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CATEGORY OF A_∞-CATEGORIES

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Abstract

We define natural A_∞-transformations and construct A_∞-category of A_∞-functors. The notion of non-strict units in an A_∞-category is introduced. The 2-category of (unital) A_∞-categories, (unital) functors and transformations is described.

The study of higher homotopy associativity conditions for topological spaces began with Stasheff's article [**Sta63**, I]. In a sequel to this paper [**Sta63**, II] Stasheff defines also A_∞-algebras and their homotopy-bar constructions. These algebras and their applications to topology were actively studied, for instance, by Smirnov [**Smi80**] and Kadeishvili [**Kad80, Kad82**]. We adopt some notations of Getzler and Jones [**GJ90**], which reduce the number of signs in formulas. The notion of an A_∞-category is a natural generalization of A_∞-algebras. It arose in connection with Floer homology in Fukaya's work [**Fuk93, Fuk**] and was related by Kontsevich to mirror symmetry [**Kon95**]. See Keller [**Kel01**] for a survey on A_∞-algebras and categories.

In the present article we show that given two A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, one can construct a third A_∞-category $A_\infty(\mathcal{A},\mathcal{B})$ whose objects are A_∞-functors $f:\mathcal{A}\to\mathcal{B}$, and morphisms are natural A_∞-transformations between such functors. This result was also obtained by Fukaya [**Fuk**] and by Kontsevich and Soibelman [**KS**], independently and, apparently, earlier. We describe compositions between such categories of A_∞-functors, which would allow us to construct a 2-category of unital A_∞-categories. The latter notion is our generalization of strictly unital A_∞-categories (cf. Keller [**Kel01**]). We also discuss unit elements in unital A_∞-categories, unital natural A_∞-transformations, and unital A_∞-functors.

Plan of the article with comments and explanations. The first section describes some notation, sign conventions, composition convention, etc. used in the article. The ground commutative ring $\Bbbk$ is not assumed to be a field. This is suggested by the development of homological algebra in [**Dri02**]. Working over a ring

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$\Bbbk$ instead of a field has strong consequences. For instance, one may not hope for Kadeishvili's theorem on minimal models [**Kad82**] to hold for all A_∞-algebras over $\Bbbk$.

In the second section we recall or give definitions of the main objects. A $\Bbbk$-quiver is such a graph that the set of arrows (morphisms) between two vertices (objects) is a $\Bbbk$-module (Definition 2.1). We view quivers as categories without multiplication and units. Cocategories are $\Bbbk$-quivers and $\Bbbk$-coalgebras with a matrix type decomposition into $\Bbbk$-submodules, indexed by pairs of objects (Definition 2.2). A_∞-categories are defined as a special kind of differential graded cocategories – the ones of the form of the tensor cocategory $T\mathcal{A}$ of a $\Bbbk$-quiver $\mathcal{A}$ (Definition 2.3). A_∞-functors are homomorphisms of cocategories that commute with the differential (Definition 2.4). A_∞-transformations between A_∞-functors are defined as coderivations (Definition 2.6). They seem to make A_∞-category theory closer to ordinary category theory. Notice, however, that A_∞-transformations are analogs of transformations between ordinary functors, which do not satisfy the naturality condition. Natural A_∞-transformations are introduced in Definition 6.4. A_∞-functors and A_∞-transformations are determined by their components.

In the third section we study tensor products of cocategories and homomorphisms between them (Section 3.2). We concentrate on homomorphisms from the tensor product of tensor cocategories to another tensor cocategory. Given $\Bbbk$-quivers $\mathcal{A}$ and $\mathcal{B}$, we consider another $\Bbbk$-quiver $\operatorname{Coder}(\mathcal{A},\mathcal{B})$, whose objects are the cocategory homomorphisms $T\mathcal{A} \to T\mathcal{B}$ and morphisms are coderivations (Section 3.2). We construct a cocategory homomorphism $\alpha : T\mathcal{A} \otimes T\operatorname{Coder}(\mathcal{A},\mathcal{B}) \to T\mathcal{B}$ (Corollary 3.3 to Proposition 3.1), based on a map $\theta : T\operatorname{Coder}(\mathcal{A},\mathcal{B}) \to \operatorname{Hom}_\Bbbk(T\mathcal{A}, T\mathcal{B})$ (3.0.1). The homomorphism α is universal (Proposition 3.4), in other words, $T\operatorname{Coder}(\mathcal{A},\mathcal{B})$ is the inner hom-object $\mathsf{Hom}(T\mathcal{A}, T\mathcal{B})$ in the monoidal category generated by tensor cocategories.

This universality is exploited in the fourth section in order to show that the category of tensor cocategories is enriched in the monoidal category of graded cocategories. That is: there exists an associative unital multiplication $M : T\operatorname{Coder}(\mathcal{A},\mathcal{B}) \otimes T\operatorname{Coder}(\mathcal{B},\mathcal{C}) \to T\operatorname{Coder}(\mathcal{A},\mathcal{C})$, which is a cocategory homomorphism (Proposition 4.1). Its explicit description uses the map θ.

The fifth section extends the results of the third section to differential graded tensor cocategories, that is, to A_∞-categories. With two A_∞-categories $\mathcal{A}$, $\mathcal{B}$ is associated a third A_∞-category $A_\infty(\mathcal{A},\mathcal{B})$ (Proposition 5.1). Its objects are A_∞-functors $\mathcal{A} \to \mathcal{B}$, and its morphisms are coderivations. To reduce the number of signs in the theory we prefer to work with grading of a graded $\Bbbk$-quiver or A_∞-category $\mathcal{A}$ shifted by 1: $s\mathcal{A} = \mathcal{A}[1]$. In this notation the quiver $A_\infty(\mathcal{A},\mathcal{B})$ is a full subquiver of the quiver $s^{-1}\operatorname{Coder}(s\mathcal{A}, s\mathcal{B})$. The proof of Proposition 5.1 consists of constructing a differential B in the tensor cocategory of $A_\infty(\mathcal{A},\mathcal{B})$. The explicit formula (5.1.3) for B uses the map θ. A cocategory homomorphism from a tensor product of differential tensor cocategories to a single such cocategory is called an A_∞-functor in the generalized sense (Section 5.3). Restricting the cocategory homomorphism of Corollary 3.3 we get a homomorphism of differential graded cocategories $Ts\mathcal{A} \otimes TsA_\infty(\mathcal{A},\mathcal{B}) \to Ts\mathcal{B} = T(s\mathcal{B})$ (Corollary 5.4). Its universality

(Proposition 5.5) may be interpreted as $TsA_\infty(\mathcal{A},\mathcal{B})$ being the inner hom-object $\mathsf{Hom}(Ts\mathcal{A},Ts\mathcal{B})$ in the monoidal category generated by differential graded tensor cocategories.

This universality is used in the sixth section to show that the category of A_∞-categories is enriched in the monoidal category of differential graded cocategories. Namely, the multiplication M of Proposition 4.1 restricted to $M : TsA_\infty(\mathcal{A},\mathcal{B}) \otimes TsA_\infty(\mathcal{B},\mathcal{C}) \to TsA_\infty(\mathcal{A},\mathcal{C})$ is an A_∞-functor, that is, it commutes with the differential (equation (6.1.1)). By universality (Proposition 5.5) M corresponds to a unique A_∞-functor

$$A_\infty(\mathcal{A},\_) : A_\infty(\mathcal{B},\mathcal{C}) \to A_\infty(A_\infty(\mathcal{A},\mathcal{B}),A_\infty(\mathcal{A},\mathcal{C})).$$

We prove that it is strict and describe it in Proposition 6.2. Natural A_∞-transformations are defined as cycles in the differential graded quiver of all A_∞-transformations (Definition 6.4).

Identifying cohomologous natural A_∞-transformations (that is, considering cohomology of the quiver of A_∞-transformations) in the seventh section, we get a non-2-unital 2-category A_∞, whose objects are A_∞-categories, 1-morphisms are A_∞-functors, and 2-morphisms are equivalence classes of natural A_∞-transformations. Here non-2-unital means that unit 2-morphisms are missing in the 2-category A_∞. However, unit 1-morphisms are present – the identity A_∞-functors. Before constructing A_∞ we construct a non-2-unital 2-category $\mathcal{K}A_\infty$ enriched in $\mathcal{K}$ – the homotopy category of differential graded complexes of $\Bbbk$-modules (Proposition 7.1). Morphisms of $\mathcal{K}$ are chain maps modulo homotopy. The notion of 2-category enriched in a symmetric monoidal category is discussed in Appendix A. The idea of the construction is that the binary operation becomes strictly associative if homotopic chain maps are identified. Similarly with other identities in a 2-category. The non-2-unital 2-category A_∞ is obtained from $\mathcal{K}A_\infty$ by taking the 0-th cohomology.

A_∞-categories are analogs of non-unital categories – categories without unit morphisms. We define a unital A_∞-category $\mathcal{C}$ so that its cohomology $H^\bullet(\mathcal{C})$ is a unital category, and for any representative $1_X \in \mathcal{C}^0(X,X)$ of the unit class $[1_X] \in H^0(\mathcal{C}(X,X))$ the binary compositions with 1_X are homotopic to identity as chain maps $\mathcal{C}(X,Y) \to \mathcal{C}(X,Y)$ or $\mathcal{C}(Y,X) \to \mathcal{C}(Y,X)$ (Definition 7.3, Lemma 7.4). We prove that for a unital A_∞-category there exists a natural A_∞-transformation $\mathbf{i}^\mathcal{C} : \mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ of the identity functor, whose square is equivalent to $\mathbf{i}^\mathcal{C}$ (Proposition 7.5). It is called a unit transformation of $\mathcal{C}$ (Definition 7.6) and, indeed, it is a unit 2-morphism in the 2-category A_∞. Moreover, $f\mathbf{i}^\mathcal{C} : f \to f$ is a unit 2-morphism of an A_∞-functor $f : \mathcal{A} \to \mathcal{C}$ (Corollary 7.9). The unit transformation $\mathbf{i}^\mathcal{C}$ is determined uniquely up to an equivalence (Corollary 7.10). If $\mathcal{C}$ is unital, then $A_\infty(\mathcal{A},\mathcal{C})$ is unital as well (Proposition 7.7).

The full 2-subcategory $\mathcal{K}^uA_\infty$ (resp. ${}^uA_\infty$) of $\mathcal{K}A_\infty$ (resp. A_∞), whose objects are unital A_∞-categories, and 1-morphisms are all A_∞-functors, is 2-unital by Corollary 7.11 (resp. Corollary 7.12). Since ${}^uA_\infty$ is an ordinary 2-category, the meaning of the statements 'a natural A_∞-transformation is invertible' and 'an A_∞-functor is an equivalence' is clear (Corollary 7.12). We show in Proposition 7.15 that a natural A_∞-transformation is invertible if and only if its 0-th component is invertible modulo boundary (in the sense of Section 7.13). The binary composition with a cycle invertible modulo boundary is homotopy invertible (Lemma 7.14).

In the eighth section we discuss unital A_∞-functors (between unital A_∞-categories). Their first components map unit elements into unit elements modulo boundary (Definition 8.1). For a unital functor $f : \mathcal{A} \to \mathcal{B}$ we have an equivalence of natural A_∞-transformations $\mathbf{i}^{\mathcal{A}} f \equiv f\mathbf{i}^{\mathcal{B}}$ (Proposition 8.2). An A_∞-functor isomorphic to a unital A_∞-functor is unital as well (8.2.4). Unital A_∞-categories, unital A_∞-functors and equivalence classes of natural A_∞-transformations form a 2-category $A_\infty^u \subset {}^uA_\infty$ (Definition 8.3), which is a close analog of the 2-category of usual categories. The construction of A_∞^u is the main point of the article. It is needed for developing a theory of free A_∞-categories, since it is expected that their universality properties are formulated in the language of 2-categories.

There is a forgetful 2-functor $\mathsf{k} : A_\infty^u \to \mathcal{K}\text{-}\mathcal{C}at$, which takes a unital A_∞-category into the same differential quiver equipped with the binary composition, viewed as a $\mathcal{K}$-category (Proposition 8.6). The 2-functor k reduces a unital A_∞-functor to its first component and a natural A_∞-transformation to its 0-th component. It turns out that many properties of an A_∞-functor are determined by its first component and many properties of a natural A_∞-transformation are determined by its 0-th component. For instance, if the first component of an A_∞-functor ϕ is homotopy invertible, then any natural A_∞-transformation $y : f\phi \to g\phi$ is equivalent to $t\phi$ for a unique (up to equivalence) natural A_∞-transformation $t : f \to g$ (Cancellation Lemma 8.7). If an A_∞-functor $\phi : \mathcal{C} \to \mathcal{B}$ to a unital A_∞-category $\mathcal{B}$ has homotopy invertible first component, and each object of $\mathcal{B}$ is "isomorphic modulo boundary" to an object from $\phi(\operatorname{Ob}\mathcal{C})$, then ϕ is a unital equivalence, and $\mathcal{C}$ is unital (Theorem 8.8). An equivalence between unital A_∞-categories is always unital (Corollary 8.9), which is not an immediate consequence of definitions.

As a first example of a unital A_∞-category we list strictly unital A_∞-categories (Section 8.11), which is a well-known notion. Other examples of unital A_∞-categories are obtained via Theorem 8.8. For instance, if an A_∞-functor $\phi : \mathcal{C} \to \mathcal{B}$ to a strictly unital A_∞-category $\mathcal{B}$ is invertible, then $\mathcal{C}$ is unital (Section 8.12). We stress again that taking the 0-th cohomology of a unital A_∞-category $\mathcal{C}$ we get a $\Bbbk$-linear category $H^0(\mathcal{C})$. This H^0 can be viewed as a 2-functor (Section 8.13).

In Appendix A we define 2-categories enriched in a symmetric monoidal category. Non-2-unital 2-categories are described in Definition Appendix A.2. 2-unital (usual) 2-categories admit a concise Definition Appendix A.1 and an expanded Definition Appendix A.3+Appendix A.2.

In Appendix B we prove that the cone of a homotopical isomorphism is contractible.

1. Conventions

We fix a universe $\mathscr{U}$ [**GV73**, Sections 0,1], [**Bou73**]. Many classes and sets in this paper will mean $\mathscr{U}$-small sets, even if not explicitly mentioned.

$\Bbbk$ denotes a ($\mathscr{U}$-small) unital associative commutative ring. By abuse of notation it denotes also a chain complex, whose 0-th component is $\Bbbk$, and the other components vanish. A $\Bbbk$-module means a $\mathscr{U}$-small $\Bbbk$-module. The tensor product $\otimes$

usually means $\otimes_{\Bbbk}$ – the tensor product of graded $\Bbbk$-modules. It turns the category of graded $\Bbbk$-modules into a closed monoidal category. We will use its standard symmetry $c : x \otimes y \mapsto (-)^{xy} y \otimes x = (-)^{\deg x \cdot \deg y} y \otimes x$. This paper contains many signs, and everywhere we abbreviate the usual $(-1)^{(\deg x)(\deg y)}$ to $(-)^{xy}$. Similarly, $(-)^x$ means $(-1)^{\deg x}$, or, simply, $(-1)^x$, if x is an integer.

It is easy to understand the line

$$\mathcal{A}(X_0, X_1) \otimes_{\Bbbk} \mathcal{A}(X_1, X_2) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathcal{A}(X_{n-1}, X_n),$$

and it is much harder to understand the order in

$$\mathcal{A}(X_{n-1}, X_n) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathcal{A}(X_1, X_2) \otimes_{\Bbbk} \mathcal{A}(X_0, X_1).$$

That is why we use the right operators: the composition of two maps (or morphisms) $f : X \to Y$ and $g : Y \to Z$ is denoted by $fg : X \to Z$. A map is written on elements as $f : x \mapsto xf = (x)f$. However, these conventions are not used systematically, and $f(x)$ might be used instead.

When $f, g : X \to Y$ are chain maps, $f \sim g$ means that they are homotopic. We denote by $\mathcal{K}$ the category of differential graded $\Bbbk$-modules, whose morphisms are chain maps modulo homotopy. A complex of $\Bbbk$-modules X is called *contractible* if $\mathrm{id}_X \sim 0$, in other words, if X is isomorphic to 0 in $\mathcal{K}$.

