx dt \tag{3.32}$$

tends to

$$\int_{\{|T_k(u) - T_k(\alpha_j^k)_\mu| < \eta\}} h_k [\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}] dx dt \tag{3.33}$$

as $n \rightarrow \infty$, since

$$a(T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \quad \text{weakly in } (L_{\overline{M}}(Q))^N \text{ for } \sigma\left(\prod L_{\overline{M}}, \prod E_M\right) \tag{3.34}$$

while $\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s} \in (E_{\overline{M}}(Q))^N$. It is obvious that

$$\int_{\{|T_k(u) - T_k(\alpha_j^k)_\mu| < \eta\}} h_k [\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(\alpha_j^k)_\mu \chi_{j,s}] dx dt \tag{3.35}$$

goes to 0 as $j \rightarrow \infty$ by using Lebesgue theorem. We deduce then that

$$\int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s} dx dt \leq C\eta + \epsilon(n, j, \mu, s). \tag{3.36}$$

Let now $0 < \delta < 1$. We have

$$\begin{aligned}
& \int_{Q_r} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)]^\delta dx dt \\
& \leq C \text{meas} \left\{ |T_k(u_n) - T_k(\alpha_j^k)_\mu| > \eta \right\}^\delta \\
& + C \left[\int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\} \cap Q_r} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] \right. \\
& \quad \times [\nabla T_k(u_n) - \nabla T_k(u)]^\delta dx dt \left. \right].
\end{aligned} \tag{3.37}$$

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On the other hand, we have for every $s \geq r, r > 0$,

$$\begin{aligned}
& \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta \cap Q^r\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] dx dt \\
& \leq \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u) \chi_s)] \\
& \quad \times [\nabla T_k(u_n) - \nabla T_k(u) \chi_s] dx dt \\
& \leq \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s})] \\
& \quad \times [\nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s}] dx dt \\
& \quad + \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(u) \chi_s] dx dt \\
& \quad + \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} [a(T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s}) - a(T_k(u_n), \nabla T_k(u) \chi_s)] \nabla T_k(u_n) dx dt \\
& \quad - \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s}) \nabla T_k(\alpha_j^k) \chi_{j,s} dx dt \\
& \quad + \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(u) \chi_s) \nabla T_k(u) \chi_s dx dt \\
& \leq I_1(n, j, \mu, s) + I_2(n, j, \mu, s) + I_3(n, j, \mu, s) + I_4(n, j, \mu, s) + I_5(n, j, \mu, s).
\end{aligned} \tag{3.38}$$

We will go to the limit as n, j, μ , and $s \rightarrow \infty$ in the last fifth integrals of the last side. Starting with I_1 , we have

$$\begin{aligned}
I_1(n, j, \mu, s) & \leq C\eta + \epsilon(n, j, \mu, s) \\
& - \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s}) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s} dx dt
\end{aligned} \tag{3.39}$$

since

$$\begin{aligned}
& a(T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s}) \chi_{\{|T_k(u) - T_k(\alpha_j^k)_\mu| < \eta\}} \\
& \longrightarrow a(T_k(u), \nabla T_k(\alpha_j^k) \chi_{j,s}) \chi_{\{|T_k(u) - T_k(\alpha_j^k)_\mu| < \eta\}} \quad \text{in } (E_{\bar{M}}(Q))^N,
\end{aligned} \tag{3.40}$$

while

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \quad \text{weakly in } (L_{\bar{M}}(\Omega))^N. \tag{3.41}$$

We deduce then that

$$\begin{aligned} & \int_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u_n), \nabla T_k(\alpha_j^k) \chi_{j,s}) \nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \chi_{j,s} dx dt \\ &= \int_{\{|T_k(u) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u), \nabla T_k(\alpha_j^k) \chi_{j,s}) \nabla T_k(u) - \nabla T_k(\alpha_j^k) \chi_{j,s} dx dt + \epsilon(n) \end{aligned} \quad (3.42)$$

which gives by letting $j \rightarrow \infty$ and using the modular convergence of $\nabla T_k(\alpha_j^k)$, that

$$\begin{aligned} & \int_{\{|T_k(u) - T_k(\alpha_j^k)_\mu| < \eta\}} a(T_k(u), \nabla T_k(\alpha_j^k) \chi_{j,s}) \nabla T_k(u) - \nabla T_k(\alpha_j^k) \chi_{j,s} dx dt \\ &= \int_Q a(T_k(u), \nabla T_k(u) \chi_s) \nabla T_k(u) - \nabla T_k(u) \chi_s dx dt + \epsilon(j) = \epsilon(j). \end{aligned} \quad (3.43)$$

