DEVELOPMENT OF LOCAL, GLOBAL, AND TRACE ESTIMATES FOR THE SOLUTIONS OF ELLIPTIC EQUATIONS

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Received 21 January 2001

1. Introduction

In this paper, we improve and extend the local, global, and trace estimates for the solutions of elliptic equations we developed in [6]. These estimates are developed for the solutions of elliptic second-order equations, with source terms, with Dirichlet-Neumann boundary conditions and Dirichlet boundary conditions. They are obtained under quasi-optimal regularity conditions on the source terms. The previous trace estimates we developed in [6] are based on our improved version of the estimates of Aleksandrov, Bakelman, and Pucci estimates [1, 2]. They are improvement in the sense that they are independent of the low, "relaxation," term of the equations. These later estimates are important in many applications, in particular, for the treatment of quasilinear and fully nonlinear equations. The trace estimates we developed are themselves of great importance in elliptic theory. They are also important in many other applications, in particular, for the study of fixed point methods, the treatment of contraction properties for some operators, and also for the analysis of algorithms in numerical analysis. We have already given some applications of the trace estimates of [6] in [4].

In Section 2, we state a local estimate for elliptic equations. In Section 3, we introduce the first and second basic problems corresponding, respectively, to elliptic equations with Dirichlet-Neumann and Dirichlet-Dirichlet boundary conditions. In Section 4, we state our trace estimates for the first and second basic problems. In Section 5, we state and prove various local and global estimates for the first basic problem. In Section 6, we prove the trace estimate for the first basic problem. In Section 7, we state and prove various local and global estimates for the second basic problem. In Section 8, we prove the trace estimate for the second basic problem. In Section 9, we introduce the third basic problem corresponding to elliptic equations with Dirichlet-Neumann boundary

conditions. In Section 10, we give the trace estimates for the third basic problem. In Section 11, we state and prove various local and global estimates for the third basic problem. In Section 12, we give indications on the proof of the trace estimates for the third basic problem.

2. A local estimate

In this section, we present a local estimate that we have developed in [6] for an arbitrary elliptic operator of second order satisfying some conditions to be precised below.

Let L be an operator of the form

$$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u, \tag{2.1}$$

for any u in $W^{2,n}(\Omega)$, with Ω a bounded domain of $\mathbb{R}^n$. The coefficients a^{ij}, b^i, and c, $i, j = 1, \ldots, n$, are defined on Ω. As usual, the repeated indices indicate a summation from 1 to n.

We suppose that the operator L is strictly elliptic in Ω in the sense that the matrix $\mathscr{A}$ of coefficients $[a^{ij}]$ is strictly positive everywhere in Ω. Let λ and Λ denote the smallest and the largest eigenvalue of $\mathscr{A}$, respectively. Let $\mathscr{D}$ denote the determinant of the matrix $\mathscr{A}$ and $\mathscr{D}^* = \mathscr{D}^{1/n}$. We have

$$0 < \lambda \leq \mathscr{D}^* \leq \Lambda. \tag{2.2}$$

We suppose, in addition, that the coefficients a^{ij}, b^i, and c are bounded in Ω, and that there exist two positive real numbers γ and δ such that

$$\frac{\Lambda}{\lambda} \leq \gamma, \quad (L \text{ is uniformly elliptic}),$$
$$\left(\frac{|b|}{\lambda}\right)^2 \leq \delta. \tag{2.3}$$

Our local estimate is stated in the following theorem.

THEOREM 2.1. *Let $u \in W^{2,n}(\Omega)$ and suppose that $Lu \geq f$ with $f \in L^n(\Omega)$ and $c \leq 0$. Then for all spheres $B = B_{2R}(y)$ of center y and radius $2R$ included in Ω and for all $p > 0$, we have*

$$\sup_{B_R(y)} u \leq C_R \left\{ \left(\frac{1}{|B|} \int_B (u^+)^p \right)^{1/p} + \frac{R}{\lambda} \|f\|_{L^n(B)} \right\}, \tag{2.4}$$

where the constant C_R depends on $(n, \gamma, \delta R^2, p)$, but is independent of c, and $u^+ = \max(u, 0)$.
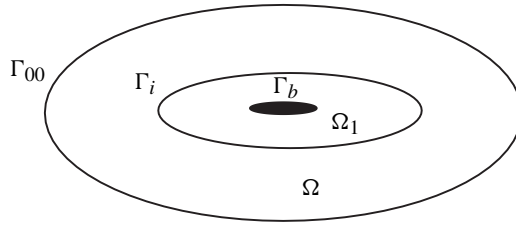
Figure 2.1. Description of the domain Ω and its splitting.

Remark 2.2. The statement of the same theorem can be found in [3], under the assumption

$$\frac{|c|}{\lambda} \leq \delta'. \tag{2.5}$$

However, in [3], the constant C_R depends indirectly on c through δ'. Our estimate is independent on the low relaxation term c. This is very important for the applications of these estimates to various problems.

Estimates of this type have many important applications to quasilinear elliptic equations (and systems) and nonlinear elliptic equations (and systems). They have also important applications to the problem of boundary regularity. We will see in the next sections some applications of these estimates to the development of new trace estimates for elliptic equations of second order. The new trace estimates of the next sections are themselves important in elliptic theory. They have important applications in proving contraction properties for some operator and in the analysis of algorithms in numerical analysis. We refer to [5, 6] for further details.

3. First and second basic problems

Let Ω and Ω_l be two connected domains of $\mathbb{R}^n$ with $\Omega_l \subset \Omega$ (Figure 2.1). Let Γ_b, Γ_i, and Γ_∞ denote the boundaries of the two domains

$$\begin{aligned}
\Gamma_b &= \partial\Omega \cap \partial\Omega_l \quad \text{(internal boundary)}, \\
\Gamma_i &= \partial\Omega_l \cap \Omega \quad \text{(interface)}, \\
\Gamma_\infty &= \partial\Omega \backslash \Gamma_b \quad \text{(farfield boundary)}.
\end{aligned} \tag{3.1}$$

Let n denote the external unit normal vector to $\partial\Omega$ or $\partial\Omega_l$. Let $V \in (L^\infty(\Omega))^n$ be a given velocity field of an inviscid incompressible flow such that

$$\operatorname{div} V = 0 \quad \text{in } \Omega, \qquad V \cdot n = 0 \quad \text{on } \Gamma_b. \tag{3.2}$$

The first basic problem is a Dirichlet-Neumann problem,

$$\mathscr{L}v = -\nu\Delta v + V\cdot\nabla v + cv = f \quad \text{in } \Omega, \tag{3.3}$$

$$v = 0 \quad \text{on } \Gamma_\infty, \qquad \frac{\partial v}{\partial n} = g \quad \text{on } \Gamma_b, \tag{3.4}$$

where g is given in $H^{-1/2}(\Gamma_b)$, the coefficient c is strictly positive, and ν is the diffusion coefficient. We assume that $f \in L^n(\Omega)\cap L^\infty(\Omega)$. Let W be the subspace of $H^1(\Omega)$ defined by

$$W = \big\{w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma_\infty\big\}. \tag{3.5}$$