If C is a $\mathbb{Z}$-graded $\Bbbk$-module, then its suspension $sC = C[1]$ is the same $\Bbbk$-module with the shifted grading $(sC)^d = C^{d+1}$. The "identity" map $C \to sC$ of degree -1 is also denoted by s. We follow the Getzler–Jones sign conventions [**GJ90**], which include the idea to apply operations to complexes with shifted grading, and Koszul's rule:

$$(x \otimes y)(f \otimes g) = (-)^{yf} xf \otimes yg = (-1)^{\deg y \cdot \deg f} xf \otimes yg.$$

It takes its origin in Koszul's note [**Kos47**]. The main notions of graded algebra were given their modern names in H. Cartan's note [**Car48**]. See Boardman [**Boa66**] for operad-like approach to signs as opposed to closed symmetric monoidal category picture of Mac Lane [**Mac63**] (standard sign commutation rule). Combined together, these sign conventions make the number of signs in this paper tolerable.

If $u : A \to C$, $a \mapsto au$, is a chain map, its cone is the complex $\mathrm{Cone}(u) = C \oplus A[1]$, $\mathrm{Cone}^k(u) = C^k \oplus A^{k+1}$, with the differential $(c, a)d = (cd^C + au, ad^{A[1]}) = (cd^C + au, -ad^A)$.

2. A_∞-categories, A_∞-functors and A_∞-transformations

2.1 Definition (Quiver). A graded $\Bbbk$-quiver $\mathcal{A}$ consists of the following data: a class of objects $\mathrm{Ob}\,\mathcal{A}$ (a $\mathscr{U}$-small set); a $\mathbb{Z}$-graded $\Bbbk$-module $\mathcal{A}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y)$ for each pair of objects X, Y. A morphism of $\Bbbk$-quivers $f : \mathcal{A} \to \mathcal{B}$ is given by a map $f : \mathrm{Ob}\,\mathcal{A} \to \mathrm{Ob}\,\mathcal{B}$, $X \mapsto Xf$ and by a $\Bbbk$-linear map $\mathcal{A}(X, Y) \to \mathcal{B}(Xf, Yf)$ for each pair of objects X, Y of $\mathcal{A}$.

To a given graded $\Bbbk$-quiver $\mathcal{A}$ we associate another graded $\Bbbk$-quiver – its tensor coalgebra $T\mathcal{A}$, which has the same class of objects as $\mathcal{A}$. For each sequence $(X_0, X_1, X_2, \ldots, X_n)$ of objects of $\mathcal{A}$ there is the $\mathbb{Z}$-graded $\Bbbk$-module $T^n\mathcal{A}$ =

$\mathcal{A}(X_0, X_1) \otimes_\Bbbk \mathcal{A}(X_1, X_2) \otimes_\Bbbk \cdots \otimes_\Bbbk \mathcal{A}(X_{n-1}, X_n)$. For the sequence (X_0) with $n = 0$ it reduces to $T^0\mathcal{A} = \Bbbk$ in degree 0. The graded $\Bbbk$-module $T\mathcal{A}(X,Y) = \oplus_{n \geqslant 0} T^n\mathcal{A}$ is the sum of the above modules over all sequences with $X_0 = X$, $X_n = Y$. The quiver $T\mathcal{A}$ is equipped with the cut comultiplication $\Delta : T\mathcal{A}(X,Y) \to \oplus_{Z \in \operatorname{Ob}\mathcal{A}} T\mathcal{A}(X,Z) \bigotimes_\Bbbk T\mathcal{A}(Z,Y)$, $h_1 \otimes h_2 \otimes \cdots \otimes h_n \mapsto \sum_{k=0}^n h_1 \otimes \cdots \otimes h_k \bigotimes h_{k+1} \otimes \cdots \otimes h_n$, and the counit $\varepsilon = \bigl(T\mathcal{A}(X,Y) \xrightarrow{\mathrm{pr}_0} T^0\mathcal{A}(X,Y) \to \Bbbk\bigr)$, where the last map is $\mathrm{id}_\Bbbk$ if $X = Y$, or 0 if $X \neq Y$ (and $T^0\mathcal{A}(X,Y) = 0$). For this article it is the main example of the following notion:

2.2 Definition (Cocategory). A graded cocategory $\mathcal{C}$ is a graded $\Bbbk$-quiver $\mathcal{C}$, equipped with a comultiplication – a $\Bbbk$-linear map $\Delta^Z_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{C}(X,Z) \otimes_\Bbbk \mathcal{C}(Z,Y)$ of degree 0 for all triples X, Y, Z of objects of $\mathcal{C}$, and with a counit – a $\Bbbk$-linear map $\varepsilon_X : \mathcal{C}(X,X) \to \Bbbk$ of degree 0 for all objects X of $\mathcal{C}$, such that the usual associativity equations and two counit equation hold.

Associated to each graded cocategory $\mathcal{C}$ is a graded $\Bbbk$-coalgebra $C = \oplus_{X,Y \in \operatorname{Ob}\mathcal{C}} \mathcal{C}(X,Y)$. Vice versa, to a graded $\Bbbk$-coalgebra, decomposed like that into $\Bbbk$-submodules $\mathcal{C}(X,Y)$, $X, Y \in \operatorname{Ob}\mathcal{C}$, for some $\mathscr{U}$-small set $\operatorname{Ob}\mathcal{C}$, so that $\Delta(\mathcal{C}(X,Y)) \subset \oplus_{Z \in \operatorname{Ob}\mathcal{C}} \mathcal{C}(X,Z) \otimes_\Bbbk \mathcal{C}(Z,Y)$ for all pairs X, Y of objects of $\mathcal{C}$, and $\varepsilon(\mathcal{C}(X,Y)) = 0$ for $X \neq Y$, we associate a graded cocategory.

This interpretation allows one to define a *cocategory homomorphism* $f : \mathcal{C} \to \mathcal{D}$ as a particular case of a coalgebra homomorphism: a map $f : \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$, and $\Bbbk$-linear maps $\mathcal{C}(X,Y) \to \mathcal{D}(Xf, Yf)$ for all pairs of objects X, Y of $\mathcal{A}$, compatible with comultiplication and counit. Given cocategory homomorphisms $f, g : \mathcal{C} \to \mathcal{D}$ we say that a system of $\Bbbk$-linear maps $r : \mathcal{C}(X,Y) \to \mathcal{D}(Xf, Yg)$, $X, Y \in \operatorname{Ob}\mathcal{C}$ is an (f,g)-*coderivation*, if the equation $r\Delta = \Delta(f \otimes r + r \otimes g)$ holds.

In particular, these definitions apply to the tensor coalgebras $Ts\mathcal{A} = T(s\mathcal{A})$ of (suspended) $\Bbbk$-quivers $s\mathcal{A}$. In this case cocategory homomorphisms and coderivations have a special form as we shall see below.

2.3 Definition (A_∞-category, Kontsevich [Kon95]). An A_∞-category $\mathcal{A}$ consists of the following data: a graded $\Bbbk$-quiver $\mathcal{A}$; a differential $b : Ts\mathcal{A} \to Ts\mathcal{A}$ of degree 1, which is a (1,1)-coderivation, such that $(T^0 s\mathcal{A})b = 0$.

The definition of a (1,1)-coderivation $b\Delta = \Delta(1 \otimes b + b \otimes 1)$ implies that a $\Bbbk$-quiver morphism b is determined by a system of $\Bbbk$-linear maps $b\,\mathrm{pr}_1 : Ts\mathcal{A} \to s\mathcal{A}$ with components of degree 1

$$b_n : s\mathcal{A}(X_0, X_1) \otimes s\mathcal{A}(X_1, X_2) \otimes \cdots \otimes s\mathcal{A}(X_{n-1}, X_n) \to s\mathcal{A}(X_0, X_n), \qquad n \geqslant 1,$$

via the formula

$$b_{kl} = (b\big|_{T^k s\mathcal{A}})\,\mathrm{pr}_l : T^k s\mathcal{A} \to T^l s\mathcal{A}, \qquad b_{kl} = \sum_{\substack{r+n+t=k \\ r+1+t=l}} 1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}.$$

Notice that the last condition of the definition implies $b_0 = 0$. In particular, $b_{k0} = 0$, and $k < l$ implies $b_{kl} = 0$. Since b^2 is a (1,1)-coderivation of degree 2, the equation

$b^2 = 0$ is equivalent to its particular case $b^2 \operatorname{pr}_1 = 0$, that is, for all $k > 0$

$$\sum_{r+n+t=k} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) b_{r+1+t} = 0 : T^k s\mathcal{A} \to s\mathcal{A}. \tag{2.3.1}$$

Using another, more traditional, form of components of b:

$$m_n = \big(\mathcal{A}^{\otimes n} \xrightarrow{s^{\otimes n}} (s\mathcal{A})^{\otimes n} \xrightarrow{b_n} s\mathcal{A} \xrightarrow{s^{-1}} \mathcal{A}\big)$$

we rewrite (2.3.1) as follows:

$$\sum_{r+n+t=k} (-)^{t+rn} (1^{\otimes r} \otimes m_n \otimes 1^{\otimes t}) m_{r+1+t} = 0 : T^k \mathcal{A} \to \mathcal{A}. \tag{2.3.2}$$

Notice that this equation differs in sign from [**Kel01**], because we are using right operators!

2.4 Definition (A_∞-functor, e.g. Keller [Kel01]). An A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ consists of the following data: A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, a cocategory homomorphism $f : Ts\mathcal{A} \to Ts\mathcal{B}$ of degree 0, which commutes with the differential b.

The definition of a cocategory homomorphism $f\Delta = \Delta(f \otimes f)$, $f\varepsilon = \varepsilon$ implies that f is determined by a map $f : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, $X \mapsto Xf$ and a system of $\Bbbk$-linear maps $f \operatorname{pr}_1 : Ts\mathcal{A} \to s\mathcal{B}$ with components of degree 0

$$f_n : s\mathcal{A}(X_0, X_1) \otimes s\mathcal{A}(X_1, X_2) \otimes \cdots \otimes s\mathcal{A}(X_{n-1}, X_n) \to s\mathcal{B}(X_0 f, X_n f),$$

$n \geqslant 1$, (note that $f_0 = 0$) via the formula

$$f_{kl} = (f\big|_{T^k s\mathcal{A}}) \operatorname{pr}_l : T^k s\mathcal{A} \to T^l s\mathcal{B}, \qquad f_{kl} = \sum_{i_1+\dots+i_l=k} f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_l}. \tag{2.4.1}$$

In particular, $f_{00} = \mathrm{id} : \Bbbk \to \Bbbk$, and $k < l$ implies $f_{kl} = 0$. Since fb and bf are both (f,f)-coderivations of degree 1, the equation $fb = bf$ is equivalent to its particular case $fb \operatorname{pr}_1 = bf \operatorname{pr}_1$, that is, for all $k > 0$

$$\sum_{l>0;\, i_1+\dots+i_l=k} (f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_l}) b_l = \sum_{r+n+t=k} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) f_{r+1+t} : T^k s\mathcal{A} \to s\mathcal{B}. \tag{2.4.2}$$

Using m_n we rewrite (2.4.2) as follows:

$$\sum_{i_1+\dots+i_l=k}^{l>0} (-)^\sigma (f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_l}) m_l =$$
$$\sum_{r+n+t=k} (-)^{t+rn} (1^{\otimes r} \otimes m_n \otimes 1^{\otimes t}) f_{r+1+t} : T^k \mathcal{A} \to \mathcal{B},$$
$$\sigma = (i_2 - 1) + 2(i_3 - 1) + \dots + (l-2)(i_{l-1} - 1) + (l-1)(i_l - 1).$$

Notice that this equation differs in sign from [**Kel01**], because we are using right operators.

2.5 Example. An A_∞-category with one object is called an *A_∞-algebra* (Stasheff [**Sta63**]). An A_∞-functor between A_∞-algebras is called an *A_∞-homomorphism*

(Kadeishvili [**Kad82**]). These notions are psychologically easier to deal with, than the general case. The following notion of A_∞-transformations also makes sense for A_∞-algebras, however, such a context seems too narrow for it, because an A_∞-transformation is an analog of a transformation between ordinary functors without the naturality condition. Needless to say, in ordinary category theory there is no reason to consider unnatural transformations. The reasons to do it for A_∞-version are given in Section 5.

2.6 Definition (A_∞-transformation). An A_∞-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}$ of degree d (pre natural transformation in terms of [**Fuk**]) consists of the following data: A_∞-categories $\mathcal{A}$ and $\mathcal{B}$; A_∞-functors $f, g : \mathcal{A} \to \mathcal{B}$; an (f,g)-coderivation $r : Ts\mathcal{A} \to Ts\mathcal{B}$ of degree d.

The definition of an (f,g)-coderivation $r\Delta = \Delta(f \otimes r + r \otimes g)$ implies that r is determined by a system of $\Bbbk$-linear maps $r\,\mathrm{pr}_1 : Ts\mathcal{A} \to s\mathcal{B}$ with components of degree d

$$r_n : s\mathcal{A}(X_0,X_1) \otimes s\mathcal{A}(X_1,X_2) \otimes \cdots \otimes s\mathcal{A}(X_{n-1},X_n) \to s\mathcal{B}(X_0f, X_ng),$$

$n \geqslant 0$, via the formula

$$\begin{aligned} r_{kl} &= (r\big|_{T^ks\mathcal{A}})\,\mathrm{pr}_l : T^ks\mathcal{A} \to T^ls\mathcal{B}, \\ r_{kl} &= \sum_{\substack{q+1+t=l \\ i_1+\cdots+i_q+n+j_1+\cdots+j_t=k}} f_{i_1} \otimes \cdots \otimes f_{i_q} \otimes r_n \otimes g_{j_1} \otimes \cdots \otimes g_{j_t}. \end{aligned} \tag{2.6.1}$$

Note that r_0 is a system of $\Bbbk$-linear maps ${}_Xr_0 : \Bbbk \to s\mathcal{B}(Xf, Xg)$, $X \in \mathrm{Ob}\,\mathcal{A}$. In fact, the terms '$A_\infty$-transformation' and 'coderivation' are synonyms.

In particular, r_{0l} vanishes unless $l = 1$, and $r_{01} = r_0$. The component r_{kl} vanishes unless $1 \leqslant l \leqslant k+1$.