Finally,

$$I_1(n, j, \mu, s) \leq C\eta + \epsilon(n, j, \mu, s) + \epsilon(n, j) = \epsilon(n, j, \mu, s, \eta). \quad (3.44)$$

For what concerns I_2 , by letting $n \rightarrow \infty$, we have

$$I_2(n, j, \mu, s) = \int_{\{|T_k(u) - T_k(\alpha_j^k)_\mu| < \eta\}} h_k [\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(u) \chi_s] dx dt + \epsilon(n) \quad (3.45)$$

since

$$a(T_k(u_n), \nabla T_k(u_n)) \chi_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} \rightharpoonup h_k \quad \text{weakly in } (L_{\overline{M}}(Q))^N \text{ for } \sigma\left(\prod L_{\overline{M}}, \prod E_{\overline{M}}\right), \quad (3.46)$$

while

$$\chi_{\{|T_k(u_n) - T_k(\alpha_j^k)_\mu| < \eta\}} [\nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(u) \chi_s] \rightharpoonup \chi_{\{|T_k(u) - T_k(\alpha_j^k)_\mu| < \eta\}} \nabla T_k(\alpha_j^k) \chi_{j,s} - \nabla T_k(u) \chi_s \quad (3.47)$$

strongly in $(E_M(Q))^N$. By letting now $j \rightarrow \infty$, and using Lebesgue theorem, we deduce then that

$$I_2(n, j, \mu, s) = \epsilon(n, j). \quad (3.48)$$

Similar tools as above give

$$\begin{aligned} I_3(n, j, \mu, s) &= \epsilon(n, j), \\ I_4(n, j, \mu, s) &= \int_Q a(T_k(u), \nabla T_k(u)) \nabla T_k(u) + \epsilon(n, j, \mu, s), \\ I_5(n, j, \mu, s) &= \int_Q a(T_k(u), \nabla T_k(u)) \nabla T_k(u) + \epsilon(n, j, \mu, s). \end{aligned} \quad (3.49)$$

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Combining (3.37)–(3.48) and (3.49), we get

$$\begin{aligned} & \int_{Q_r} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)]^\delta dx dt \\ & \leq C \text{meas} \left\{ \left| T_k(u_n) - T_k(\alpha_j^k)_\mu \right| < \eta \right\}^\delta + C(\epsilon(n, j, s, \mu, \eta))^{1-\delta}, \end{aligned} \quad (3.50)$$

and by passing to the limit sup over n, j, μ, s , and, η

$$\lim_{n \rightarrow \infty} \int_{Q_r} [a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)]^\delta dx dt = 0, \quad (3.51)$$

and thus there exists a subsequence also denoted by (u_n) such that

$$\nabla u_n \rightarrow \nabla u \quad \text{a.e. in } Q^r, \quad (3.52)$$

and since r is arbitrary, we obtain

$$\nabla u_n \rightarrow \nabla u \quad \text{a.e. in } Q. \quad (3.53)$$

Step 3. Passage to the limit.

Let $\phi \in \mathcal{H}_\psi \cap D(\bar{Q})$. Choosing now $T_k(u_n - \phi)\chi_{(0, \tau)}$ as test function in (P_n) , we get

$$\begin{aligned} & \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \phi) \right\rangle_{Q_\tau} + \int_{Q_\tau} a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - \phi) dx dt \\ & - \int_{Q_\tau} n T_n(u_n - \psi)^- T_k(u_n - \phi) dx dt = \int_{Q_\tau} f_n T_k(u_n - \phi) dx dt \end{aligned} \quad (3.54)$$

which gives, by $-\int_{Q_\tau} n T_n(u_n - \psi)^- T_k(u_n - \phi) dx dt \geq 0$,

$$\begin{aligned} & \int_{\Omega} S_k(u_n(\tau) - \phi(\tau)) dx + \left\langle \frac{\partial \phi}{\partial t}, T_k(u_n - \phi) \right\rangle_{Q_\tau} \\ & + \int_{Q_\tau} a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - \phi) dx dt \\ & \leq \int_{Q_\tau} f_n T_k(u_n - \phi) dx dt + \int_{\Omega} S_k(u_n(0) - \phi(0)) dx. \end{aligned} \quad (3.55)$$

We will show that

$$u_n \rightarrow u \quad \text{in } C([0, T], L^1(\Omega)). \quad (3.56)$$

Since $T_k(u) \in \mathcal{H}_\psi$, for every $k \geq \|\psi\|_\infty$, there exists a sequence (w_j) in $D(\bar{Q}) \cap \mathcal{H}_\phi$ such that

$$w_j \rightarrow T_k(u) \quad \text{in } W_0^{1,x} L_M(Q) \quad (3.57)$$

for the modular convergence. Choosing now $\Phi_{j,\mu}^{i,l} = T_l(w_j)_\mu + e^{-\mu t} T_l(\eta_i)$, with $\eta_i \geq 0$ converges to u_0 in $L^1(\Omega)$, as test function in (3.55),

$$\begin{aligned} & \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Phi_{j,\mu}^{i,l}) \right\rangle_{Q_\tau} + \int_{Q_\tau} a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt \\ & - \int_{Q_\tau} n T_n(u_n - \psi)^- T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt = \int_{Q_\tau} f_n T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt. \end{aligned} \quad (3.58)$$