We define two bilinear forms on W

$$a(v,w) = \int_\Omega \nu\nabla v\nabla w + \int_\Omega \text{div}(Vv)w, \quad (v,w) = \int_\Omega vw. \tag{3.6}$$

The weak formulation of the first basic problem (3.3) and (3.4) is to find $v \in W$ satisfying

$$a(v,w) + c(v,w) = \int_{\Gamma_b} \nu gw\, d\Gamma + \int_\Omega fw, \quad \forall w \in W. \tag{3.7}$$

We write c in the form $c = 1/\tau$ where τ is positive and we assume that the coefficients ν and τ satisfy

$$\nu\tau \le \frac{1}{2}, \quad \tau \le 1. \tag{3.8}$$

This hypothesis is not necessary but simplifies the proofs to come. Moreover, it is not restrictive (see [4, 6]).

The second basic problem is a Dirichlet-Dirichlet problem,

$$\mathscr{L}v = -\nu\Delta v + V\cdot\nabla v + cv = f \quad \text{in } \Omega_l, \tag{3.9}$$

$$v = h \quad \text{on } \Gamma_i, \qquad v = 0 \quad \text{on } \Gamma_b, \tag{3.10}$$

where h is given in $H^{1/2}(\Gamma_i)$, the coefficient c is strictly positive, and ν is the diffusion coefficient. We assume that $f \in L^n(\Omega_l)\cap L^\infty(\Omega_l)$ and the velocity field V is given by (3.2). We write the coefficient c in the form $c = 1/\tau$ and we assume that the coefficients ν and τ satisfy (3.8). Let W be the subspace of $H^1(\Omega_l)$ defined by

$$W = \big\{w \in H^1(\Omega_l) \mid w = 0 \text{ on } \Gamma_b\big\}. \tag{3.11}$$

We introduce two bilinear forms on W

$$a_l(v,w) = \nu\int_{\Omega_l} \nabla v\cdot\nabla w + \int_{\Omega_l} \text{div}(Vv)w, \quad (v,w) = \int_{\Omega_l} vw. \tag{3.12}$$
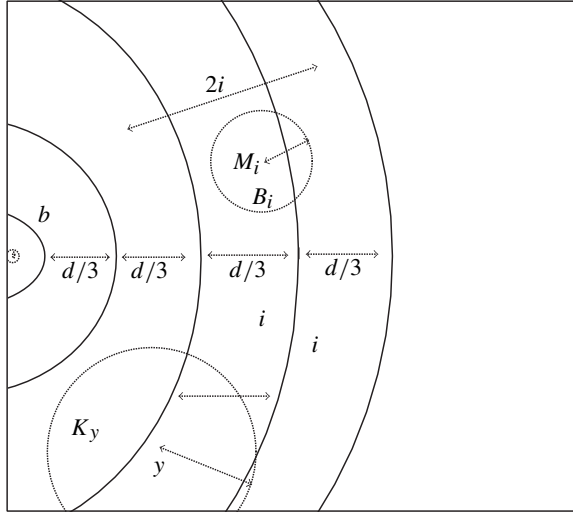
Figure 4.1. Description of the domain Ω_l and the splitting used in the
majorization of the local solution.

The weak formulation of the second basic problem (3.9) and (3.10) is to find
$v \in W$ such that

$$a_l(v, w) + \left(\frac{1}{\tau}\right)(v, w) = \int_{\Gamma_i} v \frac{\partial v}{\partial n} w + \int_{\Omega_l} f_l w, \quad \forall w \in W, \tag{3.13}$$

$$v|_{\Gamma_i} = h, \tag{3.14}$$

where h is given in $H^{1/2}(\Gamma_i)$.

For the elliptic operators (3.3) and (3.9), the coefficients introduced in
Section 2 can be expressed explicitly. We have $L = -\mathcal{L}$. Therefore, $\lambda = \Lambda = \nu$
and we choose $\gamma = 1$, $\delta = n(\|V\|_\infty/\nu)^2$.

4. Trace estimates I

We present in this section our trace estimates for the solutions of the first and
second basic problems. We first introduce some geometric notations. These are
necessary for the statement of the trace estimates and their proofs that will be
given in the next sections. Let d denote the distance between Γ_b and Γ_i. Let Ω_i
be the subdomain of Ω_l of width $d/3$ with external boundary Γ_i (Figure 4.1).
Let $y \in \Omega_i$ and $K_y = B_{d/3}(y)$ be the sphere of center y and radius $d/3$. There
exist $y_1, \ldots, y_l \in \Omega_i$ such that

$$\Omega_{2i} = \cup_{y \in \Omega_i} B_{d/6}(y) \subset \cup_{j=1}^l K_{y_j}. \tag{4.1}$$

We then define K by setting

$$K = \cup_{j=1}^l K_{y_j}. \tag{4.2}$$

Assume that d satisfies $2d < \mathrm{dist}(\Gamma_b, \Gamma_\infty)$. This assumption ensures that $B_{2d/3}(y) \subset \Omega$ for any point $y \in \Omega_i$. It is not necessary since we can modify the radius of the sphere $K_y = B_{qd/3}(y)$ such that $B_{2qd/3}(y) \subset \Omega$. However, it simplifies the notations in the proof.

Let Γ_V be the center surface of Ω_l defined as the surface whose distance from Γ_b and Γ_i is at least $d/2$. Let Ω_{il} be the subdomain of Ω_l of width $d/6$ centered at Γ_V. Let $y \in \Gamma_V$ and let $K_{ly} = B_{d/4}(y)$ be the sphere of radius $d/4$. We then introduce K_l which is constructed in the same way as the set K above. We also introduce a set Ω_b which is a subdomain of Ω_l whose internal boundary is Γ_b and such that its external boundary is of constant distance from Γ_b with a distance of $d/4$.

Let β be a real number such that

$$0 < \beta < \frac{3\sqrt{\nu}}{d}, \tag{4.3}$$

and set

$$k = \frac{\beta}{\nu\sqrt{\tau}}. \tag{4.4}$$

We now state the trace estimates.

THEOREM 4.1. *The solution v of the first basic problem (3.7) satisfies*

$$\|v\|_{1/2,\Gamma_i} \le C_1\sqrt{d}\left(d + \frac{\|V\|_\infty}{\nu}\right)^{1/2} \exp\left(-\frac{kd^2}{36}\right)$$

$$\times \left[\|g\|_{-1/2,\Gamma_b} + \frac{1}{\nu}\|f\|_{0,\Omega} + \frac{d}{\nu}\|f\|_{L^n(\Omega)} + 2\tau\|f\|_{\infty,\Omega}\right] + \frac{C_2}{\sqrt{\nu}}\|f\|_0, \tag{4.5}$$

where C_1 and C_2 are constants, with C_1 depending only on n and $(\|V\|_\infty d/\nu)^2$, but not on τ.