2.7 Examples. 1) The restriction of an A_∞-transformation r to T^1 is

$$r\big|_{T^1s\mathcal{A}} = r_1 \oplus [(f_1 \otimes r_0) + (r_0 \otimes g_1)],$$

where $r_1 : s\mathcal{A}(X,Y) \to s\mathcal{B}(Xf, Yg)$,

$$\begin{aligned} f_1 \otimes r_0 &: s\mathcal{A}(X,Y) = s\mathcal{A}(X,Y) \otimes \Bbbk \xrightarrow{f_1 \otimes r_0} s\mathcal{B}(Xf,Yf) \otimes s\mathcal{B}(Yf,Yg), \\ r_0 \otimes g_1 &: s\mathcal{A}(X,Y) = \Bbbk \otimes s\mathcal{A}(X,Y) \xrightarrow{r_0 \otimes g_1} s\mathcal{B}(Xf,Xg) \otimes s\mathcal{B}(Xg,Yg). \end{aligned}$$

2) The restriction of an A_∞-transformation r to T^2 is

$$\begin{aligned} r\big|_{T^2s\mathcal{A}} = r_2 \oplus [(f_2 \otimes r_0) + (f_1 \otimes r_1) + (r_1 \otimes g_1) + (r_0 \otimes g_2)] \oplus \\ \oplus [(f_1 \otimes f_1 \otimes r_0) + (f_1 \otimes r_0 \otimes g_1) + (r_0 \otimes g_1 \otimes g_1)], \end{aligned}$$

where $r_2 : s\mathcal{A}(X,Y) \otimes s\mathcal{A}(Y,Z) \to s\mathcal{B}(Xf, Zg)$,

$$\begin{aligned}
f_2 \otimes r_0 &: s\mathcal{A}(X,Y) \otimes s\mathcal{A}(Y,Z) \otimes \Bbbk \to s\mathcal{B}(Xf,Zf) \otimes s\mathcal{B}(Zf,Zg),\\
f_1 \otimes r_1 &: s\mathcal{A}(X,Y) \otimes s\mathcal{A}(Y,Z) \to s\mathcal{B}(Xf,Yf) \otimes s\mathcal{B}(Yf,Zg),\\
r_1 \otimes g_1 &: s\mathcal{A}(X,Y) \otimes s\mathcal{A}(Y,Z) \to s\mathcal{B}(Xf,Yg) \otimes s\mathcal{B}(Yg,Zg),\\
r_0 \otimes g_2 &: \Bbbk \otimes s\mathcal{A}(X,Y) \otimes s\mathcal{A}(Y,Z) \to s\mathcal{B}(Xf,Xg) \otimes s\mathcal{B}(Xg,Zg),\\
f_1 \otimes f_1 \otimes r_0 &: s\mathcal{A}(X,Y) \otimes s\mathcal{A}(Y,Z) \otimes \Bbbk \to s\mathcal{B}(Xf,Yf) \otimes s\mathcal{B}(Yf,Zf) \otimes s\mathcal{B}(Zf,Zg),\\
f_1 \otimes r_0 \otimes g_1 &: s\mathcal{A}(X,Y) \otimes \Bbbk \otimes s\mathcal{A}(Y,Z) \to s\mathcal{B}(Xf,Yf) \otimes s\mathcal{B}(Yf,Yg) \otimes s\mathcal{B}(Yg,Zg),\\
r_0 \otimes g_1 \otimes g_1 &: \Bbbk \otimes s\mathcal{A}(X,Y) \otimes s\mathcal{A}(Y,Z) \to s\mathcal{B}(Xf,Xg) \otimes s\mathcal{B}(Xg,Yg) \otimes s\mathcal{B}(Yg,Zg).
\end{aligned}$$

The $\Bbbk$-module of (f,g)-coderivations r is $\prod_{n=0}^{\infty} V_n$, where

$$V_n = \prod_{X_0,\dots,X_n \in \operatorname{Ob}\mathcal{A}} \operatorname{Hom}_{\Bbbk}\big(s\mathcal{A}(X_0,X_1) \otimes \dots \otimes s\mathcal{A}(X_{n-1},X_n), s\mathcal{B}(X_0f, X_ng)\big) \tag{2.7.1}$$

is the graded $\Bbbk$-module of n-th components r_n. It is equipped with the differential $d : V_n \to V_n$, given by the following formula

$$r_n d = r_n b_1 - (-)^{r_n} \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta}) r_n. \tag{2.7.2}$$

3. Coderivations and cocategory homomorphisms

Let $\mathcal{A}$, $\mathcal{B}$ be graded $\Bbbk$-quivers, and let $f^0, f^1, \dots, f^n : T\mathcal{A} \to T\mathcal{B}$ be cocategory homomorphisms. Consider n coderivations $r_1, \dots, r_n$ as in

$$f^0 \xrightarrow{r^1} f^1 \xrightarrow{r^2} \dots f^{n-1} \xrightarrow{r^n} f^n : T\mathcal{A} \to T\mathcal{B}.$$

We construct the following system of $\Bbbk$-linear maps from these data: $\theta = (r^1 \otimes \dots \otimes r^n)\theta : T\mathcal{A}(X,Y) \to T\mathcal{B}(Xf^0, Yf^n)$ of degree $\deg r^1 + \dots + \deg r^n$. Its components $\theta_{kl} = \theta\big|_{T^k\mathcal{A}} \operatorname{pr}_l : T^k\mathcal{A} \to T^l\mathcal{B}$ are given by the following formula

$$\theta_{kl} = \sum f^0_{i^0_1} \otimes \dots \otimes f^0_{i^0_{m_0}} \otimes r^1_{j_1} \otimes f^1_{i^1_1} \otimes \dots \otimes f^1_{i^1_{m_1}} \otimes \dots \otimes r^n_{j_n} \otimes f^n_{i^n_1} \otimes \dots \otimes f^n_{i^n_{m_n}}, \tag{3.0.1}$$

where summation is taken over all terms with

$$m_0+m_1+\dots+m_n+n = l, \quad i^0_1+\dots+i^0_{m_0}+j_1+i^1_1+\dots+i^1_{m_1}+\dots+j_n+i^n_1+\dots+i^n_{m_n} = k.$$

The component θ_{kl} vanishes unless $n \leqslant l \leqslant k+n$. If $n = 0$, we set $()\theta = f^0$. If $n = 1$, the formula gives $(r^1)\theta = r^1$.

3.1 Proposition. *For each $n \geqslant 0$ the map θ satisfies the equation*

$$(r^1 \otimes r^2 \otimes \dots \otimes r^n)\theta\Delta = \Delta \sum_{k=0}^{n} (r^1 \otimes \dots \otimes r^k)\theta \otimes (r^{k+1} \otimes \dots \otimes r^n)\theta. \tag{3.1.1}$$

Proof. Let us write down the required equation using $\bigotimes$ to separate the two copies of $T\mathcal{B}$ in $T\mathcal{B} \bigotimes T\mathcal{B}$, and $\otimes$ to denote the multiplication in $T\mathcal{B}$. We have to prove that for each $n, x, y, z \geqslant 0$

$$
\begin{aligned}
&\bigl(T^x\mathcal{A} \xrightarrow{(r^1\otimes\cdots\otimes r^n)\theta_{x,y+z}} T^{y+z}\mathcal{B} \xrightarrow{\Delta_{yz}} T^y\mathcal{B} \bigotimes T^z\mathcal{B}\bigr) \\
&= \sum_{k=0}^{n} \sum_{u+v=x} \bigl(T^x\mathcal{A} \xrightarrow{\Delta_{uv}} T^u\mathcal{A} \bigotimes T^v\mathcal{A} \xrightarrow{(r^1\otimes\cdots\otimes r^k)\theta_{uy} \bigotimes (r^{k+1}\otimes\cdots\otimes r^n)\theta_{vz}} T^y\mathcal{B} \bigotimes T^z\mathcal{B}\bigr).
\end{aligned}
$$

Substituting (3.0.1) for θ in the above equation we come to an identity, which is proved by inspection. Indeed, skipping all the intermediate steps, we get the following equation:

$$
\begin{aligned}
&\sum_{k=0}^{n} \sum_{\substack{m_0+m_1+\dots+m_n+n=y+z \\ m_0+m_1+\dots+m_{k-1}+k-1<y\leqslant m_0+m_1+\dots+m_k+k \\ i_1^0+\dots+i_{m_0}^0+j_1+i_1^1+\dots+i_{m_1}^1+\dots+j_n+i_1^n+\dots+i_{m_n}^n=x}} \\
&f_{i_1^0}^0\otimes\cdots\otimes f_{i_{m_0}^0}^0\otimes\cdots\otimes r_{j_k}^k\otimes f_{i_1^k}^k\otimes\cdots\otimes f_{i_w^k}^k \bigotimes f_{i_{w+1}^k}^k\otimes\cdots\otimes f_{i_{m_k}^k}^k\otimes r_{j_{k+1}}^{k+1}\otimes\cdots\otimes f_{i_1^n}^n\otimes\cdots\otimes f_{i_{m_n}^n}^n \\
&\qquad = \sum_{k=0}^{n} \sum_{u+v=x} \sum_{\substack{m_0+m_1+\dots+m_{k-1}+k+w=y \\ i_1^0+\dots+i_{m_0}^0+\dots+j_k+i_1^k+\dots+i_w^k=u}} \sum_{\substack{t+m_{k+1}+\dots+m_n+n-k=z \\ l_1+\dots+l_t+j_{k+1}+\dots+i_1^n+\dots+i_{m_n}^n=v}} \\
&f_{i_1^0}^0\otimes\cdots\otimes f_{i_{m_0}^0}^0\otimes\cdots\otimes r_{j_k}^k\otimes f_{i_1^k}^k\otimes\cdots\otimes f_{i_w^k}^k \bigotimes f_{l_1}^k\otimes\cdots\otimes f_{l_t}^k\otimes r_{j_{k+1}}^{k+1}\otimes\cdots\otimes f_{i_1^n}^n\otimes\cdots\otimes f_{i_{m_n}^n}^n .
\end{aligned}
$$

In the left hand side w denotes the expression $y-(m_0+m_1+\dots+m_{k-1}+k)$ and lies in the interval $0 \leqslant w \leqslant m_k$. Identifying m_k in the left hand side with $w+t$ in the right hand side, we deduce that the both sides are equal. $\square$

3.2. Cocategory homomorphisms. Graded $\Bbbk$-coalgebras form a symmetric monoidal category. The tensor product $C\otimes_\Bbbk D$ of $\Bbbk$-coalgebras C, D is equipped with the comultiplication $C\otimes D \xrightarrow{\Delta\otimes\Delta} C\otimes C\otimes D\otimes D \xrightarrow{1\otimes c\otimes 1} C\otimes D\otimes C\otimes D$, using the standard symmetry c of graded $\Bbbk$-modules. Since graded cocategories are, in fact, graded coalgebras with a special decomposition, they also form a symmetric monoidal category gCoCat. If $\mathcal{C}$ and $\mathcal{D}$ are cocategories, then the class of objects of their tensor product $\mathcal{C}\otimes\mathcal{D}$ is $\operatorname{Ob}\mathcal{C}\times\operatorname{Ob}\mathcal{D}$, and $\mathcal{C}\otimes\mathcal{D}(X\times U, Y\times W) = \mathcal{C}(X,Y)\otimes_\Bbbk \mathcal{D}(U,W)$.

Let $\phi : T\mathcal{A}\otimes T\mathcal{C} \to T\mathcal{B}$ be a cocategory homomorphism of degree 0. It is determined uniquely by its composition with pr_1, that is, by a family $\phi\,\mathrm{pr}_1 = (\phi_{nm})_{n,m\geqslant 0}$, $\phi_{nm} : T^n\mathcal{A}\otimes T^m\mathcal{C} \to \mathcal{B}$, $\phi_{00} = 0$, with the same underlying map of objects $\operatorname{Ob}\mathcal{A}\times\operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{B}$. Indeed, for given families of composable arrows $f^0 \xrightarrow{p^1} f^1 \xrightarrow{p^2} \dots f^{n-1} \xrightarrow{p^n} f^n$ of $\mathcal{A}$ and $g^0 \xrightarrow{t^1} g^1 \xrightarrow{t^2} \dots g^{m-1} \xrightarrow{t^m} g^m$ of $\mathcal{C}$ we have

$$
\begin{aligned}
&(p^1 \otimes \cdots \otimes p^n \otimes t^1 \otimes \cdots \otimes t^m)\phi \\
&\qquad = \sum_{\substack{i_1+\cdots+i_k=n \\ j_1+\cdots+j_k=m}} (-)^\sigma (p^1 \otimes \cdots \otimes p^{i_1} \otimes t^1 \otimes \cdots \otimes t^{j_1})\phi_{i_1 j_1} \\
&\qquad\qquad \otimes (p^{i_1+1} \otimes \cdots \otimes p^{i_1+i_2} \otimes t^{j_1+1} \otimes \cdots \otimes t^{j_1+j_2})\phi_{i_2 j_2} \\
&\otimes \cdots \otimes (p^{i_1+\cdots+i_{k-1}+1} \otimes \cdots \otimes p^{i_1+\cdots+i_k} \otimes t^{j_1+\cdots+j_{k-1}+1} \otimes \cdots \otimes t^{j_1+\cdots+j_k})\phi_{i_k j_k}.
\end{aligned} \tag{3.2.1}
$$

The sign depends on the parity of an integer

$$
\begin{aligned}
\sigma = {} & (t^1 + \cdots + t^{j_1})(p^{i_1+1} + \cdots + p^{i_1+\cdots+i_k}) + \\
& (t^{j_1+1} + \cdots + t^{j_1+j_2})(p^{i_1+i_2+1} + \cdots + p^{i_1+\cdots+i_k}) \\
& + \cdots + (t^{j_1+\cdots+j_{k-2}+1} + \cdots + t^{j_1+\cdots+j_{k-1}})(p^{i_1+\cdots+i_{k-1}+1} + \cdots + p^{i_1+\cdots+i_k}),
\end{aligned} \tag{3.2.2}
$$

where each coderivation has to be replaced with its degree. Recall that in our notation (Section 1) we abbreviate $(-1)^{(\deg t)(\deg p)}$ to $(-)^{tp}$. By definition the homomorphism ϕ satisfies the equation

$$
\begin{tikzcd}
T\mathcal{A} \otimes T\mathcal{C} \arrow[r, "\phi"] \arrow[d, "\Delta\otimes\Delta"'] & T\mathcal{B} \arrow[r, "\Delta"] & T\mathcal{B} \otimes T\mathcal{B} \\
T\mathcal{A} \otimes T\mathcal{A} \otimes T\mathcal{C} \otimes T\mathcal{C} \arrow[rr, "1\otimes c\otimes 1"] & & T\mathcal{A} \otimes T\mathcal{C} \otimes T\mathcal{A} \otimes T\mathcal{C} \arrow[u, "\phi\otimes\phi"']
\end{tikzcd} \tag{3.2.3}
$$

Introduce $\Bbbk$-linear maps $(t^1 \otimes \cdots \otimes t^m)\chi : T\mathcal{A} \to T\mathcal{B}$ by the formula $a[(t^1 \otimes \cdots \otimes t^m)\chi] = (a \otimes t^1 \otimes \cdots \otimes t^m)\phi$, $a \in T\mathcal{A}$. Then the above equation is equivalent to

$$
(t^1 \otimes t^2 \otimes \cdots \otimes t^m)\chi\Delta = \Delta \sum_{k=0}^{m} (t^1 \otimes \cdots \otimes t^k)\chi \otimes (t^{k+1} \otimes \cdots \otimes t^m)\chi. \tag{3.2.4}
$$

for all $m \geqslant 0$.

When $\mathcal{A}$, $\mathcal{B}$ are graded $\Bbbk$-quivers, we define a new $\Bbbk$-quiver $\operatorname{Coder}(\mathcal{A}, \mathcal{B})$, whose objects are cocategory homomorphisms $f : T\mathcal{A} \to T\mathcal{B}$. These homomorphisms are determined by a system $f\operatorname{pr}_1 = (f_n)_{n\geqslant 1}$ of morphisms of $\Bbbk$-quivers $f_n : T^n\mathcal{A} \to \mathcal{B}$ of degree 0 with the same underlying map $\operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, see (2.4.1). The $\Bbbk$-module of morphisms between $f, g : T\mathcal{A} \to T\mathcal{B}$ consists of (f, g)-coderivations:

$$
[\operatorname{Coder}(\mathcal{A}, \mathcal{B})(f, g)]^d = \{r : T\mathcal{A} \to T\mathcal{B} \mid r\Delta = \Delta(f \otimes r + r \otimes g), \quad \deg r = d\}, \quad d \in \mathbb{Z}.
$$

Such a coderivation r is determined by a system of $\Bbbk$-linear maps $r\operatorname{pr}_1 = (r_n)_{n\geqslant 0}$, $r_n : T^n\mathcal{A}(X, Y) \to \mathcal{B}(Xf, Yg)$ of degree d as in (2.6.1).