On the one hand, we have

$$\left\langle (\Phi_{j,\mu}^{i,l})', T_k(u_n - \Phi_{j,\mu}^{i,l}) \right\rangle_{Q_\tau} = \mu \int_{Q_\tau} (T_l(w_j) - \Phi_{j,\mu}^{i,l}) T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt \geq \epsilon(n, j, \mu, l); \quad (3.59)$$

on the other hand, by using the monotonicity of a and the fact that $-\int_{Q_\tau} n T_n(u_n - \psi)^- T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt \geq 0$, we deduce that

$$\begin{aligned} & \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Phi_{j,\mu}^{i,l}) \right\rangle_{Q_\tau} + \int_{Q_\tau} a(x, t, u_n, \nabla \Phi_{j,\mu}^{i,l}) \nabla T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt \\ & \leq \int_{Q_\tau} f_n T_k(u_n - \Phi_{j,\mu}^{i,l}) dx dt. \end{aligned} \quad (3.60)$$

Since, for every $\epsilon > 0$,

$$\begin{aligned} & |\chi_{Q_\tau} a(x, t, u_n, \nabla \Phi_{j,\mu}^{i,l}) \nabla T_k(u_n - \Phi_{j,\mu}^{i,l})| \\ & \leq \epsilon \bar{M}(a(x, t, T_{k+\|l\|_\infty}(u_n), \nabla \Phi_{j,\mu}^{i,l})) + M \left(\frac{|\nabla T_k(u_n - \Phi_{j,\mu}^{i,l})|}{\epsilon} \right), \end{aligned} \quad (3.61)$$

we have by using Vitali's theorem

$$\limsup_{l \rightarrow \infty} \limsup_{i \rightarrow \infty} \limsup_{\mu \rightarrow \infty} \limsup_{j \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Phi_{j,\mu}^{i,l}) \right\rangle_{Q_\tau} \leq 0 \quad (3.62)$$

uniformly on τ . Therefore, by writing

$$\begin{aligned} \int_{\Omega} S_k(u_n(\tau) - \Phi_{j,\mu}^{i,l}) dx &= \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \Phi_{j,\mu}^{i,l}) \right\rangle_{Q_\tau} - \left\langle (\Phi_{j,\mu}^{i,l})', T_k(u_n - \Phi_{j,\mu}^{i,l}) \right\rangle_{Q_\tau} \\ &+ \int_{\Omega} S_k(u_0 - T_l(\eta_i)) dx \end{aligned} \quad (3.63)$$

and using (3.55) and (3.59), we see that

$$\int_{\Omega} S_k(u_n(\tau) - \Phi_{j,\mu}^{i,l}) dx \leq \epsilon(n, j, \mu, i, l) \quad (3.64)$$

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which implies, by writing

$$\int_{\Omega} S_k \left(\frac{u_n(\tau) - u_m(\tau)}{2} \right) dx \leq \frac{1}{2} \left(\int_{\Omega} S_k (u_n(\tau) - \Phi_{j,\mu}^{i,l}) dx + \int_{\Omega} S_k (u_m(\tau) - \Phi_{j,\mu}^{i,l}) dx \right), \quad (3.65)$$

that

$$\int_{\Omega} S_k \left(\frac{u_n(\tau) - u_m(\tau)}{2} \right) dx \leq \epsilon_1(n, m), \quad (3.66)$$

we deduce then that

$$\int_{\Omega} |u_n(\tau) - u_m(\tau)| dx \leq \epsilon_2(n, m), \quad \text{not depending on } \tau, \quad (3.67)$$

and thus (u_n) is a Cauchy sequence in $C([0, T], L^1(\Omega))$, and since $u_n \rightarrow u$, a.e. in Q , we deduce that

$$u_n \rightarrow u \quad \text{in } C([0, T], L^1(\Omega)). \quad (3.68)$$

Go back now to (3.48) and pass to the limit to obtain

$$\begin{aligned} & \int_{\Omega} S_k (u(\tau) - \phi(\tau)) dx + \left\langle \frac{\partial \phi}{\partial t}, T_k(u - \phi) \right\rangle_{Q_\tau} + \int_{Q_\tau} a(x, t, u, \nabla u) \nabla T_k(u - \phi) dx dt \\ & \leq \int_{Q_\tau} f T_k(u - \phi) dx dt + \int_{\Omega} S_k (u(0) - \phi(0)) dx \end{aligned} \quad (3.69)$$