THEOREM 4.2. *The solution v of the second basic problem (3.13) and (3.14) satisfies*

$$\left\|\frac{\partial v}{\partial n}\right\|_{-1/2,\Gamma_b} \le C_1\alpha_1^2\alpha_2 \exp\left(-\frac{kd^2}{36}\right)\|h\|_{1/2,\Gamma_i}$$

$$+ \alpha_1\left(C_1\alpha_2 \exp\left(-\frac{kd^2}{36}\right)\frac{1}{\nu} + C_2\frac{1}{\sqrt{\nu}}\right)\|f_l\|_{0,\Omega_l}$$

$$+ C_1\alpha_1\alpha_2 \exp\left(-\frac{kd^2}{36}\right)\frac{d}{\nu}\|f_l\|_{L^n(\Omega_l)} \tag{4.6}$$

$$+ C_1\alpha_1\alpha_2 \exp\left(-\frac{kd^2}{36}\right)\tau\|f_l\|_{\infty,\Omega_i},$$

where C_1 and C_2 are constants with C_1 depending only on n and $(n\|V\|_\infty d/\nu)^2$, $\alpha_1 = [1 + (1/\nu)\|V\|_{\infty,\Omega_l} + 1/\nu\tau]$ and $\alpha_2 = \sqrt{d}(d + \|V\|_\infty/\nu)^{1/2}$.

In these estimates we observe that the dependence in terms of the low relaxation terms $(1/\tau)$ is completely explicit. The practical applications of these estimates can be for example the study of approximations in time methods when the time step (that can be taken here to be τ) goes to 0. They can also be applied to the study of the convergence properties of operators [4, 5]. Moreover, they can be used for other purposes such as proving existence results. We refer to [4, 5] for more details.

5. Local and global estimates for the first basic problem

The first basic result states a global H^1 estimate of the solution v of the first basic problem (3.7) in terms of the boundary data g and the data f.

LEMMA 5.1. *There exists a constant c_0 such that*

$$\|v\|_{1,\Omega} \le c_0\|g\|_{-1/2,\Gamma_b} + \frac{1}{\nu}\|f\|_0. \tag{5.1}$$

Proof. Proceeding as in the proof of Lemma 3.1 of [6], we obtain

$$\int_\Omega \left\{ \nu|\nabla v|^2 + \left(\frac{1}{\tau}\right)v^2 \right\} = \int_{\Gamma_b} \nu g v + \int_\Omega f v. \tag{5.2}$$

Using Cauchy-Schwarz inequality, the trace theorem, and (3.8) we obtain

$$\begin{aligned}
\|v\|_{1,\Omega}^2 &\le \|g\|_{-1/2,\Gamma_b}\|v\|_{1/2,\Gamma_b} + \frac{1}{\nu}\|f\|_0\|v\|_0 \\
&\le C(\Omega)\|g\|_{-1/2,\Gamma_b}\|v\|_{1,\Omega} + \frac{1}{\nu}\|f\|_0\|v\|_0.
\end{aligned} \tag{5.3}$$

Hence we have

$$\|v\|_{1,\Omega} \le c_0\|g\|_{-1/2,\Gamma_b} + \frac{1}{\nu}\|f\|_0. \tag{5.4}$$

$\square$

The next lemma states a local estimate of the solution v of the first basic problem (3.7).

LEMMA 5.2. *There exists a constant c_1 such that*

$$\|v\|_{\infty,K} \le c_1\|v\|_{0,\Omega} + c_1\frac{d}{\nu}\|f\|_{L^n(\Omega)}, \tag{5.5}$$

where c_1 depends only on n and $(\|V\|_\infty d/\nu)^2$.

Proof. The operator

$$L = -\mathcal{L} \tag{5.6}$$

satisfies the assumptions of Theorem 2.1 with $c = -1/\tau$. Applying this theorem with $p = 2$, $y \in \Omega_i$, and $K_y = B_{d/3}(y)$ ($B_{2d/3}(y) \subset \Omega$ since $2d < \text{dist}(\Gamma_b, \Gamma_\infty)$), we obtain

$$\|v\|_{\infty, K_y} \leq c_1 \|v\|_{0, B_{2d/3}(y)} + c_1 \frac{d}{v} \|f\|_{L^n(\Omega)}. \tag{5.7}$$

Therefore we obtain

$$\|v\|_{\infty, K_y} \leq c_1 \|v\|_{0, \Omega} + c_1 \frac{d}{v} \|f\|_{L^n(\Omega)}, \tag{5.8}$$

where c_1 is a constant depending only on n and $(\|V\|_\infty d/v)^2$.

Applying (5.8) to each K_{y_j} we obtain

$$\|v\|_{\infty, K} \leq \sup_{j=1,\dots,l} c_{1j} \left(\|v\|_{0, \Omega} + \frac{d}{v} \|f\|_{L^n(\Omega)} \right). \tag{5.9}$$

Setting $c_1 = \sup_{j=1,\dots,l} c_{1j}$, we finally obtain

$$\|v\|_{\infty, K} \leq c_1 \|v\|_{0, \Omega} + c_1 \frac{d}{v} \|f\|_{L^n(\Omega)}. \tag{5.10}$$

And the lemma is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now establish other local estimates for the solution v of the first basic problem. For any M_i in Ω_i, we introduce (see Figure 4.1) a ball B_i centered at M_i of radius $d/6$ and $v_i = \exp[k(r^2 - d^2/36)](\|v\|_{\infty, \partial B_i} + 2\tau \|f\|_{\infty, B_i})$.

We have the following result.

LEMMA 5.3. *The solution v of the first basic problem satisfies*

$$\|v\|_{\infty, \Omega_i} \leq \exp\left(-\frac{kd^2}{36} \right) \left(\|v\|_{\infty, \partial B_i} + 2\tau \|f\|_{\infty, B_i} \right). \tag{5.11}$$

Proof. To prove this lemma we use the same ideas we introduced to derive Lemma 3.3 in [6]. For the convenience of the reader we give detailed proof.