3.3 Corollary (to Proposition 3.1). *A map $\alpha : T\mathcal{A} \otimes T\operatorname{Coder}(\mathcal{A}, \mathcal{B}) \to T\mathcal{B}$, $a \otimes r^1 \otimes \cdots \otimes r^n \mapsto a[(r^1 \otimes \cdots \otimes r^n)\theta]$, is a cocategory homomorphism of degree 0.*

Proof. Equation (3.1.1) means that equation (3.2.4) holds for $\chi = \theta$, which is equivalent to (3.2.3) for $\phi = \alpha$, $\mathcal{C} = \operatorname{Coder}(\mathcal{A}, \mathcal{B})$. ☐

3.4 Proposition. *For any cocategory homomorphism $\phi : T\mathcal{A} \otimes T\mathcal{C}^1 \otimes T\mathcal{C}^2 \otimes \cdots \otimes T\mathcal{C}^q \to T\mathcal{B}$ of degree 0 there is a unique cocategory homomorphism $\psi : T\mathcal{C}^1 \otimes T\mathcal{C}^2 \otimes \cdots \otimes T\mathcal{C}^q \to T\,\mathrm{Coder}(\mathcal{A}, \mathcal{B})$ of degree 0, such that*

$$\phi = \big(T\mathcal{A} \otimes T\mathcal{C}^1 \otimes T\mathcal{C}^2 \otimes \cdots \otimes T\mathcal{C}^q \xrightarrow{1\otimes\psi} T\mathcal{A} \otimes T\,\mathrm{Coder}(\mathcal{A}, \mathcal{B}) \xrightarrow{\alpha} T\mathcal{B}\big).$$

Proof. Let us start with a simple case $q = 1$, $\mathcal{C} = \mathcal{C}^1$. Each object g of $\mathcal{C}$ induces a cocategory morphism $g\psi : a \mapsto (a \otimes g)\phi$. We set $\psi_0 = 0$. Each element $p \in \mathcal{C}(g, h)$ induces a coderivation $(p)\psi_1 = p\psi : a \mapsto (a \otimes p)\phi$. Suppose that ψ_i are already found for $0 \leqslant i < n$. Then we find ψ_n from the sought identity $\chi = \psi\theta$. Namely, for $g^0 \xrightarrow{p^1} g^1 \xrightarrow{p^2} \dots g^{n-1} \xrightarrow{p^n} g^n$ we have to satisfy the identity

$$(p^1\otimes\cdots\otimes p^n)\chi = (p^1\otimes\cdots\otimes p^n)\psi_n + \sum_{l=2}^{n} \sum_{i_1+\cdots+i_l=n} [(p^1\otimes\cdots\otimes p^n).(\psi_{i_1}\otimes\psi_{i_2}\otimes\cdots\otimes\psi_{i_l})]\theta,$$

which expresses ψ via its components ψ_k. Notice that the unknown ψ_n occurs only in the singled out summand, corresponding to $l = 1$. The factors ψ_i in the sum are already known, since $i < n$. So we define $(p^1 \otimes \cdots \otimes p^n)\psi_n : T\mathcal{A} \to T\mathcal{B}$ as the difference of $(p^1 \otimes \cdots \otimes p^n)\chi$ and the sum in the right hand side. Assume that ψ is a cocategory homomorphism up to the level n, that is,

$$(p^1 \otimes \cdots \otimes p^m)\psi = \sum_{l=1}^{m} \sum_{i_1+\cdots+i_l=m} (p^1 \otimes \cdots \otimes p^m).(\psi_{i_1} \otimes \psi_{i_2} \otimes \cdots \otimes \psi_{i_l}) \tag{3.4.1}$$

for all $0 \leqslant m \leqslant n$. Taking into account equations (3.1.1), we see that (3.2.4) is equivalent to an equation of the form

$$(p^1 \otimes \cdots \otimes p^n)\psi_n\Delta = \Delta[g^0\psi \otimes (p^1 \otimes \cdots \otimes p^n)\psi_n + (p^1 \otimes \cdots \otimes p^n)\psi_n \otimes g^n\psi + \mu].$$

Moreover, if $(p^1 \otimes\cdots\otimes p^n)\psi_n$ were a $(g^0\psi, g^n\psi)$-coderivation, it would imply (3.2.4) by Section 3.2. We deduce that, indeed, the above $\mu = 0$, and $(p^1 \otimes \cdots \otimes p^n)\psi_n$ is a $(g^0\psi, g^n\psi)$-coderivation. Thus, we have found a unique $(p^1 \otimes \cdots \otimes p^n)\psi_n \in \mathrm{Coder}(\mathcal{A}, \mathcal{B})$ and (3.4.1) for $m = n$ defines uniquely an element $(p^1 \otimes \cdots \otimes p^n)\psi \in T\,\mathrm{Coder}(\mathcal{A}, \mathcal{B})$.

The case $q > 1$ is similar to the case $q = 1$, however, the reasoning is slightly obstructed by a big amount of indices. So we explain in detail the case $q = 2$ only, and in the general case no new phenomena occur. Further we shall use the obtained formulas in the case $q = 2$.

Let $\mathcal{C} = \mathcal{C}^1$, $\mathcal{D} = \mathcal{C}^2$. We consider a cocategory homomorphism $\phi : T\mathcal{A} \otimes T\mathcal{C} \otimes T\mathcal{D} \to T\mathcal{B}$, $a \mapsto a[(c \otimes d)\chi]$. We have to obtain from it a unique cocategory homomorphism $\psi : T\mathcal{C} \otimes T\mathcal{D} \to T\,\mathrm{Coder}(\mathcal{A}, \mathcal{B})$.

A pair of objects $f \in \mathrm{Ob}\,\mathcal{C}$, $g \in \mathrm{Ob}\,\mathcal{D}$ induces a cocategory morphism $(f, g)\psi : a \mapsto (a\otimes 1_f \otimes 1_g)\phi$. We set $\psi_{00} = 0$. An object $f \in \mathrm{Ob}\,\mathcal{C}$ and an element $t \in \mathcal{D}(g^0, g^1)$ induce an $((f, g^0)\psi, (f, g^1)\psi)$-coderivation $(f \otimes t)\psi_{01} = (f \otimes t)\psi : a \mapsto (a\otimes 1_f \otimes t)\phi$. An element $p \in \mathcal{C}(f^0, f^1)$ and an object $g \in \mathrm{Ob}\,\mathcal{D}$ induce an $((f^0, g)\psi, (f^1, g)\psi)$-coderivation $(p \otimes g)\psi_{10} = (p \otimes g)\psi : a \mapsto (a \otimes p \otimes 1_g)\phi$. Suppose that ψ_{ij} are already found for $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant m$, $(i, j) \neq (n, m)$. Then we find ψ_{nm} from the sought identity $\chi = \psi\theta$. Namely, for $f^0 \xrightarrow{p^1} f^1 \xrightarrow{p^2} \dots f^{n-1} \xrightarrow{p^n} f^n$ in $\mathcal{C}$ and

$g^0 \xrightarrow{t^1} g^1 \xrightarrow{t^2} \dots g^{m-1} \xrightarrow{t^m} g^m$ in $\mathcal{D}$ we have to satisfy the identity

$$\begin{aligned}(p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\chi &= (p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\psi_{nm} \\ &+ \sum_{\substack{i_1+\dots+i_k=n\\ j_1+\dots+j_k=m}}^{k>1} (-)^\sigma \big[(p^1 \otimes \dots \otimes p^{i_1} \otimes t^1 \otimes \dots \otimes t^{j_1})\psi_{i_1 j_1} \\ &\otimes (p^{i_1+1} \otimes \dots \otimes p^{i_1+i_2} \otimes t^{j_1+1} \otimes \dots \otimes t^{j_1+j_2})\psi_{i_2 j_2} \\ \otimes \dots \otimes (p^{i_1+\dots+i_{k-1}+1} &\otimes \dots \otimes p^{i_1+\dots+i_k} \otimes t^{j_1+\dots+j_{k-1}+1} \otimes \dots \otimes t^{j_1+\dots+j_k})\psi_{i_k j_k}\big]\theta,\end{aligned} \tag{3.4.2}$$

which is nothing else but (3.2.1). The sign is determined by (3.2.2). All terms of the sum are already known. So we define a map $(p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\psi_{nm} : T\mathcal{A} \to T\mathcal{B}$ as the difference of the left hand side and the sum in the right hand side. The fact that ϕ is a homomorphism is equivalent to the identity for the map χ for all $n, m \geqslant 0$:

$$(p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\chi\Delta = \Delta \sum_{k=0}^{n} \sum_{l=0}^{m} (-)^{(p^{k+1}+\dots+p^n)(t^1+\dots+t^l)}$$
$$(p^1 \otimes \dots \otimes p^k \otimes t^1 \otimes \dots \otimes t^l)\chi \otimes (p^{k+1} \otimes \dots \otimes p^n \otimes t^{l+1} \otimes \dots \otimes t^m)\chi, \tag{3.4.3}$$

similarly to Section 3.2. From it we get an equation for $(p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\psi_{nm}$

$$\begin{aligned}(p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\psi_{nm}\Delta = \Delta\big[(f^0, g^0)\psi \otimes (p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\psi_{nm} \\ + (p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\psi_{nm} \otimes (f^n, g^m)\psi + \mu\big].\end{aligned}$$

Notice that if ψ is indeed a homomorphism and ψ_{nm} is its component, then ϕ is a homomorphism, hence, (3.4.3) holds. Thus, the above equation with $\mu = 0$ implies (3.4.3) (and it happens only for one value of μ). Since we know that (3.4.3) holds, it implies $\mu = 0$. Therefore, $(p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\psi_{nm}$ is a $((f^0, g^0)\psi, (f^n, g^m)\psi)$-coderivation. Thus, we have found a unique $(p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\psi_{nm} \in \mathrm{Coder}(\mathcal{A}, \mathcal{B})$, and

$$\begin{aligned}(p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\psi &= (p^1 \otimes \dots \otimes p^n \otimes t^1 \otimes \dots \otimes t^m)\psi_{nm} \\ &+ \sum_{\substack{i_1+\dots+i_k=n\\ j_1+\dots+j_k=m}}^{k>1} (-)^\sigma (p^1 \otimes \dots \otimes p^{i_1} \otimes t^1 \otimes \dots \otimes t^{j_1})\psi_{i_1 j_1} \\ &\otimes (p^{i_1+1} \otimes \dots \otimes p^{i_1+i_2} \otimes t^{j_1+1} \otimes \dots \otimes t^{j_1+j_2})\psi_{i_2 j_2} \\ \otimes \dots \otimes (p^{i_1+\dots+i_{k-1}+1} &\otimes \dots \otimes p^{i_1+\dots+i_k} \otimes t^{j_1+\dots+j_{k-1}+1} \otimes \dots \otimes t^{j_1+\dots+j_k})\psi_{i_k j_k}\end{aligned}$$

defines uniquely an element of $T\,\mathrm{Coder}(\mathcal{A}, \mathcal{B})$. The above formula implies that ψ is a homomorphism.

A generalization to $q > 2$ is straightforward. □

We interpret the above proposition as the existence of inner hom-objects $\mathsf{Hom}(T\mathcal{A}, T\mathcal{B}) = T\,\mathrm{Coder}(\mathcal{A}, \mathcal{B})$ in the monoidal category of cocategories of the form $T\mathcal{C}^1 \otimes T\mathcal{C}^2 \otimes \dots \otimes T\mathcal{C}^r$.

4. A category enriched in cocategories

Let us show that the category of tensor coalgebras of graded $\Bbbk$-quivers is enriched in gCoCat.

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be graded $\Bbbk$-quivers. Consider the cocategory homomorphism given by the upper right path in the diagram

$$\begin{tikzcd} T\mathcal{A} \otimes T\operatorname{Coder}(\mathcal{A},\mathcal{B}) \otimes T\operatorname{Coder}(\mathcal{B},\mathcal{C}) \arrow[r, "\alpha\otimes 1"] \arrow[d, "1\otimes M"'] \arrow[dr, phantom, "="] & T\mathcal{B} \otimes T\operatorname{Coder}(\mathcal{B},\mathcal{C}) \arrow[d, "\alpha"] \\ T\mathcal{A} \otimes T\operatorname{Coder}(\mathcal{A},\mathcal{C}) \arrow[r, "\alpha"] & T\mathcal{C} \end{tikzcd} \tag{4.0.1}$$

By Proposition 3.4 there is a graded cocategory morphism of degree 0

$$M : T\operatorname{Coder}(\mathcal{A},\mathcal{B}) \otimes T\operatorname{Coder}(\mathcal{B},\mathcal{C}) \to T\operatorname{Coder}(\mathcal{A},\mathcal{C}).$$

Denote by $\mathbb{1}$ a graded 1-object-0-morphisms $\Bbbk$-quiver, that is, $\operatorname{Ob}\mathbb{1} = \{*\}$, $\mathbb{1}(*,*) = 0$. Then $T\mathbb{1} = \Bbbk$ is a unit object of the monoidal category of graded cocategories. Denote by $\mathbf{r} : T\mathcal{A} \otimes T\mathbb{1} \to T\mathcal{A}$ and $\mathbf{l} : T\mathbb{1} \otimes T\mathcal{A} \to T\mathcal{A}$ the corresponding natural cocategory isomorphisms. By Proposition 3.4 there exists a unique cocategory morphism $\eta_\mathcal{A} : T\mathbb{1} \to T\operatorname{Coder}(\mathcal{A},\mathcal{A})$, such that

$$\mathbf{r} = \bigl(T\mathcal{A} \otimes T\mathbb{1} \xrightarrow{1\otimes\eta_\mathcal{A}} T\mathcal{A} \otimes T\operatorname{Coder}(\mathcal{A},\mathcal{A}) \xrightarrow{\alpha} T\mathcal{A}\bigr).$$

Namely, the object $* \in \operatorname{Ob}\mathbb{1}$ goes to the identity homomorphism $\mathrm{id}_\mathcal{A} : \mathcal{A} \to \mathcal{A}$, which acts as the identity map on objects, and has only one non-vanishing component $(\mathrm{id}_\mathcal{A})_1 = \mathrm{id} : \mathcal{A}(X,Y) \to \mathcal{A}(X,Y)$.

4.1 Proposition (See also Kontsevich and Soibelman [KS]). *The multiplication M is associative and η is its two-sided unit:*

$$\begin{tikzcd} T\operatorname{Coder}(\mathcal{A},\mathcal{B}) \otimes T\operatorname{Coder}(\mathcal{B},\mathcal{C}) \otimes T\operatorname{Coder}(\mathcal{C},\mathcal{D}) \arrow[r, "M\otimes 1"] \arrow[d, "1\otimes M"'] & T\operatorname{Coder}(\mathcal{A},\mathcal{C}) \otimes T\operatorname{Coder}(\mathcal{C},\mathcal{D}) \arrow[d, "M"] \\ T\operatorname{Coder}(\mathcal{A},\mathcal{B}) \otimes T\operatorname{Coder}(\mathcal{B},\mathcal{D}) \arrow[r, "M"] & T\operatorname{Coder}(\mathcal{A},\mathcal{D}) \end{tikzcd}$$

Proof. The cocategory homomorphism

$$\begin{aligned} &T\mathcal{A} \otimes T\operatorname{Coder}(\mathcal{A},\mathcal{B}) \otimes T\operatorname{Coder}(\mathcal{B},\mathcal{C}) \otimes T\operatorname{Coder}(\mathcal{C},\mathcal{D}) \\ &\xrightarrow{\alpha\otimes1\otimes1} T\mathcal{B} \otimes T\operatorname{Coder}(\mathcal{B},\mathcal{C}) \otimes T\operatorname{Coder}(\mathcal{C},\mathcal{D}) \xrightarrow{\alpha\otimes 1} T\mathcal{C} \otimes T\operatorname{Coder}(\mathcal{C},\mathcal{D}) \xrightarrow{\alpha} T\mathcal{D} \end{aligned}$$

can be written down as

$$(\alpha\otimes1\otimes1)(1\otimes M)\alpha = (1\otimes1\otimes M)(\alpha\otimes1)\alpha = (1\otimes1\otimes M)(1\otimes M)\alpha,$$

or as

$$(1\otimes M\otimes1)(\alpha\otimes1)\alpha = (1\otimes M\otimes1)(1\otimes M)\alpha.$$

The uniqueness part of Proposition 3.4 implies that $(1\otimes M)M = (M\otimes1)M$.

Similarly one proves that η is a unit for M. $\square$

By (2.4.1) we find

$$\mathrm{gCoCat}(T\mathbb{1}, T\,\mathrm{Coder}(\mathcal{A},\mathcal{B})) = \mathrm{Maps}(\{*\}, \mathrm{Ob}\,\mathrm{Coder}(\mathcal{A},\mathcal{B})) = \mathrm{gCoCat}(T\mathcal{A}, T\mathcal{B}).$$

Thus we can interpret Proposition 4.1 as saying that the category of tensor coalgebras of graded $\Bbbk$-quivers admits an enrichment in gCoCat.