since for every $v \in \mathcal{K}_\psi \cap L^\infty(Q)$, there exists $v_j \in \mathcal{K}_\psi \cap D(\overline{Q})$ such that

$$\begin{aligned} v_j & \rightarrow v \quad \text{for the modular convergence in } W_0^{1,x} L_M(Q), \\ \frac{\partial v_j}{\partial t} & \rightarrow \frac{\partial v}{\partial t} \quad \text{for the modular in } W^{-1,x} L_{\overline{M}}(Q) + L^1(Q), \end{aligned} \quad (3.70)$$

we deduce then that

$$\begin{aligned} & \int_{\Omega} S_k (u(\tau) - v(\tau)) dx + \left\langle \frac{\partial v}{\partial t}, T_k(u - v) \right\rangle_{Q_\tau} + \int_{Q_\tau} a(x, t, u, \nabla u) \nabla T_k(u - v) dx dt \\ & \leq \int_{Q_\tau} f T_k(u - v) dx dt + \int_{\Omega} S_k (u(0) - v(0)) dx \end{aligned} \quad (3.71)$$

which completes the proof. \square

Remark 3.3. A similar result can be proved when dealing with the right-hand side in $L^1(Q) + W^{-1,x}E_{\bar{M}}(Q)$ or replacing the assumption (3.1) by the general one:

$$|a(x, t, s, \zeta)| \leq b(|s|)(h(x, t) + \bar{M}^{-1}M(k_4|\zeta|)), \quad (3.72)$$

where $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is an increasing continuous function. Indeed, we consider the following approximate problems:

$$\begin{aligned} \frac{\partial u_n}{\partial t} - \operatorname{div}(a(x, t, T_n(u_n), \nabla u_n)) - nT_n(u_n - \psi)^- &= f_n, \\ u_n \in W_0^{1,x}L_M(Q), \quad u_n(x, 0) &= u_0^n, \end{aligned} \quad (P_n)$$

and we conclude by adapting the same steps.

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References

- [1] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, vol. 65, Academic Press, New York, 1975.
- [2] T. Donaldson, *Inhomogeneous Orlicz-Sobolev spaces and nonlinear parabolic initial value problems*, Journal of Differential Equations **16** (1974), no. 2, 201–256.
- [3] A. Elmahi and D. Meskine, *Parabolic equations in Orlicz spaces*, Journal of the London Mathematical Society. Second Series **72** (2005), no. 2, 410–428.
- [4] ———, *Strongly nonlinear parabolic equations with natural growth terms and L^1 data in Orlicz spaces*, Portugaliae Mathematica. Nova Série **62** (2005), no. 2, 143–183.
- [5] ———, *Strongly nonlinear parabolic equations with natural growth terms in Orlicz spaces*, Nonlinear Analysis. Theory, Methods & Applications **60** (2005), no. 1, 1–35.
- [6] J.-P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Transactions of the American Mathematical Society **190** (1974), 163–205.
- [7] ———, *Some approximation properties in Orlicz-Sobolev spaces*, Studia Mathematica **74** (1982), no. 1, 17–24.
- [8] ———, *A strongly nonlinear elliptic problem in Orlicz-Sobolev spaces*, Nonlinear Functional Analysis and Its Applications, Part 1 (Berkeley, Calif, 1983), Proc. Sympos. Pure Math., vol. 45, American Mathematical Society, Rhode Island, 1986, pp. 455–462.
- [9] J.-P. Gossez and V. Mustonen, *Variational inequalities in Orlicz-Sobolev spaces*, Nonlinear Analysis. Theory, Methods & Applications **11** (1987), no. 3, 379–392.
- [10] V. K. Le and K. Schmitt, *Quasilinear elliptic equations and inequalities with rapidly growing coefficients*, Journal of the London Mathematical Society. Second Series **62** (2000), no. 3, 852–872.
- [11] J. Robert, *Inéquations variationnelles paraboliques fortement non linéaires*, Journal de Mathématiques Pures et Appliquées. Neuvième Série **53** (1974), 299–320.
- [12] M. Rudd, *Nonlinear constrained evolution in Banach spaces*, Ph.D. thesis, University of Utah, Utah, 2003.

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- [13] ———, *Weak and strong solvability of parabolic variational inequalities in Banach spaces*, Journal of Evolution Equations 4 (2004), no. 4, 497–517.
- [14] M. Rudd and K. Schmitt, *Variational inequalities of elliptic and parabolic type*, Taiwanese Journal of Mathematics 6 (2002), no. 3, 287–322.

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