The operator $\mathcal{L}$ applied to v_i can be written in polar coordinates (with $r = M_i M$)

$$\mathcal{L} v_i = 4 \left(-k^2 v r^2 - kv + \frac{k}{2} V \cdot e_r r + \frac{1}{4\tau} \right) v_i. \tag{5.12}$$

Therefore

$$\mathcal{L} v_i \geq 4 \left(-k^2 v r^2 - \frac{k}{2} |V \cdot e_r| r + \left(\frac{1}{4\tau} - kv \right) \right) v_i. \tag{5.13}$$

We introduce the function

$$\varphi(r,k) = a(k)r^2 + b(k)r + c(k), \tag{5.14}$$

with

$$a(k) = -k^2\nu, \qquad b(k) = -\frac{k}{2}|V \cdot e_r|, \qquad c(k) = \frac{1}{8\tau} - k\nu. \tag{5.15}$$

We seek to satisfy the following relation:

$$0 \le \inf \varphi(r,k) \quad \text{for } 0 \le r \le \frac{d}{6}. \tag{5.16}$$

As $\varphi(r,k)$ decreases on $\mathbb{R}^+$, this will be satisfied if and only if

$$\varphi\left(\frac{d}{6}\right) \ge 0, \tag{5.17}$$

that is, if and only if

$$-\frac{k^2\nu d^2}{36} - \frac{kd\|V\|}{12} + \frac{1}{8\tau} - k\nu \ge 0. \tag{5.18}$$

Replacing k by its value, this becomes

$$-\frac{\beta^2 d^2}{(36\nu\tau)} - \frac{\beta d\|V\|}{12\nu\sqrt{\tau}} + \frac{1}{8\tau} - \frac{\beta\nu}{\nu\sqrt{\tau}} \ge 0. \tag{5.19}$$

Multiplying by $\sqrt{\tau}$, it follows that

$$\frac{1}{4\sqrt{\tau}}\left(\frac{1}{2} - \frac{\beta^2 d^2}{9\nu}\right) \ge \beta\left(1 + \frac{d\|V\|}{12\nu}\right). \tag{5.20}$$

The constraint $\beta < 3\sqrt{\nu}/d$ finally yields after division

$$\varphi(r,k) \ge 0 \quad \text{iff} \quad \frac{1}{4\sqrt{\tau}} \ge \beta\left[1 + \frac{d\|V\|}{12\nu}\right]\left[\frac{1}{2} - \frac{\beta^2 d^2}{9\nu}\right]^{-1}. \tag{5.21}$$

From the relation (5.13), we deduce that for $\beta < 3\sqrt{\nu}/d$ and τ satisfying (5.21), we have

$$L v_i \ge \frac{1}{2\tau} v_i \ge L v. \tag{5.22}$$

In addition, by construction,

$$v_i \ge v \quad \text{on } \partial B_i. \tag{5.23}$$

Consequently, using the maximum principle we obtain

$$v \le v_i \quad \text{in } B_i. \tag{5.24}$$

In particular

$$v(M_i) \le \exp\left(-\frac{kd^2}{36}\right)\left[\|v\|_{\infty,\partial B_i} + 2\tau\|f\|_{\infty,B_i}\right]. \tag{5.25}$$

Repeating the same process for $-v$, we finally obtain

$$\left|v(M_i)\right| \le \exp\left(-\frac{kd^2}{36}\right)\left[\|v\|_{\infty,\partial B_i} + 2\tau\|f\|_{\infty,B_i}\right], \quad \forall M_i \in \Omega_i. \tag{5.26}$$

We then obtain the estimate (5.11). $\square$

We now give an H^1 estimate of the solution v of the first basic problem on the domain $\Omega_\infty = \Omega \setminus \Omega_l$.

LEMMA 5.4. *There exist constants c_2 and c_3 such that*

$$\|v\|_{1,\Omega_\infty} \le c_2\|v\|_{\infty,\Omega_i}\sqrt{d}\left(d + \frac{\|V\|_\infty}{\nu}\right)^{1/2} + \frac{c_3}{\sqrt{\nu}}\|f\|_0. \tag{5.27}$$

Proof. Let $\xi \in H^1(\Omega)$ be such that

$$\xi = 1 \text{ in } \Omega_\infty, \quad \text{supp}\,\xi \subset \Omega_i \cup \Omega_\infty. \tag{5.28}$$

Proceeding as in the proof of Lemma 3.4 in [6], we obtain

$$\int_\Omega f\xi^2 v = \int_{\Omega_\infty} v\left(|\nabla v|^2 + |v|^2\right) + \int_{\Omega_i} v\left(|\nabla(\xi v)|^2 + |\xi v|^2\right)$$
$$+ \int_\Omega \left(\frac{1}{\tau} - v\right)\xi^2 v^2 - \int_{\Omega_i} \left(vv^2|\nabla\xi|^2 + v^2\xi V\cdot\nabla\xi\right). \tag{5.29}$$

Hence, we obtain

$$v\|v\|_{1,\Omega_\infty}^2 + \int_{\Omega_i} v\left(|\nabla(\xi v)|^2 + |\xi v|^2\right) + \int_\Omega \left(\frac{1}{\tau} - v\right)\xi^2 v^2$$
$$\le \int_{\Omega_i} \left(vv^2|\nabla\xi|^2 + v^2\xi V\cdot\nabla\xi\right) + \frac{1}{2}\int_\Omega \xi^2 f^2 + \frac{1}{2}\int_\Omega \xi^2 v^2. \tag{5.30}$$

The relation (3.8) then yields

$$v\|\xi v\|_{1,\Omega_\infty\cup\Omega_i}^2 \le \|v\|_{\infty,\Omega_i}^2 |\xi|_{1,\Omega_i}^2\left(v + \frac{\|\xi\|_{0,\Omega_i}\|V\|_\infty}{|\xi|_{1,\Omega_i}}\right) + \frac{1}{2}\int_\Omega \xi^2 f^2. \tag{5.31}$$

If we take ξ bounded such that

$$\|\xi\|_{0,\Omega_i} \le 1, \qquad |\xi|_{1,\Omega_i} = c_2 d, \tag{5.32}$$

where c_2 is a constant, (5.31) then becomes

$$\|v\|_{1,\Omega_\infty} \le c_2 \|v\|_{\infty,\Omega_i} \sqrt{d} \left(d + \frac{\|V\|_\infty}{\nu} \right)^{1/2} + \frac{c_3}{\sqrt{\nu}} \|f\|_0 \qquad (5.33)$$

and this concludes the proof. □

6. Proof of Theorem 4.1

In this section we prove Theorem 4.1. Since $\partial B_i \subset K$, we have

$$\|v\|_{\infty,\partial B_i} \le \|v\|_{\infty,K}. \qquad (6.1)$$

Lemma 5.2 then yields

$$\|v\|_{\infty,\partial B_i} \le c_1 \|v\|_{0,\Omega} + c_1 \frac{d}{\nu} \|f\|_{L^n(\Omega)}. \qquad (6.2)$$

Using Lemma 5.3 we obtain

$$\|v\|_{\infty,\Omega_i} \le \exp\left(-\frac{kd^2}{36} \right) \left[\|v\|_{\infty,\partial B_i} + 2\tau \|f\|_{\infty,B_i} \right]. \qquad (6.3)$$

Combining the last two estimates, we obtain

$$\|v\|_{\infty,\Omega_i} \le c_1 \exp\left(-\frac{kd^2}{36} \right) \left[\|v\|_{0,\Omega} + \frac{d}{\nu} \|f\|_{L^n(\Omega)} + 2\tau \|f\|_{\infty,\Omega_i} \right]. \qquad (6.4)$$