Let us find explicit formulas for M. It is defined on objects as composition: if $f:\mathcal{A}\to\mathcal{B}$ and $g:\mathcal{B}\to\mathcal{C}$ are cocategory morphisms, then $(f,g)M = fg:\mathcal{A}\to\mathcal{C}$. On coderivations M is specified by its composition with $\mathrm{pr}_1 : T\,\mathrm{Coder}(\mathcal{A},\mathcal{C}) \to \mathrm{Coder}(\mathcal{A},\mathcal{C})$. Let us write in this section (h,k) as a shorthand for $\mathrm{Coder}(\mathcal{A},\mathcal{C})(h,k)$, the $\Bbbk$-module of (h,k)-coderivations. The components of M are

$$M_{nm} = M\big|_{T^n\otimes T^m}\mathrm{pr}_1 : T^n\,\mathrm{Coder}(\mathcal{A},\mathcal{B})\otimes T^m\,\mathrm{Coder}(\mathcal{B},\mathcal{C}) \to \mathrm{Coder}(\mathcal{A},\mathcal{C}),$$
$$M_{nm} : (f^0,f^1)\otimes\cdots\otimes(f^{n-1},f^n)\otimes(g^0,g^1)\otimes\cdots\otimes(g^{m-1},g^m) \to (f^0g^0, f^ng^m),$$

where $f^0,\dots,f^n:\mathcal{A}\to\mathcal{B}$ and $g^0,\dots,g^m:\mathcal{B}\to\mathcal{C}$ are cocategory morphisms. We have $M_{00}=0$.

According to proof of Proposition 3.4 the component M_{nm} is determined recursively from equation (3.4.2):

$$\begin{aligned}
(p^1\otimes\cdots\otimes p^n)\theta(t^1\otimes\cdots\otimes t^m)\theta &= (p^1\otimes\cdots\otimes p^n\otimes t^1\otimes\cdots\otimes t^m)M_{nm} \\
&+ \sum_{\substack{i_1+\cdots+i_k=n\\ j_1+\cdots+j_k=m}}^{k>1} (-)^\sigma\big[(p^1\otimes\cdots\otimes p^{i_1}\otimes t^1\otimes\cdots\otimes t^{j_1})M_{i_1j_1} \\
&\otimes(p^{i_1+1}\otimes\cdots\otimes p^{i_1+i_2}\otimes t^{j_1+1}\otimes\cdots\otimes t^{j_1+j_2})M_{i_2j_2} \\
\otimes\cdots\otimes(p^{i_1+\cdots+i_{k-1}+1}&\otimes\cdots\otimes p^{i_1+\cdots+i_k}\otimes t^{j_1+\cdots+j_{k-1}+1}\otimes\cdots\otimes t^{j_1+\cdots+j_k})M_{i_kj_k}\big]\theta.
\end{aligned} \tag{4.1.1}$$

Since $(p^1\otimes\cdots\otimes p^n\otimes t^1\otimes\cdots\otimes t^m)M_{nm}$ is a coderivation, it is determined by its composition with projection pr_1. Composing (4.1.1) with pr_1 we get

$$(p^1\otimes\cdots\otimes p^n)\theta(t^1\otimes\cdots\otimes t^m)\theta\,\mathrm{pr}_1 = (p^1\otimes\cdots\otimes p^n\otimes t^1\otimes\cdots\otimes t^m)M_{nm}\,\mathrm{pr}_1\,. \tag{4.1.2}$$

Therefore, if $m=0$ and n is positive, M_{n0} is given by the formula:

$$\begin{aligned}
M_{n0} : (f^0,f^1)\otimes\cdots\otimes(f^{n-1},f^n)\otimes\Bbbk_{g^0} &\to (f^0g^0, f^ng^0), \\
r^1\otimes\cdots\otimes r^n\otimes 1 &\mapsto (r^1\otimes\cdots\otimes r^n \mid g^0)M_{n0},
\end{aligned}$$

$$[(r^1\otimes\cdots\otimes r^n\mid g^0)M_{n0}]\,\mathrm{pr}_1 = (r^1\otimes\cdots\otimes r^n)\theta g^0\,\mathrm{pr}_1,$$

where $\mid$ separates the arguments in place of $\otimes$. If $m=1$, then M_{n1} is given by the formula:

$$\begin{aligned}
M_{n1} : (f^0,f^1)\otimes\cdots\otimes(f^{n-1},f^n)\otimes(g^0,g^1) &\to (f^0g^0, f^ng^1), \\
r^1\otimes\cdots\otimes r^n\otimes t^1 &\mapsto (r^1\otimes\cdots\otimes r^n\otimes t^1)M_{n1},
\end{aligned}$$

$$[(r^1\otimes\cdots\otimes r^n\otimes t^1)M_{n1}]\,\mathrm{pr}_1 = (r^1\otimes\cdots\otimes r^n)\theta t^1\,\mathrm{pr}_1\,.$$

Explicitly we write

$$[(r^1 \otimes \dots \otimes r^n \mid g^0)M_{n0}]_k = \sum_l (r^1 \otimes \dots \otimes r^n)\theta_{kl} g^0_l, \tag{4.1.3}$$

$$[(r^1 \otimes \dots \otimes r^n \otimes t^1)M_{n1}]_k = \sum_l (r^1 \otimes \dots \otimes r^n)\theta_{kl} t^1_l. \tag{4.1.4}$$

Finally, $M_{nm} = 0$ for $m > 1$, since the left hand side of (4.1.2) vanishes.

4.2 Examples. 1) The component M_{01} is the composition: $(f^0 \mid t^1)M_{01} = f^0 t^1$.

2) The component M_{10} is the composition: $(r^1 \mid g^0)M_{10} = r^1 g^0$.

3) If $r : f \to g : \mathcal{A} \to \mathcal{B}$ and $p : h \to k : \mathcal{B} \to \mathcal{C}$ are A_∞-transformations, then $(r \otimes p)M_{11} : fh \to gk : \mathcal{A} \to \mathcal{C}$ has the following components:

$$[(r \otimes p)M_{11}]_0 = r_0 p_1,$$
$$[(r \otimes p)M_{11}]_1 = r_1 p_1 + (f_1 \otimes r_0)p_2 + (r_0 \otimes g_1)p_2, \text{ etc.}$$

4) If $f \xrightarrow{r} g \xrightarrow{p} h : \mathcal{A} \to \mathcal{B}$ are A_∞-transformations, and $k : \mathcal{B} \to \mathcal{C}$ is an A_∞-functor, then $(r \otimes p \mid k)M_{20} : fk \to hk : \mathcal{A} \to \mathcal{C}$ has the following components:

$$[(r \otimes p \mid k)M_{20}]_0 = (r_0 \otimes p_0)k_2,$$
$$[(r \otimes p \mid k)M_{20}]_1 = (r_1 \otimes p_0)k_2 + (r_0 \otimes p_1)k_2 \\ + (r_0 \otimes p_0 \otimes h_1)k_3 + (r_0 \otimes g_1 \otimes p_0)k_3 + (f_1 \otimes r_0 \otimes p_0)k_3, \text{ etc.}$$

5) If $f \xrightarrow{r} g \xrightarrow{p} h : \mathcal{A} \to \mathcal{B}$ and $t : k \to l : \mathcal{B} \to \mathcal{C}$ are A_∞-transformations, then $(r \otimes p \otimes t)M_{21} : fk \to hl : \mathcal{A} \to \mathcal{C}$ has the following components:

$$[(r \otimes p \otimes t)M_{21}]_0 = (r_0 \otimes p_0)t_2,$$
$$[(r \otimes p \otimes t)M_{21}]_1 = (r_1 \otimes p_0)t_2 + (r_0 \otimes p_1)t_2 \\ + (r_0 \otimes p_0 \otimes h_1)t_3 + (r_0 \otimes g_1 \otimes p_0)t_3 + (f_1 \otimes r_0 \otimes p_0)t_3, \text{ etc.}$$

5. A_∞-category of A_∞-functors

Let us construct a new A_∞-category $A_\infty(\mathcal{A}, \mathcal{B})$ out of given two $\mathcal{A}$ and $\mathcal{B}$. Its underlying graded $\Bbbk$-quiver is a full subquiver of $s^{-1}\operatorname{Coder}(s\mathcal{A}, s\mathcal{B})$. The objects of $A_\infty(\mathcal{A}, \mathcal{B})$ are A_∞-functors $f : \mathcal{A} \to \mathcal{B}$. Given two such functors $f, g : \mathcal{A} \to \mathcal{B}$ we define the graded $\Bbbk$-module $A_\infty(\mathcal{A}, \mathcal{B})(f, g)$ as the space of all A_∞-transformations $r : f \to g$, namely,

$$[A_\infty(\mathcal{A}, \mathcal{B})(f, g)]^{d+1} = \\ \{r : f \to g \mid A_\infty\text{-transformation } r : Ts\mathcal{A} \to Ts\mathcal{B} \text{ has degree } d\}.$$

In this section we use the notation $(f, g) = sA_\infty(\mathcal{A}, \mathcal{B})(f, g) = \operatorname{Coder}(s\mathcal{A}, s\mathcal{B})(f, g)$ for the sake of brevity. The degree of r as an element of (f, g) will be exactly d:

$$(f, g)^d = \{r : f \to g \mid A_\infty\text{-transformation } r : Ts\mathcal{A} \to Ts\mathcal{B} \text{ has degree } d\}.$$

We will use only this (natural) degree of r in order to permute it with other things by Koszul's rule.

Notice that even if $\mathcal{A}$, $\mathcal{B}$ have one object (and are A_∞-algebras), the quiver $A_\infty(\mathcal{A},\mathcal{B})$ has several objects. Thus theory of A_∞-algebras leads to the theory of A_∞-categories.

5.1 Proposition (See also Fukaya [Fuk], Kontsevich and Soibelman [KS02, KS] and Lefèvre-Hasegawa [LH02]). *Let $\mathcal{A}$, $\mathcal{B}$ be A_∞-categories. Then there exists a unique (1,1)-coderivation $B : TsA_\infty(\mathcal{A},\mathcal{B}) \to TsA_\infty(\mathcal{A},\mathcal{B})$ of degree 1, such that $B_0 = 0$ and*

$$(r^1\otimes\cdots\otimes r^n)\theta b = [(r^1\otimes\cdots\otimes r^n)B]\theta + (-)^{r^1+\cdots+r^n} b(r^1\otimes\cdots\otimes r^n)\theta \qquad (5.1.1)$$

for all $n \geqslant 0$, $r^1\otimes\cdots\otimes r^n \in (f^0,f^1)\otimes\cdots\otimes(f^{n-1},f^n)$. It satisfies $B^2 = 0$, thus, it gives an A_∞-structure of $A_\infty(\mathcal{A},\mathcal{B})$.

Proof. For $n = 0$ (5.1.1) reads as $f^0b = (f^0)B + bf^0$, hence, $(f^0)B = f^0b - bf^0 = 0$. In particular, we may set $B_0 = 0$. Assume that the coderivation components B_j for $j < n$ are already found, so that (5.1.1) is satisfied up to $n-1$ arguments. Let us determine a $\Bbbk$-linear map $(r^1\otimes\cdots\otimes r^n)B_n : Ts\mathcal{A} \to Ts\mathcal{B}$ from equation (5.1.1), rewritten as follows:

$$\begin{aligned}(r^1\otimes\cdots\otimes r^n)B_n &= (r^1\otimes\cdots\otimes r^n)\theta b - (-)^{r^1+\cdots+r^n} b(r^1\otimes\cdots\otimes r^n)\theta \\ &\quad - \sum_{q+j+t=n}^{j<n} [(r^1\otimes\cdots\otimes r^n)(1^{\otimes q}\otimes B_j\otimes 1^{\otimes t})]\theta. \qquad (5.1.2)\end{aligned}$$

Let us show that $(r^1\otimes\cdots\otimes r^n)B_n$ is a (f^0,f^n)-coderivation. Indeed,

$$\begin{aligned}(r^1\otimes\cdots\otimes r^n)B_n\Delta_{\mathcal{B}} &= (r^1\otimes\cdots\otimes r^n)\theta b\Delta_{\mathcal{B}} - (-)^{r^1+\cdots+r^n} b(r^1\otimes\cdots\otimes r^n)\theta\Delta_{\mathcal{B}} \\ &\quad - \sum_{q+j+t=n}^{j<n} [(r^1\otimes\cdots\otimes r^n)(1^{\otimes q}\otimes B_j\otimes 1^{\otimes t})]\theta\Delta_{\mathcal{B}} \\ &= (r^1\otimes\cdots\otimes r^n)\theta\Delta_{\mathcal{B}}(1\otimes b + b\otimes 1) \\ &\quad - (-)^{r^1+\cdots+r^n} b\Delta_{\mathcal{A}} \sum_{k=0}^{n} (r^1\otimes\cdots\otimes r^k)\theta\otimes(r^{k+1}\otimes\cdots\otimes r^n)\theta \\ &\quad - \sum_{q+j+t=n}^{j<n} \Delta_{\mathcal{A}}\{[(r^1\otimes\cdots\otimes r^n)(1^{\otimes q}\otimes B_j\otimes 1^{\otimes t})]\Delta_{A_\infty(\mathcal{A},\mathcal{B})}(\theta\otimes\theta)\}\end{aligned}$$

$$
\begin{aligned}
&= \Delta_{\mathcal{A}} \Big\{ \sum_{k=0}^{n} (r^1 \otimes \cdots \otimes r^k)\theta \otimes (r^{k+1} \otimes \cdots \otimes r^n)\theta(1 \otimes b + b \otimes 1) \\
&\quad - (-)^{r^1+\cdots+r^n}(1 \otimes b + b \otimes 1) \sum_{k=0}^{n} (r^1 \otimes \cdots \otimes r^k)\theta \otimes (r^{k+1} \otimes \cdots \otimes r^n)\theta \\
&\quad - \sum_{k+v+j+t=n}^{j<n} [(r^1 \otimes \cdots \otimes r^n)(1^{\otimes k} \bigotimes 1^{\otimes v} \otimes B_j \otimes 1^{\otimes t})](\theta \bigotimes \theta) \\
&\quad - \sum_{q+j+w+u=n}^{j<n} [(r^1 \otimes \cdots \otimes r^n)(1^{\otimes q} \otimes B_j \otimes 1^{\otimes w} \bigotimes 1^{\otimes u})](\theta \bigotimes \theta) \Big\} \\
&= \Delta_{\mathcal{A}} \Big\{ \sum_{k=0}^{n} (r^1 \otimes \cdots \otimes r^k)\theta \otimes (r^{k+1} \otimes \cdots \otimes r^n)\theta b \\
&\quad - \sum_{k=0}^{n} (-)^{r^{k+1}+\cdots+r^n} (r^1 \otimes \cdots \otimes r^k)\theta \otimes b(r^{k+1} \otimes \cdots \otimes r^n)\theta \\
&\quad - \sum_{k+v+j+t=n}^{j<n} (r^1 \otimes \cdots \otimes r^k)\theta \otimes (r^{k+1} \otimes \cdots \otimes r^n)(1^{\otimes v} \otimes B_j \otimes 1^{\otimes t})\theta \\
&\quad + \sum_{k=0}^{n} (-)^{r^{k+1}+\cdots+r^n} (r^1 \otimes \cdots \otimes r^k)\theta b \otimes (r^{k+1} \otimes \cdots \otimes r^n)\theta \\
&\quad - \sum_{k=0}^{n} (-)^{r^1+\cdots+r^n} b(r^1 \otimes \cdots \otimes r^k)\theta \otimes (r^{k+1} \otimes \cdots \otimes r^n)\theta \\
&\quad - \sum_{k=0}^{n} (-)^{r^{k+1}+\cdots+r^n} \sum_{q+j+w=k}^{j<n} (r^1 \otimes \cdots \otimes r^k)(1^{\otimes q} \otimes B_j \otimes 1^{\otimes w})\theta \\
&\quad \otimes (r^{k+1} \otimes \cdots \otimes r^n)\theta \Big\} \\
&= \Delta_{\mathcal{A}} \big[f^0 \otimes (r^1 \otimes \cdots \otimes r^n)B_n + (r^1 \otimes \cdots \otimes r^n)B_n \otimes f^n \big].
\end{aligned}
$$

The last three sums cancel out for all $k < n$ due to (5.1.2), and for $k = n$ they give $(r^1 \otimes \cdots \otimes r^n)B_n \otimes f^n$ due to the same equation. Similarly for the previous three sums. Therefore, (5.1.2) is, indeed, a recursive definition of components B_n of a coderivation B. The uniqueness of B is obvious.