Applying Lemma 5.1 we get

$$\|v\|_{\infty,\Omega_i} \le c_1 \exp\left(-\frac{kd^2}{36} \right) \left[c_0 \|g\|_{-1/2,\Gamma_b} + \frac{1}{\nu} \|f\|_0 + \frac{d}{\nu} \|f\|_{L^n(\Omega)} + 2\tau \|f\|_{\infty,\Omega_i} \right]. \qquad (6.5)$$

An application of Lemma 5.4 yields

$$\|v\|_{1,\Omega_\infty} \le c_2 \sqrt{d} \left(d + \frac{\|V\|_\infty}{\nu} \right)^{1/2} c_1 \exp\left(-\frac{kd^2}{36} \right)$$

$$\times \left[c_0 \|g\|_{-1/2,\Gamma_b} + \frac{1}{\nu} \|f\|_0 + \frac{d}{\nu} \|f\|_{L^n(\Omega)} + 2\tau \|f\|_{\infty,\Omega_i} \right] + c_3 \frac{1}{\sqrt{\nu}} \|f\|_0. \qquad (6.6)$$

To conclude we use the trace theorem which yields

$$\|v\|_{1/2,\Gamma_i} \le c_4 \|v\|_{1,\Omega_\infty}$$

$$\le c_0 c_1 c_4 c_2 \sqrt{d} \left(d + \frac{\|V\|_\infty}{\nu} \right)^{1/2} \exp\left(-\frac{kd^2}{36} \right)$$

$$\times \left[\|g\|_{-1/2,\Gamma_b} + \frac{1}{\nu} \|f\|_{0,\Omega} + \frac{d}{\nu} \|f\|_{L^n(\Omega)} + 2\tau \|f\|_{\infty,\Omega} \right] + \frac{c_3 c_4}{\sqrt{\nu}} \|f\|_0 \qquad (6.7)$$

which corresponds to our theorem with $C_1 = c_0 c_1 c_4 c_2$ and $C_2 = c_3 c_4$. □

We can obtain the explicit dependence of C_1 in terms of n and $(n\|V\|_\infty d/\nu)^2$. Since in our Lemmas 5.1, 5.2, 5.3, and 5.4 the constants are explicit, all we need is to make explicit the dependence of C_R in terms of $(n, \gamma, \delta R^2, p)$ in Theorem 2.1 of [6]. This is possible by looking more closely at the proof of this theorem [6]. However, our main goal here is to derive a trace estimate in which the dependence in terms of τ only is explicit. We observe that the trace estimate and the other local and global estimates are obtained under no restrictions on τ or any other parameters (except those appearing in the statement of the first and second basic problems).

In the case where $f \equiv 0$ we obtain estimates which are improved versions of the estimates we have developed in [6]. For $f \neq 0$, the estimates we obtain here represent an extension of our previous estimates [6] to elliptic equations with source terms.

7. Local and global estimates for the second basic problem

We first state a global estimate for the solution v of the second basic problem (3.13) and (3.14).

LEMMA 7.1. *The solution v of the second basic problem (3.13) and (3.14) satisfies*

$$\|v\|_{1,\Omega_l} \leq \left[1 + \frac{c_1}{\nu}\|V\|_{\infty,\Omega_l} + \frac{1}{\nu\tau}\right]\|h\|_{1/2,\Gamma_i} + \frac{c_1}{\nu}\|f\|_{0,\Omega_l}. \tag{7.1}$$

Proof. Choosing $w = v$ in (3.13) we obtain

$$\int_{\Omega_l} |\nabla v|^2 + \frac{1}{\nu\tau}\int_{\Omega_l} v^2 = \int_{\Gamma_i} \frac{\partial v}{\partial n}h + \frac{1}{\nu}\int_{\Omega_l} fv - \frac{1}{\nu}\int_{\Omega_l} v\,\mathrm{div}(Vv). \tag{7.2}$$

Using Cauchy-Schwarz inequality, (3.2), and (3.10), we obtain

$$\int_{\Omega_l} |\nabla v|^2 + \frac{1}{\nu\tau}\int_{\Omega_l} v^2$$

$$\leq \left\|\frac{\partial v}{\partial n}\right\|_{-1/2,\Gamma_i}\|h\|_{1/2,\Gamma_i} + \frac{1}{\nu}\|f\|_{0,\Omega_l}\|v\|_{0,\Omega_l} + \frac{1}{2\nu}\left|\int_{\Gamma_i} V\cdot nh^2\right|$$

$$\leq \left\|\frac{\partial v}{\partial n}\right\|_{-1/2,\Gamma_i}\|h\|_{1/2,\Gamma_i} + \frac{1}{\nu}\|f\|_{0,\Omega_l}\|v\|_{0,\Omega_l} + \frac{1}{2\nu}\|V\cdot n\|_{\infty,\Gamma_i}\|h\|_{1/2,\Gamma_i}^2. \tag{7.3}$$

The term $\|\partial v/\partial n\|_{-1/2,\Gamma_i}$ is estimated as follows. Using (3.2), (3.10), and (3.13), we obtain

$$\int_{\Gamma_i} \frac{\partial v}{\partial n}w = \int_{\Omega_l} \nabla v\nabla w + \frac{1}{\nu}\int_{\Omega_l} wV\cdot\nabla v - \frac{1}{\nu}\int_{\Omega_l} fw + \frac{1}{\nu\tau}\int_{\Omega_l} vw. \tag{7.4}$$

Using the trace theorem and (3.2), we obtain

$$\left| \int_{\Gamma_i} \frac{\partial v}{\partial n} w \right| \le \left(\left(1 + \frac{1}{\nu} \|V\|_{\infty, \Omega_l} \right) \|\nabla v\|_{0, \Omega_l} + \frac{1}{\nu} \|f\|_{0, \Omega_l} + \frac{1}{\nu \tau} \|v\|_{0, \Omega_l} \right) \|w\|_{1, \Omega_l}. \tag{7.5}$$

Therefore, we have

$$\left\| \frac{\partial v}{\partial n} \right\|_{-1/2, \Gamma_i} \le \left(1 + \frac{1}{\nu} \|V\|_{\infty, \Omega_l} \right) \|\nabla v\|_{0, \Omega_l} + \frac{1}{\nu \tau} \|v\|_{0, \Omega_l} + \frac{1}{\nu} \|f\|_{0, \Omega_l}. \tag{7.6}$$

Combining (7.3) and (7.6), and using (3.8) and the trace theorem, we obtain

$$\begin{aligned}
\|v\|_{1, \Omega_l}^2 \le & \left\| \frac{\partial v}{\partial n} \right\|_{-1/2, \Gamma_i} \|h\|_{1/2, \Gamma_i} \\
& + \frac{1}{\nu} \|f\|_{0, \Omega_l} \|v\|_{0, \Omega_l} + \frac{1}{2\nu} \|V \cdot n\|_{\infty, \Gamma_i} \|h\|_{1/2, \Gamma_i}^2 \\
\le & \left[\left(1 + \frac{1}{\nu} \|V\|_{\infty, \Omega_l} + \frac{1}{\nu \tau} \right) \|h\|_{1/2, \Gamma_i} \right. \\
& \left. + \frac{c_1}{\nu} \|V\|_{\infty, \Omega_l} \|h\|_{1/2, \Gamma_i} + \frac{c_1}{\nu} \|f\|_{0, \Omega_l} \right] \|v\|_{1, \Omega_l}.
\end{aligned} \tag{7.7}$$

We finally obtain

$$\|v\|_{1, \Omega_l} \le \left[1 + \frac{c_1}{\nu} \|V\|_{\infty, \Omega_l} + \frac{1}{\nu \tau} \right] \|h\|_{1/2, \Gamma_i} + \frac{c_1}{\nu} \|f\|_{0, \Omega_l} \tag{7.8}$$

which is the required estimate. □

Remark 7.2. Lemma 7.1 represents a major improvement over Lemmas 4.1 and 4.2 of [6] which required the following condition on the diffusion coefficient ν, the velocity field V and the coefficient τ: $1/\tau \ge \nu/2 + (1/2\nu)\|V\|_{\infty}^2$.

Let $K_{ly} = B_{d/4}(y)$ be the sphere centered at y and of radius $d/4$, with y belonging to Γ_V (see Figure 7.1). By construction, Γ_V is the center surface of Ω_l, and Ω_{il} is the subdomain of width $d/6$ centered at Γ_V. We have the following lemma.

LEMMA 7.3. *There exists a constant c_2 such that*

$$\|v\|_{\infty, K_l} \le c_2 \|v\|_{0, \Omega_l} + c_2 \frac{d}{\nu} \|f\|_{L^n(\Omega_l)}, \tag{7.9}$$

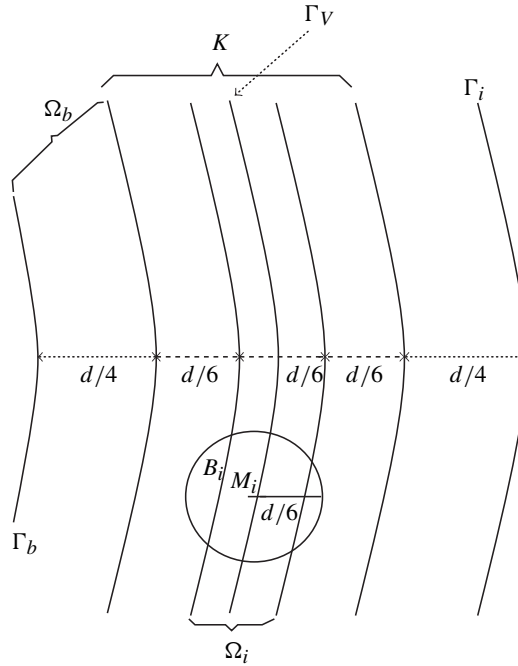*where c_2 depends only on n and $(\|V\|_{\infty} d/\nu)^2$.*

Figure 7.1. Description of the local domain Ω_l and the splitting used
in the majorization of the global solution.

Proof of Lemma 7.3. See the proof of Lemma 5.2 in Section 5.   □

Next we establish another local estimate for the solution of the second basic
problem (3.13) and (3.14). For any $M_i \in \Omega_{il}$, we introduce (see Figure 7.1)
a ball B_i centered at M_i and of radius $d/6$ and the function $v_i = \exp[k(r^2 - d^2/36)][\|v\|_{\infty,\partial B_i} + 2\tau\|f\|_{\infty,B_i}]$.

We then have the following result.

LEMMA 7.4. *The solution v of the second basic problem (3.13) and (3.14)
satisfies*

$$\|v\|_{\infty,\Omega_{il}} \leq \exp\left[-\frac{kd^2}{36}\right]\left[\|v\|_{\infty,\partial B_i} + 2\tau\|f\|_{\infty,B_i}\right]. \qquad (7.10)$$

Proof of Lemma 7.4. See the proof of Lemma 5.3 in Section 5.   □

Let Ω_b be the subdomain of Ω_l introduced in Section 4 and described in
Figure 7.1. The next result states an H^1 global estimate of the solution of the
second basic problem.

LEMMA 7.5. *The solution v of the second basic problem (3.13) and (3.14) satisfies*

$$\|v\|_{1,\Omega_b\cup\Omega_{il}} \le c_3\|v\|_{\infty,\Omega_{il}}\sqrt{d}\left(d+\frac{\|V\|_\infty}{\nu}\right)^{1/2}+\frac{c_3}{\sqrt{\nu}}\|f\|_{0,\Omega_l}. \qquad (7.11)$$

Proof of Lemma 7.5. See the proof of Lemma 5.4 in Section 5. □

8. Proof of Theorem 4.2

In this section we prove Theorem 4.2. Since $\partial B_i \subset K_l$ by construction, Lemmas 7.3 and 7.4 imply

$$\|v\|_{\infty,\Omega_{il}} \le c_2\exp\left(-\frac{kd^2}{36}\right)\left[\|v\|_{0,\Omega_l}+\frac{d}{\nu}\|f\|_{L^n(\Omega_l)}+2\tau\|f\|_{\infty,B_i}\right]. \quad (8.1)$$

Furthermore, Lemma 7.1 yields

$$\begin{aligned}
\|v\|_{\infty,\Omega_{il}} \le {}&c_2\exp\left(-\frac{kd^2}{36}\right)\left(\left[1+\frac{c_1}{\nu}\|V\|_{\infty,\Omega_l}+\frac{1}{\nu\tau}\right]\|h\|_{1/2,\Gamma_i}\right.\\
&\left.+\frac{c_1}{\nu}\|f\|_{0,\Omega_l}+\frac{d}{\nu}\|f\|_{L^n(\Omega_l)}+2\tau\|f\|_{\infty,\Omega_{il}}\right).
\end{aligned} \qquad (8.2)$$

Using Lemma 7.5 we obtain

$$\begin{aligned}
\|v\|_{1,\Omega_b\cup\Omega_{il}} \le {}&c_3\|v\|_{\infty,\Omega_{il}}\sqrt{d}\left(d+\frac{\|V\|_\infty}{\nu}\right)^{1/2}+\frac{c_3}{\sqrt{\nu}}\|f\|_{0,\Omega_l}\\
\le {}&c_2c_3\sqrt{d}\left(d+\frac{\|V\|_\infty}{\nu}\right)^{1/2}\exp\left(-\frac{kd^2}{36}\right)\\
&\times\left(\left[1+\frac{c_1}{\nu}\|V\|_{\infty,\Omega_l}+\frac{1}{\nu\tau}\right]\|h\|_{1/2,\Gamma_i}\right.\\
&\left.+\frac{c_1}{\nu}\|f\|_{0,\Omega_l}+\frac{d}{\nu}\|f\|_{L^n(\Omega_l)}+2\tau\|f\|_{\infty,\Omega_i}\right)\\
&+\frac{c_3}{\sqrt{\nu}}\|f\|_{0,\Omega_l}.
\end{aligned} \qquad (8.3)$$