Clearly, $B^2 : TsA_\infty(\mathcal{A}, \mathcal{B}) \to TsA_\infty(\mathcal{A}, \mathcal{B})$ is a (1,1)-coderivation of degree 2. From (5.1.1) we find

$$
\begin{aligned}
[(r^1 \otimes \cdots \otimes r^n)B^2]\theta &= [(r^1 \otimes \cdots \otimes r^n)B]\theta b - (-)^{r^1+\cdots+r^n+1} b[(r^1 \otimes \cdots \otimes r^n)B]\theta \\
&= (r^1 \otimes \cdots \otimes r^n)\theta b^2 - (-)^{r^1+\cdots+r^n} b[(r^1 \otimes \cdots \otimes r^n)\theta]b \\
&\quad - (-)^{r^1+\cdots+r^n+1} b[(r^1 \otimes \cdots \otimes r^n)\theta]b - b^2(r^1 \otimes \cdots \otimes r^n)\theta = 0.
\end{aligned}
$$

Composing this equation with $\mathrm{pr}_1 : Ts\mathcal{B} \to s\mathcal{B}$ we get

$$
0 = [(r^1 \otimes \cdots \otimes r^n)B^2]\theta \, \mathrm{pr}_1 = (r^1 \otimes \cdots \otimes r^n)[B^2]_n \, \mathrm{pr}_1 .
$$

Therefore, all components of the (f^0, f^n)-coderivation $(r^1 \otimes \cdots \otimes r^n)[B^2]_n$ vanish. We deduce that the coderivations $(r^1 \otimes \cdots \otimes r^n)[B^2]_n$ vanish, hence, all $[B^2]_n = 0$. Finally, $B^2 = 0$. □

Let us find explicitly the components of B, composing (5.1.1) with $\mathrm{pr}_1 : Ts\mathcal{B} \to s\mathcal{B}$:

$$B_1 : (f, g) \to (f, g), \quad r \mapsto (r)B_1 = [r, b] = rb - (-)^r br,$$
$$B_n : (f^0, f^1) \otimes \cdots \otimes (f^{n-1}, f^n) \to (f^0, f^n),\ r^1 \otimes \cdots \otimes r^n \mapsto (r^1 \otimes \cdots \otimes r^n)B_n, \quad \text{for } n > 1,$$

where the last transformation is defined by its composition with pr_1:

$$[(r^1 \otimes \cdots \otimes r^n)B_n]\,\mathrm{pr}_1 = [(r^1 \otimes \cdots \otimes r^n)\theta]b\,\mathrm{pr}_1\,.$$

In the other terms, for $n > 1$

$$[(r^1 \otimes \cdots \otimes r^n)B_n]_k = \sum_l (r^1 \otimes \cdots \otimes r^n)\theta_{kl}\, b_l. \tag{5.1.3}$$

Since $B^2 = 0$, we have, in particular,

$$\sum_{r+n+t=k} (1^{\otimes r} \otimes B_n \otimes 1^{\otimes t})B_{r+1+t} = 0 : T^k sA_\infty(\mathcal{A}, \mathcal{B}) \to sA_\infty(\mathcal{A}, \mathcal{B}).$$

5.2 Examples. 1) When $n = 1$, $r : f \to g : \mathcal{A} \to \mathcal{B}$, we find the components of the A_∞-transformation $(r)B_1 : f \to g : \mathcal{A} \to \mathcal{B}$ as follows (see Examples 2.7):

$$\begin{aligned}
[(r)B_1]_0 &= r_0 b_1, \\
[(r)B_1]_1 &= r_1 b_1 + (f_1 \otimes r_0)b_2 + (r_0 \otimes g_1)b_2 - (-)^r b_1 r_1, \\
[(r)B_1]_2 &= r_2 b_1 + (f_2 \otimes r_0)b_2 + (f_1 \otimes r_1)b_2 + (r_1 \otimes g_1)b_2 + (r_0 \otimes g_2)b_2 \\
&\quad + (f_1 \otimes f_1 \otimes r_0)b_3 + (f_1 \otimes r_0 \otimes g_1)b_3 + (r_0 \otimes g_1 \otimes g_1)b_3 - (-)^r b_2 r_1 \\
&\quad - (-)^r (1 \otimes b_1)r_2 - (-)^r (b_1 \otimes 1)r_2.
\end{aligned}$$

2) When $n = 2$, $f \xrightarrow{r} g \xrightarrow{p} h : \mathcal{A} \to \mathcal{B}$, we find the components of the A_∞-transformation $(r \otimes p)B_2 : f \to h : \mathcal{A} \to \mathcal{B}$ as follows:

$$\begin{aligned}
[(r \otimes p)B_2]_0 &= (r_0 \otimes p_0)b_2, \\
[(r \otimes p)B_2]_1 &= (r_1 \otimes p_0)b_2 + (r_0 \otimes p_1)b_2 \\
&\quad + (r_0 \otimes p_0 \otimes h_1)b_3 + (r_0 \otimes g_1 \otimes p_0)b_3 + (f_1 \otimes r_0 \otimes p_0)b_3, \\
[(r \otimes p)B_2]_2 &= (r_2 \otimes p_0)b_2 + (r_1 \otimes p_1)b_2 + (r_0 \otimes p_2)b_2 + (r_1 \otimes p_0 \otimes h_1)b_3 \\
&\quad + (r_0 \otimes p_1 \otimes h_1)b_3 + (r_1 \otimes g_1 \otimes p_0)b_3 + (r_0 \otimes g_1 \otimes p_1)b_3 \\
&\quad + (f_1 \otimes r_1 \otimes p_0)b_3 + (f_1 \otimes r_0 \otimes p_1)b_3 + (r_0 \otimes p_0 \otimes h_2)b_3 \\
&\quad + (r_0 \otimes g_2 \otimes p_0)b_3 + (f_2 \otimes r_0 \otimes p_0)b_3 + (r_0 \otimes p_0 \otimes h_1 \otimes h_1)b_4 \\
&\quad + (r_0 \otimes g_1 \otimes g_1 \otimes p_0)b_4 + (r_0 \otimes g_1 \otimes p_0 \otimes h_1)b_4 \\
&\quad + (f_1 \otimes r_0 \otimes p_0 \otimes h_1)b_4 + (f_1 \otimes r_0 \otimes g_1 \otimes p_0)b_4 \\
&\quad + (f_1 \otimes f_1 \otimes r_0 \otimes p_0)b_4.
\end{aligned}$$

3) When $n = 3$, $f \xrightarrow{r} g \xrightarrow{p} h \xrightarrow{t} k : \mathcal{A} \to \mathcal{B}$, we find the components of the A_∞-transformation $(r \otimes p \otimes t)B_3 : f \to k : \mathcal{A} \to \mathcal{B}$ as follows:

$$
\begin{aligned}
[(r \otimes p \otimes t)B_3]_0 &= (r_0 \otimes p_0 \otimes t_0)b_3, \\
[(r \otimes p \otimes t)B_3]_1 &= (r_1 \otimes p_0 \otimes t_0)b_3 + (r_0 \otimes p_1 \otimes t_0)b_3 + (r_0 \otimes p_0 \otimes t_1)b_3 \\
&\quad + (r_0 \otimes p_0 \otimes t_0 \otimes k_1)b_4 + (r_0 \otimes p_0 \otimes h_1 \otimes t_0)b_4 \\
&\quad + (r_0 \otimes g_1 \otimes p_0 \otimes t_0)b_4 + (f_1 \otimes r_0 \otimes p_0 \otimes t_0)b_4.
\end{aligned}
$$

5.3. Differentials. Let $\mathcal{A}$, $\mathcal{C}^1$, $\mathcal{C}^2$, ..., $\mathcal{C}^q$, $\mathcal{B}$ be A_∞-categories. Let $\phi : Ts\mathcal{A} \otimes Ts\mathcal{C}^1 \otimes Ts\mathcal{C}^2 \otimes \cdots \otimes Ts\mathcal{C}^q \to Ts\mathcal{B}$, $(a \otimes c^1 \otimes c^2 \otimes \cdots \otimes c^q) \mapsto a.(c^1 \otimes c^2 \otimes \cdots \otimes c^q)\chi$ be a cocategory homomorphism of degree 0. If the homomorphism ϕ commutes with the differential:

$$\phi b = \Big(\sum_{r+t=q} 1^{\otimes r} \otimes b \otimes 1^{\otimes t} \Big) \phi,$$

then ϕ is called an A_∞-functor (in a generalized sense, extending Definition 2.4). This condition is fulfilled if and only if χ commutes with the differential:

$$
\begin{aligned}
&(c^1 \otimes c^2 \otimes \cdots \otimes c^q)\chi b \\
&= \sum_{k=1}^{q} (-)^{c^{k+1}+\cdots+c^q} (c^1 \otimes \cdots \otimes c^k b \otimes \cdots \otimes c^q)\chi + (-)^{c^1+\cdots+c^q} b(c^1 \otimes c^2 \otimes \cdots \otimes c^q)\chi.
\end{aligned}
\tag{5.3.1}
$$

In particular, for $q = 1$ we get the equation

$$(c)\chi b = (cb)\chi + (-)^c b(c)\chi.$$

5.4 Corollary (to Proposition 5.1). *There is a unique A_∞-category structure for $A_\infty(\mathcal{A}, \mathcal{B})$, such that the action homomorphism $\alpha : Ts\mathcal{A} \otimes TsA_\infty(\mathcal{A}, \mathcal{B}) \to Ts\mathcal{B}$ is an A_∞-functor.*

Proof. The homomorphism $\alpha : a \otimes r^1 \otimes \cdots \otimes r^n \mapsto a[(r^1 \otimes \cdots \otimes r^n)\theta]$ of Corollary 3.3 uses $\chi = \theta$. Hence, α is an A_∞-functor if and only if $(r)\theta b = (rB)\theta + (-)^r b(r)\theta$ for $r = r^1 \otimes \cdots \otimes r^n$, $n \geqslant 0$, that is, if and only if equations (5.1.1) hold. □

5.5 Proposition. *For any A_∞-functor $\phi : Ts\mathcal{A} \otimes Ts\mathcal{C}^1 \otimes Ts\mathcal{C}^2 \otimes \cdots \otimes Ts\mathcal{C}^q \to Ts\mathcal{B}$ there is a unique A_∞-functor $\psi : Ts\mathcal{C}^1 \otimes Ts\mathcal{C}^2 \otimes \cdots \otimes Ts\mathcal{C}^q \to TsA_\infty(\mathcal{A}, \mathcal{B})$, such that*

$$\phi = \big(Ts\mathcal{A} \otimes Ts\mathcal{C}^1 \otimes Ts\mathcal{C}^2 \otimes \cdots \otimes Ts\mathcal{C}^q \xrightarrow{1 \otimes \psi} Ts\mathcal{A} \otimes TsA_\infty(\mathcal{A}, \mathcal{B}) \xrightarrow{\alpha} Ts\mathcal{B} \big). \tag{5.5.1}$$

Proof. If $f^i \in \operatorname{Ob} \mathcal{C}^i$, then (5.3.1) implies that the cocategory homomorphism $(f^1, f^2, \ldots, f^q)\psi = (f^1, f^2, \ldots, f^q)\chi$ of degree 0 commutes with the differential b. Hence, it is an A_∞-functor, that is, an object of $A_\infty(\mathcal{A}, \mathcal{B})$. By Proposition 3.4 there exists a unique cocategory homomorphism $\psi : Ts\mathcal{C}^1 \otimes Ts\mathcal{C}^2 \otimes \cdots \otimes Ts\mathcal{C}^q \to TsA_\infty(\mathcal{A}, \mathcal{B})$, of degree 0, such that (5.5.1) holds. We have to prove that ψ commutes with the differential.

Using (5.1.1) and (5.3.1) we find for $\chi = \psi\theta$ the equation

$$(c^1 \otimes c^2 \otimes \cdots \otimes c^q)\psi B\theta = (c^1 \otimes c^2 \otimes \cdots \otimes c^q)\psi\theta b - (-)^{c^1+\cdots+c^q} b(c^1 \otimes c^2 \otimes \cdots \otimes c^q)\psi\theta$$
$$= (c^1 \otimes c^2 \otimes \cdots \otimes c^q) \sum_{k=1}^{q} (1^{\otimes k-1} \otimes b \otimes 1^{\otimes q-k})\psi\theta. \quad (5.5.2)$$

Notice that ψB and $\kappa \overset{\text{def}}{=} \sum_{k=1}^{q}(1^{\otimes k-1} \otimes b \otimes 1^{\otimes q-k})\psi$ are both (ψ,ψ)-coderivations. Let c^i be an element of $T^{n^i} s\mathcal{C}^i = s\mathcal{C}^i(f^i,-) \otimes \cdots \otimes s\mathcal{C}^i(-,g^i)$ for $1 \leqslant i \leqslant q$. Composing (5.5.2) with $\mathrm{pr}_1 : Ts\mathcal{B} \to s\mathcal{B}$ we get

$$(c^1 \otimes c^2 \otimes \cdots \otimes c^q)[\psi B]_{n^1\dots n^q} \mathrm{pr}_1 = (c^1 \otimes c^2 \otimes \cdots \otimes c^q)\psi B\theta \,\mathrm{pr}_1$$
$$= (c^1 \otimes c^2 \otimes \cdots \otimes c^q)\kappa\theta \,\mathrm{pr}_1 = (c^1 \otimes c^2 \otimes \cdots \otimes c^q)\kappa_{n^1\dots n^q} \mathrm{pr}_1 .$$

Since all components of $((f^1,f^2,\dots,f^q)\psi,(g^1,g^2,\dots,g^q)\psi)$-coderivations $(c^1 \otimes c^2 \otimes \cdots \otimes c^q)[\psi B]_{n^1\dots n^q}$ and $(c^1 \otimes c^2 \otimes \cdots \otimes c^q)\kappa_{n^1\dots n^q}$ coincide, these coderivations coincide as well. Therefore, all the components of (ψ,ψ)-coderivations ψB and κ coincide. We conclude that these coderivations coincide as well: $\psi B = \kappa = \sum_{k=1}^{q}(1^{\otimes k-1} \otimes b \otimes 1^{\otimes q-k})\psi$. □

6. Enriched category of A_∞-categories

6.1 Definition (Differential graded cocategory). A differential graded cocategory $\mathcal{C}$ is a graded cocategory equipped with a (1,1)-coderivation $b = \bigl(b : \mathcal{C}(X,Y) \to \mathcal{C}(X,Y)\bigr)_{X,Y\in\operatorname{Ob}\mathcal{C}}$ of degree 1, such that $b^2 = 0$.

As in Section 2.2 a differential graded cocategory can be identified with a differential graded $\Bbbk$-coalgebra, decomposed in a special way. An example of a differential graded cocategory is given by $Ts\mathcal{A}$, where $\mathcal{A}$ is an A_∞-category.

Differential graded cocategories form a symmetric monoidal category dgCoCat. If $\mathcal{C}$ and $\mathcal{D}$ are differential graded cocategories then their tensor product is the graded cocategory $\mathcal{C} \otimes \mathcal{D}$, equipped with the differential $1 \otimes b + b \otimes 1$. We want to show that the category of A_∞-categories is enriched in dgCoCat.

Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be A_∞-categories. There is a graded cocategory morphism of degree 0

$$M : TsA_\infty(\mathcal{A},\mathcal{B}) \otimes TsA_\infty(\mathcal{B},\mathcal{C}) \to TsA_\infty(\mathcal{A},\mathcal{C}),$$

defined in Section 4 via diagram (4.0.1). Since all cocategory morphisms α, $\alpha \otimes 1$ in this diagram commute with the differential by Corollary 5.4, the cocategory morphism M also commutes with the differential:

$$(1 \otimes B + B \otimes 1)M = MB \quad (6.1.1)$$

by Proposition 5.5. Therefore, M is an A_∞-functor. The unit $\eta_{\mathcal{A}} : T\mathbb{1} \to T\,\mathrm{Coder}(s\mathcal{A},s\mathcal{A})$, $1 \mapsto \mathrm{id}_{\mathcal{A}}$ also is an A_∞-functor for trivial reasons. The set

$$\mathrm{dgCoCat}(T\mathbb{1},TsA_\infty(\mathcal{A},\mathcal{B})) = \mathrm{Maps}(\{*\},\operatorname{Ob}A_\infty(\mathcal{A},\mathcal{B})) = \mathrm{dgCoCat}(Ts\mathcal{A},Ts\mathcal{B})$$

is the set of A_∞-functors $\mathcal{A} \to \mathcal{B}$. We summarize the above statements as follows: the category of A_∞-categories is enriched in dgCoCat. Moreover, it is enriched in the monoidal subcategory of dgCoCat generated by $Ts\mathcal{C}$, where $\mathcal{C}$ are A_∞-categories.