Before concluding we establish an estimate of the term

$$\left\|\frac{\partial v}{\partial n}\right\|_{-1/2,\Gamma_b}. \qquad (8.4)$$

Choosing w such that

$$w \in H^1(\Omega_b) \quad \text{with } w=0 \text{ on } \partial\Omega_b\setminus\partial\Gamma_b, \qquad (8.5)$$

and using (3.13) we obtain

$$\int_{\Omega_b} \left(-\nu \Delta v + \mathrm{div}(Vv) + \frac{v}{\tau} \right) w = \int_{\Omega_b} f w. \tag{8.6}$$

Applying Green's formula and using (3.2), we obtain

$$\int_{\Gamma_b} \frac{\partial v}{\partial n} w = \int_{\Omega_b} \left(\nabla v \nabla w + \left(\frac{1}{\nu} \right) V \cdot \nabla v w + \frac{1}{\nu \tau} v w \right) - \frac{1}{\nu} \int_{\Omega_b} f w. \tag{8.7}$$

As in the proof of Lemma 7.1, we obtain

$$\left\| \frac{\partial v}{\partial n} \right\|_{-1/2, \Gamma_b} \leq \left[1 + \frac{1}{\nu} \|V\|_{\infty, \Omega_b} + \frac{1}{\nu \tau} \right] \|v\|_{1, \Omega_b} + \frac{1}{\nu} \|f\|_{0, \Omega_b}. \tag{8.8}$$

The completion of the proof of the theorem results from the combination of the relations (8.3) with (8.8). We obtain

$$
\begin{aligned}
\left\| \frac{\partial v}{\partial n} \right\|_{-1/2, \Gamma_b} &\leq c_1 c_2 c_3 \alpha_1 \alpha_2 \exp \left(-\frac{kd^2}{36} \right) \\
&\quad \times \left(\alpha_1 \|h\|_{1/2, \Gamma_i} + \frac{1}{\nu} \|f\|_{0, \Omega_l} + \frac{d}{\nu} \|f\|_{L^n(\Omega_l)} + 2\tau \|f\|_{\infty, \Omega_i} \right) \\
&\quad + c_3 \alpha_1 \frac{1}{\sqrt{\nu}} \|f\|_{0, \Omega_l} + \frac{1}{\nu} \|f\|_{0, \Omega_b} \\
&\leq C_1 \alpha_1^2 \alpha_2 \exp \left(-\frac{kd^2}{36} \right) \|h\|_{1/2, \Gamma_i} \\
&\quad + \alpha_1 \left(C_1 \alpha_2 \exp \left(-\frac{kd^2}{36} \right) \frac{1}{\nu} + C_2 \frac{1}{\sqrt{\nu}} \right) \|f\|_{0, \Omega_l} \\
&\quad + C_1 \alpha_1 \alpha_2 \exp \left(-\frac{kd^2}{36} \right) \frac{d}{\nu} \|f\|_{L^n(\Omega_l)} \\
&\quad + C_1 \alpha_1 \alpha_2 \exp \left(-\frac{kd^2}{36} \right) \tau \|f\|_{\infty, \Omega_i}.
\end{aligned} \tag{8.9}
$$

We then obtain the theorem with $C_1 = c_1 c_2 c_3$ and $C_2 = c_3$ constants with C_1 depending only on n and $(\|V\|_\infty d/\nu)^2$, $\alpha_1 = [1 + (1/\nu)\|V\|_{\infty, \Omega_l} + 1/\nu\tau]$ and $\alpha_2 = \sqrt{d}(d + \|V\|_\infty/\nu)^{1/2}$. □

We notice that this theorem represents a major improvement as compared with Theorem 4.1 of [6] which required the assumption $1/\tau \geq \nu/2 + (1/2\nu)\|V\|_\infty^2$.

In the case where $f \equiv 0$ we obtain estimates which are improved versions of the estimates we have developed in [6]. For $f \neq 0$, the estimates we obtain here are extensions of our previous estimates [6] to elliptic equations with source terms.

9. Third basic problem

Let Ω be a connected bounded domain of $\mathbb{R}^n$, such that its boundary $\partial\Omega$ is Lipschitzian. Let Ω_g and Ω_l be connected domains of $\mathbb{R}^n$ with $\Omega_g \cup \Omega_l = \Omega$ and $\Omega_g \cap \Omega_l = \emptyset$. Let $\Gamma_b = \partial\Omega \cap \partial\Omega_l$, (internal boundary), $\Gamma_i = \partial\Omega_l \cap \Omega = \partial\Omega_g \cap \Omega$, (interface), $\Gamma_\infty = \partial\Omega \backslash \Gamma_b = \partial\Omega_g \cap \partial\Omega$ (farfield boundary). We denote by n the external unit normal vector to $\partial\Omega$ or $\partial\Omega_l$. Let $V \in (L^\infty(\Omega))^n$ be a velocity field of an inviscid incompressible flow given by (3.2).

The third basic problem is a Dirichlet-Neumann problem

$$\mathscr{L}v = -\nu\Delta v + V \cdot \nabla v + cv = f \quad \text{in } \Omega_g, \tag{9.1}$$

$$v = 0 \quad \text{on } \Gamma_\infty, \qquad \frac{\partial v}{\partial n} = g \quad \text{on } \Gamma_i, \tag{9.2}$$

where g is given in $H^{-1/2}(\Gamma_i)$, the coefficient c is strictly positive, and ν is the diffusion coefficient. We assume that $f \in L^n(\Omega_g) \cap L^\infty(\Omega_g)$. Let W be the subspace of $H^1(\Omega_g)$ defined by

$$W = \left\{ w \in H^1(\Omega_g) \mid w = 0 \text{ on } \Gamma_\infty \right\}. \tag{9.3}$$

We define two bilinear forms on W

$$a(v, w) = \int_{\Omega_g} \nu\nabla v\nabla w + \int_{\Omega_g} \text{div}(Vv)w, \quad (v, w) = \int_{\Omega_g} vw. \tag{9.4}$$

The weak formulation of the third basic problem (9.1) and (9.2) is to find $v \in W$ satisfying

$$a(v, w) + c(v, w) = \int_{\Gamma_i} vgw\,d\Gamma + \int_{\Omega_g} fw, \quad \forall w \in W. \tag{9.5}$$

As in the previous sections we write c in the form $c = 1/\tau$ where τ is positive and we assume that the coefficients ν and τ satisfy (3.8). This hypothesis is not necessary but simplifies the proofs to come. Moreover, it is not restrictive (see [4, 6]).