Let us apply Proposition 5.5 to the A_∞-functor M. From that result we deduce the existence of a unique A_∞-functor

$$A_\infty(\mathcal{A},\_) : A_\infty(\mathcal{B},\mathcal{C}) \to A_\infty(A_\infty(\mathcal{A},\mathcal{B}), A_\infty(\mathcal{A},\mathcal{C})),$$

such that

$$\begin{aligned} M = \big[&TsA_\infty(\mathcal{A},\mathcal{B}) \otimes TsA_\infty(\mathcal{B},\mathcal{C}) \xrightarrow{1\otimes A_\infty(\mathcal{A},\_)} \\ &TsA_\infty(\mathcal{A},\mathcal{B}) \otimes TsA_\infty(A_\infty(\mathcal{A},\mathcal{B}), A_\infty(\mathcal{A},\mathcal{C})) \xrightarrow{\alpha} TsA_\infty(\mathcal{A},\mathcal{C})\big]. \end{aligned} \tag{6.1.2}$$

Let us find the components of $A_\infty(\mathcal{A},\_)$.

6.2 Proposition. *The A_∞-functor $A_\infty(\mathcal{A},\_)$ is strict. It maps an object of $A_\infty(\mathcal{B},\mathcal{C})$, an A_∞-functor $g : \mathcal{B} \to \mathcal{C}$, to the object of the target A_∞-category $(1 \otimes g)M : A_\infty(\mathcal{A},\mathcal{B}) \to A_\infty(\mathcal{A},\mathcal{C})$ (which is also an A_∞-functor). The first component $A_\infty(\mathcal{A},\_)_1$ maps an element t of $sA_\infty(\mathcal{B},\mathcal{C})(g,h)$, a (g,h)-coderivation $t : Ts\mathcal{B} \to Ts\mathcal{C}$, to the $((1\otimes g)M,(1\otimes h)M)$-coderivation $(1\otimes t)M : TsA_\infty(\mathcal{A},\mathcal{B}) \to TsA_\infty(\mathcal{A},\mathcal{C})$, an element of $sA_\infty(A_\infty(\mathcal{A},\mathcal{B}), A_\infty(\mathcal{A},\mathcal{C}))((1 \otimes g)M,(1 \otimes h)M)$.*

Proof. Clearly, $A_\infty(\mathcal{A},\_)$ gives the mapping of objects $g \mapsto (1 \otimes g)M$ as described. To prove that $A_\infty(\mathcal{A},\_)_1 : t \mapsto (1 \otimes t)M$ and $A_\infty(\mathcal{A},\_)_k = 0$ for $k > 1$ it suffices to substitute a cocategory homomorphism with those components into (6.1.2) and to check this equation (see Proposition 3.4). Let

$$\begin{aligned} &p^1 \otimes \cdots \otimes p^n \in sA_\infty(\mathcal{A},\mathcal{B})(f^0, f^1) \otimes \cdots \otimes sA_\infty(\mathcal{A},\mathcal{B})(f^{n-1}, f^n), \\ &t^1 \otimes \cdots \otimes t^m \in sA_\infty(\mathcal{B},\mathcal{C})(g^0, g^1) \otimes \cdots \otimes sA_\infty(\mathcal{B},\mathcal{C})(g^{m-1}, g^m). \end{aligned} \tag{6.2.1}$$

The equation to check is

$$(p^1 \otimes \cdots \otimes p^n \otimes t^1 \otimes \cdots \otimes t^m)M = (p^1 \otimes \cdots \otimes p^n).[(t^1 \otimes \cdots \otimes t^m)A_\infty(\mathcal{A},\_)]\theta,$$

that is,

$$(p^1 \otimes \cdots \otimes p^n \otimes t^1 \otimes \cdots \otimes t^m)M = (p^1 \otimes \cdots \otimes p^n).[(1 \otimes t^1)M \otimes \cdots \otimes (1 \otimes t^m)M]\theta.$$

The left hand side is a cocategory homomorphism. Let us prove that the right hand side

$$(p^1 \otimes \cdots \otimes p^n \otimes t^1 \otimes \cdots \otimes t^m)L \overset{\text{def}}{=} (p^1 \otimes \cdots \otimes p^n).[(1 \otimes t^1)M \otimes \cdots \otimes (1 \otimes t^m)M]\theta$$

is also a cocategory homomorphism. Indeed,

$$
\begin{aligned}
&(p^1 \otimes \cdots \otimes p^n \otimes t^1 \otimes \cdots \otimes t^m)L\Delta \\
&= (p^1 \otimes \cdots \otimes p^n).[(1 \otimes t^1)M \otimes \cdots \otimes (1 \otimes t^m)M]\theta\Delta \\
&= (p^1 \otimes \cdots \otimes p^n)\Delta \sum_{k=0}^{m} [(1 \otimes t^1)M \otimes \cdots \otimes (1 \otimes t^k)M]\theta \\
&\qquad\qquad\qquad\qquad \otimes [(1 \otimes t^{k+1})M \otimes \cdots \otimes (1 \otimes t^m)M]\theta \\
&= \sum_{l=0}^{n} \sum_{k=0}^{m} (-)^{(p^{l+1}+\cdots+p^n)(t^1+\cdots+t^k)} (p^1 \otimes \cdots \otimes p^l).[(1 \otimes t^1)M \otimes \cdots \otimes (1 \otimes t^k)M]\theta \\
&\qquad\qquad\qquad\qquad \otimes (p^{l+1} \otimes \cdots \otimes p^n).[(1 \otimes t^{k+1})M \\
&\qquad\qquad\qquad\qquad \otimes \cdots \otimes (1 \otimes t^m)M]\theta \\
&= [(p^1 \otimes \cdots \otimes p^n)\Delta \otimes (t^1 \otimes \cdots \otimes t^m)\Delta](1 \otimes c \otimes 1)(L \otimes L)
\end{aligned}
$$

by Proposition 3.1.

Let us prove that the components of M and L coincide. For $m = 0$ and any $n > 0$ we have

$$(p^1 \otimes \cdots \otimes p^n \mid g^0)L_{n0} = (p^1 \otimes \cdots \otimes p^n).(1 \otimes g^0)M \operatorname{pr}_1 = (p^1 \otimes \cdots \otimes p^n \mid g^0)M_{n0},$$

hence, $L_{n0} = M_{n0}$. For $m = 1$ and any $n \geqslant 0$ we have

$$(p^1 \otimes \cdots \otimes p^n \otimes t^1)L_{n1} = (p^1 \otimes \cdots \otimes p^n).(1 \otimes t^1)M \operatorname{pr}_1 = (p^1 \otimes \cdots \otimes p^n \otimes t^1)M_{n1},$$

hence, $L_{n1} = M_{n1}$. For $m > 1$ and any $n \geqslant 0$ we have $L_{nm} = 0$ and $M_{nm} = 0$. Therefore, $L = M$ and the proposition is proved. $\square$

6.3 Corollary. *For all $m > 0$ and all $t^1 \otimes \cdots \otimes t^m$ as in (6.2.1) we have*

$$[(1 \otimes t^1)M \otimes \cdots \otimes (1 \otimes t^m)M]\tilde{B}_m = [1 \otimes (t^1 \otimes \cdots \otimes t^m)B_m]M, \tag{6.3.1}$$

where $\tilde{B}$ denotes the differential in $TsA_\infty(A_\infty(\mathcal{A}, \mathcal{B}), A_\infty(\mathcal{A}, \mathcal{C}))$.

Indeed, the general property of an A_∞-functor $A_\infty(\mathcal{A}, \_)\tilde{B} = BA_\infty(\mathcal{A}, \_)$ reduces to the above formula, since $A_\infty(\mathcal{A}, \_)$ is strict.

In the following definition we introduce A_∞-analogs of natural transformations.

6.4 Definition (ω-globular set of A_∞-categories). A *natural A_∞-transformation* $r : f \to g : \mathcal{A} \to \mathcal{B}$ (natural transformation in terms of [**Fuk**]) is an A_∞-transformation of degree -1 such that $rb + br = 0$ (that is, $(r)B_1 = 0$). The ω-globular set [**Bat98**] A_ω of A_∞-categories is defined as follows: objects (0-morphisms) are A_∞-categories $\mathcal{A}$; 1-morphisms are A_∞-functors $f : \mathcal{A} \to \mathcal{B}$; 2-morphisms are natural A_∞-transformations $r : f \to g : \mathcal{A} \to \mathcal{B}$; 3-morphisms $\lambda : r \to s : f \to g : \mathcal{A} \to \mathcal{B}$ are (f, g)-coderivations of degree -2, such that $r - s = [\lambda, b]$; for $n \geqslant 3$ n-morphisms $\lambda_n : \lambda_{n-1} \to \mu_{n-1} : \cdots : r \to s : f \to g : \mathcal{A} \to \mathcal{B}$ are (f, g)-coderivations of degree $1 - n$, such that $\lambda_{n-1} - \mu_{n-1} = [\lambda_n, b]$ (notice that the both sides are (f, g)-coderivations of degree $2 - n$).

6.5 Remark. Let us notice that the A_∞-functor $A_\infty(\mathcal{A}, \_)$ from Proposition 6.2 defines a map of the ω-globular set A_ω into itself. Indeed, objects $\mathcal{B}$ of A_ω are mapped into objects $A_\infty(\mathcal{A}, \mathcal{B})$, 1-morphisms $g : \mathcal{B} \to \mathcal{C}$ are mapped into 1-morphisms $(1 \otimes g)M : A_\infty(\mathcal{A}, \mathcal{B}) \to A_\infty(\mathcal{A}, \mathcal{C})$ and the first component $A_\infty(\mathcal{A}, \_)_1$ maps (g, h)-coderivations into $((1 \otimes g)M, (1 \otimes h)M)$-coderivations. Moreover, if the equation $\lambda_{n-1} - \mu_{n-1} = \lambda_n B_1$ holds for (g, h)-coderivations, then $(1 \otimes \lambda_{n-1})M - (1 \otimes \mu_{n-1})M = (1 \otimes \lambda_n)M\tilde{B}_1$ by (6.3.1), so the sources and the targets are preserved.

It might be useful to turn the ω-globular set A_ω into a weak non-unital ω-category in the sense of some of the existing definitions of the latter. Plenty of such definitions including [**Bat98**] are listed in Leinster's survey [**Lei02**]. We do not try to proceed in this direction. Instead we truncate the ω-globular set to a 2-globular set (that is, we deal with 0-, 1- and 2-morphisms) and we make a 2-category out of it.

7. 2-categories of A_∞-categories

Let $\mathcal{K}$ denote the category $\mathsf{K}(\Bbbk\text{-mod}) = H^0(\mathsf{C}(\Bbbk\text{-mod}))$ of differential graded complexes of $\Bbbk$-modules, whose morphisms are chain maps modulo homotopy. Equipped with the usual tensor product, the unit object $\Bbbk$ and the standard symmetry, $\mathcal{K}$ becomes a $\Bbbk$-linear closed monoidal symmetric category. The inner hom-object is the usual $\mathrm{Hom}^\bullet_\Bbbk(\text{-}, \text{-})$. There is a notion of a category $\mathcal{C}$ enriched in $\mathcal{K}$ ($\mathcal{K}$-categories, $\mathcal{K}$-functors, $\mathcal{K}$-natural transformations), see Kelly [**Kel82**]: for all objects X, Y of $\mathcal{C}$ $\mathcal{C}(X, Y)$ is an object of $\mathcal{K}$. There is a similar notion of a 2-category enriched in $\mathcal{K}$, or a $\mathcal{K}$-2-category: it consist of a class of objects $\mathrm{Ob}\,\mathcal{C}$, a class of 1-morphisms $\mathcal{C}(X, Y)$ for each pair of objects X, Y of $\mathcal{A}$, an object of 2-morphisms $\mathcal{C}(X, Y)(f, g) \in \mathrm{Ob}\,\mathcal{K}$ for each pair of 1-morphisms $f, g \in \mathcal{C}(X, Y)$ and other data. We shall consider 1-unital, non-2-unital $\mathcal{K}$-2-categories. They are equipped with the following operations: associative composition of 1-morphisms, commuting left and right associative actions of 1-morphisms on 2-morphisms (these actions are morphisms in $\mathcal{K}$), 1-units, associative vertical composition of 2-morphisms (a morphism in $\mathcal{K}$) compatible with the left and right actions of 1-morphisms on 2-morphisms and such that the both ways to obtain the horizontal composition coincide. Precise definitions are given in Appendix A.

7.1 Proposition. *The following data define a 1-unital, non-2-unital $\mathcal{K}$-2-category $\mathcal{K}A_\infty$:*

- *objects are A_∞-categories;*
- *1-morphisms are A_∞-functors;*
- *an object of 2-morphisms between $f, g : \mathcal{A} \to \mathcal{B}$ is $(A_\infty(\mathcal{A}, \mathcal{B})(f, g), m_1) \in \mathrm{Ob}\,\mathcal{K}$, $m_1 = sB_1s^{-1}$;*
- *the composition of 1-morphisms is the composition of A_∞-functors;*
- *unit 1-morphisms are identity A_∞-functors;*
- *the right action of a 1-morphism $k : \mathcal{B} \to \mathcal{C}$ on 2-morphisms is the chain map $(A_\infty(\mathcal{A}, \mathcal{B})(f, g), m_1) \to (A_\infty(\mathcal{A}, \mathcal{C})(fk, gk), m_1)$, $rs^{-1} \mapsto (rs^{-1})\cdot k = (rk)s^{-1}$, where r is an (f, g)-coderivation;*

- *the left action of a 1-morphism $e : \mathcal{D} \to \mathcal{A}$ on 2-morphisms is the chain map $(A_\infty(\mathcal{A},\mathcal{B})(f,g), m_1) \to (A_\infty(\mathcal{D},\mathcal{B})(ef,eg), m_1)$, $rs^{-1} \mapsto e\cdot(rs^{-1}) = (er)s^{-1}$, where r is an (f,g)-coderivation;*
- *the vertical composition is the chain map $m_2 = (s\otimes s)B_2 s^{-1} : A_\infty(\mathcal{A},\mathcal{B})(f,g)\otimes A_\infty(\mathcal{A},\mathcal{B})(g,h) \to A_\infty(\mathcal{A},\mathcal{B})(f,h)$.*

Proof. Clearly, the composition of 1-morphisms and the actions of 1-morphisms on 2-morphisms are associative. The right and the left actions are unital, and commute with each other. The equation $-(1\otimes m_1 + m_1\otimes 1)m_2 + m_2 m_1 = 0$ (see (2.3.2) for $k=2$) shows that m_2 is a chain map. The equation

$$m_3 m_1 + (1\otimes 1\otimes m_1 + 1\otimes m_1\otimes 1 + m_1\otimes 1\otimes 1)m_3 - (m_2\otimes 1)m_2 + (1\otimes m_2)m_2 = 0 \quad (7.1.1)$$

(see (2.3.2) for $k=3$) shows that m_2 is associative in $\mathcal{K}$.

Let us check that the vertical composition is compatible with the actions of 1-morphisms on 2-morphisms. Applying equation (6.1.1) to $r\otimes p\otimes 1 \in sA_\infty(\mathcal{A},\mathcal{B})(f,g)\otimes sA_\infty(\mathcal{A},\mathcal{B})(g,h)\otimes \Bbbk \subset T^2 sA_\infty(\mathcal{A},\mathcal{B})(f,h)\otimes T^0 sA_\infty(\mathcal{B},\mathcal{C})(k,k)$ we find that

$$(r\otimes p \mid k)M_{20}B_1 + (rk\otimes pk)B_2 = [(r\otimes p)(1\otimes B_1 + B_1\otimes 1) \mid k]M_{20} + (r\otimes p)B_2 k. \quad (7.1.2)$$

One deduces that $(rs^{-1}\cdot k\otimes ps^{-1}\cdot k)m_2 = (rs^{-1}\otimes ps^{-1})m_2\cdot k$ in $\mathcal{K}$. Applying equation (6.1.1) to $1\otimes r\otimes p \in \Bbbk\otimes sA_\infty(\mathcal{A},\mathcal{B})(f,g)\otimes sA_\infty(\mathcal{A},\mathcal{B})(g,h) \subset T^0 sA_\infty(\mathcal{D},\mathcal{A})(e,e)\otimes T^2 sA_\infty(\mathcal{A},\mathcal{B})(f,h)$ we find that

$$\begin{aligned}(er\otimes ep)B_2 &= (e \mid r\otimes p)M_{02}B_1 + (er\otimes ep)B_2 \\ &= [e \mid (r\otimes p)(1\otimes B_1 + B_1\otimes 1)]M_{02} + e(r\otimes p)B_2 = e(r\otimes p)B_2 \quad (7.1.3)\end{aligned}$$

(notice that $M_{02}=0$). Therefore, $(e\cdot rs^{-1}\otimes e\cdot ps^{-1})m_2 = e\cdot(rs^{-1}\otimes ps^{-1})m_2$.