For the elliptic operators (9.1), the coefficients introduced in Section 2 can be expressed explicitly. We have $L = -\mathscr{L}$. Therefore $\lambda = \Lambda = \nu$ and we choose $\gamma = 1$, $\delta = n(\|V\|_\infty/\nu)^2$.

10. Trace estimates II

We present in this section our trace estimates for the solutions of the third basic problems. We first introduce some geometric notations. These are necessary for the statement of the trace estimates and their proofs that will be given in the next sections.

Let d denote the distance between Γ_b and Γ_i. Let Ω_i be the subdomain of Ω_l of width $d/3$ with external boundary Γ_i (Figure 4.1). Let $y \in \Omega_i$ and

$K_y = B_{d/3}(y)$ be the sphere of center y and radius $d/3$. There exist $y_1, \ldots, y_l \in \Omega_i$ such that

$$\Omega_{2i} = \cup_{y \in \Omega_i} B_{d/6}(y) \subset \cup_{j=1}^{l} K_{y_j}. \tag{10.1}$$

We then define K by setting

$$K = \cup_{j=1}^{l} K_{y_j}. \tag{10.2}$$

Assume that d satisfies $2d < \text{dist}(\Gamma_b, \Gamma_\infty)$. This assumption ensures that $B_{2d/3}(y) \subset \Omega$ for any point $y \in \Omega_i$. It is not necessary since we can modify the radius of the sphere $K_y = B_{qd/3}(y)$ such that $B_{2qd/3}(y) \subset \Omega$. However, it simplifies the notations in the proof.

Let β be a real number such that

$$0 < \beta < \frac{3\sqrt{\nu}}{d}, \tag{10.3}$$

and set

$$k = \frac{\beta}{\nu\sqrt{\tau}}. \tag{10.4}$$

We now state the trace estimates.

THEOREM 10.1. *The solution v of the third basic problem (9.5) satisfies*

$$\|v\|_{1/2,\Gamma_i} \leq C_1 \sqrt{d} \left(d + \frac{\|V\|_\infty}{\nu} \right)^{1/2} \exp\left(-\frac{kd^2}{36} \right)$$

$$\times \left[\|g\|_{-1/2,\Gamma_i} + \frac{2}{\nu} \|f\|_{0,\Omega} + \frac{d}{\nu} \|f\|_{L^n(\Omega)} + 2\tau \|f\|_{\infty,\Omega} \right] + \frac{C_2}{\sqrt{\nu}} \|f\|_0, \tag{10.5}$$

where C_1 and C_2 are constants, with C_1 depending only on n and $(\|V\|_\infty d/\nu)^2$, but not on τ.

In these estimates we observe that the dependence in terms of the low relaxation terms $(1/\tau)$ is completely explicit. The practical applications of these estimates can be for example the study of approximations in time methods when the time step (that can be taken here to be τ) goes to 0. They can also be applied to the study of the convergence properties of operators such as those in [4, 5]. Moreover, they can be used for other purposes such as proving existence results. We refer to [4, 5] for more details.

11. Local and global estimates for the third basic problem

For the third basic problem we obtain similar local and global estimates as in the case of the first basic problem. The proofs of these estimates are similar to those for the first basic problem. The only exception is the global H^1 estimate stated in Lemma 11.1 below.

LEMMA 11.1. *For τ small there is*

$$\|v\|_{1,\Omega} \leq c_0 \|g\|_{-1/2,\Gamma_i} + \frac{1}{\nu}\|f\|_0, \tag{11.1}$$

where c_0 is a constant.

Proof. Proceeding as in the proof of Lemma 3.1 of [6], we obtain

$$\int_\Omega \left\{ \nu|\nabla v|^2 + V \cdot \nabla v v + \left(\frac{1}{\tau}\right)v^2 \right\} = \int_{\Gamma_i} \nu g v + \int_\Omega f v. \tag{11.2}$$

We then obtain

$$\frac{\nu}{2}\int_\Omega |\nabla v|^2 + \frac{2\nu - \tau\|V\|_\infty^2}{2\nu\tau}\int_\Omega v^2 = \int_{\Gamma_i} \nu g v + \int_\Omega f v. \tag{11.3}$$

Using Cauchy-Schwarz inequality, the trace theorem, and (3.8) we obtain

$$\begin{aligned}
\|v\|_{1,\Omega}^2 &\leq 2\|g\|_{-1/2,\Gamma_i}\|v\|_{1/2,\Gamma_i} + \frac{2}{\nu}\|f\|_0\|v\|_0 \\
&\leq C(\Omega)\|g\|_{-1/2,\Gamma_i}\|v\|_{1,\Omega} + \frac{2}{\nu}\|f\|_0\|v\|_0.
\end{aligned} \tag{11.4}$$

Hence we have

$$\|v\|_{1,\Omega} \leq c_0\|g\|_{-1/2,\Gamma_i} + \frac{2}{\nu}\|f\|_0. \tag{11.5}$$

$\square$

The next lemma states a local estimate of the solution v of the third basic problem (3.7).

LEMMA 11.2. *There exists a constant c_1 such that*

$$\|v\|_{\infty,K} \leq c_1\|v\|_{0,\Omega} + c_1\frac{d}{\nu}\|f\|_{L^n(\Omega)}, \tag{11.6}$$

where c_1 depends only on n and $(\|V\|_\infty d/\nu)^2$.

Proof. See the proof of Lemma 5.2. $\square$

We also have other local estimates for the solution v of the third basic problem. For any M_i in Ω_i, we introduce (see Figure 4.1) a ball B_i centered at M_i of radius $d/6$ and $v_i = \exp[k(r^2 - d^2/36)](\|v\|_{\infty,\partial B_i} + 2\tau\|f\|_{\infty,B_i})$.

We have the following result.

LEMMA 11.3. *The solution v of the first basic problem satisfies*

$$\|v\|_{\infty,\Omega_i} \leq \exp\left(-\frac{kd^2}{36}\right)\left(\|v\|_{\infty,\partial B_i} + 2\tau\|f\|_{\infty,B_i}\right). \tag{11.7}$$

Proof. See the proof of Lemma 5.3.   □

We now give an H^1 estimate of the solution v of the third basic problem on the domain $\Omega_\infty = \Omega \setminus \Omega_l$.

LEMMA 11.4. *There exist constants c_2 and c_3 such that*

$$\|v\|_{1,\Omega_\infty} \le c_2 \|v\|_{\infty,\Omega_i} \sqrt{d}\left(d + \frac{\|V\|_\infty}{\nu}\right)^{1/2} + \frac{c_3}{\sqrt{\nu}}\|f\|_0. \qquad (11.8)$$

Proof. See the proof of Lemma 5.4.   □

12. Proof of Theorem 10.1

The proof of Theorem 10.1 can be obtained following the same ideas as in the proof of Theorem 4.1. It relies on the various local and global estimates derived in the last section.

Acknowledgement

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