Now let us prove distributivity. Applying equation (6.1.1) to $r\otimes p \in sA_\infty(\mathcal{A},\mathcal{B})(f,g)\otimes sA_\infty(\mathcal{B},\mathcal{C})(h,k)$ we find that

$$(rh\otimes gp)B_2 + (-)^{rp}(fp\otimes rk)B_2 + (r\otimes p)M_{11}B_1 = (r\otimes p)(1\otimes B_1 + B_1\otimes 1)M_{11}.$$

Thus, $(rh\otimes gp)B_2 + (-)^{rp}(fp\otimes rk)B_2 = 0$ in $\mathcal{K}$. We deduce that modulo homotopy

$$\begin{aligned}(rs^{-1}\cdot h\otimes g\cdot ps^{-1})m_2 s &= (rhs^{-1}\otimes gps^{-1})(s\otimes s)B_2 = (-)^{p+1}(rh\otimes gp)B_2 \\ &= (-)^{rp+p}(fp\otimes rk)B_2 = (-)^{rp+p}(fps^{-1}s\otimes rks^{-1}s)B_2 \\ &= (-)^{rp+p+r+1}(f\cdot ps^{-1}\otimes rs^{-1}\cdot k)m_2 s.\end{aligned}$$

Therefore, $(rs^{-1}\cdot h\otimes g\cdot ps^{-1})m_2 = (-)^{(r+1)(p+1)}(f\cdot ps^{-1}\otimes rs^{-1}\cdot k)m_2$ in $\mathcal{K}$, as stated. □

The 0-th cohomology functor $H^0 = \mathcal{K}(\Bbbk,\_) : \mathcal{K} \to \Bbbk\text{-mod}$, $X \mapsto H^0(X) = \mathcal{K}(\Bbbk, X)$ is lax monoidal symmetric, since the complex $\Bbbk$ concentrated in degree 0 is the unit object of $\mathcal{K}$. It determines a functor $H^0 : \mathcal{K}\text{-}\mathcal{C}at \to \Bbbk\text{-}\mathcal{C}at$. To a $\mathcal{K}$-category $\mathcal{C}$ it assigns a $\Bbbk$-linear category $H^0(\mathcal{C})$ with the same class of objects, and for each pair X, Y of objects of $\mathcal{C}$ the $\Bbbk$-module $H^0(\mathcal{C})(X,Y) = H^0(\mathcal{C}(X,Y))$. The functor $H^0 : \mathcal{K}\text{-}\mathcal{C}at \to \Bbbk\text{-}\mathcal{C}at$ is also lax monoidal symmetric. Therefore, there is a functor $\mathcal{K}\text{-}\mathcal{C}at\text{-}\mathcal{C}at \to \mathcal{C}at\text{-}\mathcal{C}at$, again denoted H^0 by abuse of notation, and the corresponding functor $\mathcal{K}\text{-}2\text{-}\mathcal{C}at^{nu} \to 2\text{-}\mathcal{C}at^{nu}$. See Appendix A for the definition of

1-unital, non-2-unital $\mathcal{K}$- or $\Bbbk$- 2-categories. To $\mathcal{K}A_\infty$ the functor assigns a 1-unital, non-2-unital $\Bbbk$-linear 2-category A_∞. Let us describe it in detail. Objects of A_∞ are A_∞-categories, 1-morphisms are A_∞-functors, and 2-morphisms are elements of

$$H^0(A_\infty(\mathcal{A},\mathcal{B})(f,g),m_1) \xrightarrow[\sim]{s} H^{-1}(sA_\infty(\mathcal{A},\mathcal{B})(f,g),B_1),$$

that is, equivalence classes of natural A_∞-transformations $r : f \to g : \mathcal{A} \to \mathcal{B}$. Natural A_∞-transformations $r,t : f \to g : \mathcal{A} \to \mathcal{B}$ are *equivalent*, if they are connected by a 3-morphism $\lambda : r \to t$, that is, $r - t = \lambda B_1$. Both compositions of 1-morphisms with 2-morphisms $\operatorname{Mor}_2(\mathcal{A},\mathcal{B}) \times \operatorname{Mor}_1(\mathcal{B},\mathcal{C}) \to \operatorname{Mor}_2(\mathcal{A},\mathcal{C})$, $(r,h) \mapsto rh$ and $\operatorname{Mor}_1(\mathcal{A},\mathcal{B}) \times \operatorname{Mor}_2(\mathcal{B},\mathcal{C}) \to \operatorname{Mor}_2(\mathcal{A},\mathcal{C})$, $(f,p) \mapsto fp$ are compositions of $\Bbbk$-linear maps $Ts\mathcal{A} \to Ts\mathcal{B} \to Ts\mathcal{C}$. The vertical composition m_2 of 2-morphisms, translated to $H^{-1}(sA_\infty)$ assigns the natural A_∞-transformation $r \cdot p = (r \otimes p)B_2$ to natural A_∞-transformations $r : f \to g$ and $p : g \to h$. Indeed, $(rs^{-1} \otimes ps^{-1})m_2 s = (rs^{-1} \otimes ps^{-1})(s \otimes s)B_2 = (r \otimes p)B_2$. Compatibility of these constructions with the equivalence relation is obvious from the construction, and can be verified directly.

7.2 Remark. Let $\mathcal{A}$ be an A_∞-category. It determines a map $A_\omega \to A_\omega$, described in Remark 6.5. This map restricts to a map $A_\infty(\mathcal{A},\_) : A_\infty \to A_\infty$. It takes an A_∞-category $\mathcal{B}$ to the A_∞-category $A_\infty(\mathcal{A},\mathcal{B})$, an A_∞-functor $g : \mathcal{B} \to \mathcal{C}$ to the A_∞-functor $(1 \otimes g)M : A_\infty(\mathcal{A},\mathcal{B}) \to A_\infty(\mathcal{A},\mathcal{C})$, and an equivalence class of a natural A_∞-transformation $t : g \to h : \mathcal{A} \to \mathcal{B}$ to the equivalence class of the natural A_∞-transformation $(1 \otimes t)M : (1 \otimes g)M \to (1 \otimes h)M : A_\infty(\mathcal{A},\mathcal{B}) \to A_\infty(\mathcal{A},\mathcal{C})$, see Remark 6.5. Let us prove that $A_\infty(\mathcal{A},\_) : A_\infty \to A_\infty$ is a strict 2-functor. Indeed, it preserves the composition of 1-morphisms, $(1 \otimes f)M(1 \otimes g)M = (1 \otimes fg)M$, and the both compositions of 1-morphisms and 2-morphisms, $(1 \otimes f)M(1 \otimes t)M = (1 \otimes ft)M$, $(1 \otimes t)M(1 \otimes f)M = (1 \otimes tf)M$, due to associativity of M. The vertical composition of 2-morphisms is preserved due to (6.3.1) for $m = 2$:

$$[(1 \otimes t^1)M \otimes (1 \otimes t^2)M]\tilde{B}_2 = [1 \otimes (t^1 \otimes t^2)B_2]M.$$

7.3 Definition (Unital A_∞-categories). Let $\mathcal{C}$ be an A_∞-category. It is called *unital* if for each object X of $\mathcal{C}$ there is a *unit element* – a $\Bbbk$-linear map ${}_X\mathbf{i}_0^{\mathcal{C}} : \Bbbk \to (s\mathcal{C})^{-1}(X,X)$ such that ${}_X\mathbf{i}_0^{\mathcal{C}} b_1 = 0$, $({}_X\mathbf{i}_0^{\mathcal{C}} \otimes {}_X\mathbf{i}_0^{\mathcal{C}})b_2 - {}_X\mathbf{i}_0^{\mathcal{C}} \in \operatorname{Im} b_1$, and for all pairs X, Y of objects of $\mathcal{C}$ the chain maps $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2$, $({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 : s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y)$ are homotopy invertible.

In particular, an A_∞-algebra $\mathcal{C}$ is unital if it has an element $\mathbf{i}_0^{\mathcal{C}} \in (s\mathcal{C})^{-1}$ such that $\mathbf{i}_0^{\mathcal{C}} b_1 = 0$, $(\mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}})b_2 - \mathbf{i}_0^{\mathcal{C}} \in \operatorname{Im} b_1$, and the chain maps $(1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2$, $(\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 : s\mathcal{C} \to s\mathcal{C}$ are homotopy invertible. Our definition differs from a that of a homological unit (e.g. [**LH02**]) by the last invertibility condition. It produces rather a homotopical unit:

7.4 Lemma. *Let ${}_X\mathbf{i}_0^{\mathcal{C}}$ be as in Definition 7.3 of a unital A_∞-category $\mathcal{C}$, then for each pair X, Y of objects of $\mathcal{C}$ we have*

$$(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2 \sim 1 : s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y),$$
$$({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 \sim -1 : s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y).$$

Proof. For each object X of $\mathcal{C}$ there is a $\Bbbk$-linear map ${}_Xv_0 : \Bbbk \to (s\mathcal{C})^{-2}(X, X)$ such that $({}_X\mathbf{i}_0^{\mathcal{C}} \otimes {}_X\mathbf{i}_0^{\mathcal{C}})b_2 - {}_X\mathbf{i}_0^{\mathcal{C}} = {}_Xv_0b_1$. Hence,

$$({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 = ({}_X\mathbf{i}_0^{\mathcal{C}} \otimes {}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)(1 \otimes b_2)b_2 \overset{(2.3.1)}{\underset{k=3}{\sim}} -({}_X\mathbf{i}_0^{\mathcal{C}} \otimes {}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)(b_2 \otimes 1)b_2$$
$$= -[({}_X\mathbf{i}_0^{\mathcal{C}} \otimes {}_X\mathbf{i}_0^{\mathcal{C}})b_2 \otimes 1]b_2 = -({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 - ({}_Xv_0b_1 \otimes 1)b_2$$
$$= -({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 + b_1({}_Xv_0 \otimes 1)b_2 + ({}_Xv_0 \otimes 1)b_2b_1 \sim -({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2,$$

$$(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2 = -(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}} \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})(b_2 \otimes 1)b_2 \overset{(2.3.1)}{\underset{k=3}{\sim}} (1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}} \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})(1 \otimes b_2)b_2$$
$$= [1 \otimes ({}_Y\mathbf{i}_0^{\mathcal{C}} \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2]b_2 = (1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2 + (1 \otimes {}_Yv_0b_1)b_2$$
$$= (1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2 - b_1(1 \otimes {}_Yv_0)b_2 - (1 \otimes {}_Yv_0)b_2b_1 \sim (1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2.$$

We see that $-({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2$ and $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2$ are invertible idempotents in $\mathcal{K}$. Therefore, these maps are both homotopic to the identity map. □

This lemma shows that a unital A_∞-algebra may be defined as an A_∞-algebra $\mathcal{C}$, such that the graded associative $\Bbbk$-algebra $H^\bullet(\mathcal{C}, m_1)$ has a unit $1 \in H^0(\mathcal{C}, m_1)$ and for some/any representative $1^{\mathcal{C}} \in \mathcal{C}^0$ of the class $1 \in H^0(\mathcal{C}, m_1)$ the chain maps $(\mathrm{id} \otimes 1^{\mathcal{C}})m_2$, $(1^{\mathcal{C}} \otimes \mathrm{id})m_2 : \mathcal{C} \to \mathcal{C}$ are homotopic to $\mathrm{id}_{\mathcal{C}}$. A unit element $\mathbf{i}_0^{\mathcal{C}} \in (s\mathcal{C})^{-1}$ corresponds to a unit $1^{\mathcal{C}} \in \mathcal{C}^0$ via $1^{\mathcal{C}}s = \mathbf{i}_0^{\mathcal{C}}$.

7.5 Proposition. *Let $\mathcal{C}$ be a unital A_∞-category. Then the collection ${}_X\mathbf{i}_0^{\mathcal{C}}$ extends to a natural A_∞-transformation $\mathbf{i}^{\mathcal{C}} : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ such that $(\mathbf{i}^{\mathcal{C}} \otimes \mathbf{i}^{\mathcal{C}})B_2 \equiv \mathbf{i}^{\mathcal{C}}$.*

Proof. Let $\Bbbk$-linear maps ${}_Xv_0 : \Bbbk \to (s\mathcal{C})^{-2}(X, X)$ satisfy the equations $({}_X\mathbf{i}_0^{\mathcal{C}} \otimes {}_X\mathbf{i}_0^{\mathcal{C}})b_2 - {}_X\mathbf{i}_0^{\mathcal{C}} = {}_Xv_0b_1$. We will prove that given ${}_X\mathbf{i}_0^{\mathcal{C}}$, ${}_Xv_0$ (with ${}_X\mathbf{i}_0^{\mathcal{C}}b_1 = 0$) are 0-th components of a natural A_∞-transformation $\mathbf{i}^{\mathcal{C}}$ and a 3-morphism v as follows:

$$\mathbf{i}^{\mathcal{C}} : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C},$$
$$v : (\mathbf{i}^{\mathcal{C}} \otimes \mathbf{i}^{\mathcal{C}})B_2 \to \mathbf{i}^{\mathcal{C}} : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}.$$

That is, we will prove the existence of $(1,1)$-coderivations $\mathbf{i}^{\mathcal{C}}, v : Ts\mathcal{C} \to Ts\mathcal{C}$ of degree -1 (resp. -2) such that

$$\mathbf{i}^{\mathcal{C}}b + b\mathbf{i}^{\mathcal{C}} = 0,$$
$$(\mathbf{i}^{\mathcal{C}} \otimes \mathbf{i}^{\mathcal{C}})B_2 - \mathbf{i}^{\mathcal{C}} = vb - bv.$$

We already have the 0-th components $\mathbf{i}_0^{\mathcal{C}}$, v_0. Let us construct the other components of $\mathbf{i}^{\mathcal{C}}$ and v by induction. Given a positive n, assume that we have already found components $\mathbf{i}_m^{\mathcal{C}}$, v_m of the sought $\mathbf{i}^{\mathcal{C}}$, v for $m < n$, such that the equations

$$(\mathbf{i}^{\mathcal{C}}b)_m + (b\mathbf{i}^{\mathcal{C}})_m = 0 : s\mathcal{C}(X_0, X_1) \otimes \cdots \otimes s\mathcal{C}(X_{m-1}, X_m) \to s\mathcal{C}(X_0, X_m), \quad (7.5.1)$$

$$[(\mathbf{i}^{\mathcal{C}} \otimes \mathbf{i}^{\mathcal{C}})B_2]_m - \mathbf{i}_m^{\mathcal{C}} = (vb)_m - (bv)_m : s\mathcal{C}(X_0, X_1) \otimes \cdots \otimes s\mathcal{C}(X_{m-1}, X_m) \to s\mathcal{C}(X_0, X_m) \quad (7.5.2)$$

are satisfied for all $m < n$. Here $(f)_m = \bigl(T^ms\mathcal{C} \hookrightarrow Ts\mathcal{C} \xrightarrow{f} Ts\mathcal{C} \xrightarrow{\mathrm{pr}_1} s\mathcal{C}\bigr)$ for an arbitrary morphism of quivers $f : Ts\mathcal{C} \to Ts\mathcal{C}$.

Introduce $(1,1)$-coderivations $\tilde{\mathbf{i}}, v : Ts\mathcal{C} \to Ts\mathcal{C}$ of degree -1 and -2 by their components $(\mathbf{i}_0^{\mathcal{C}}, \mathbf{i}_1^{\mathcal{C}}, \dots, \mathbf{i}_{n-1}^{\mathcal{C}}, 0, 0, \dots)$ (resp. $(v_0, v_1, \dots, v_{n-1}, 0, 0, \dots)$). Define $(1,1)$-coderivations $\lambda = \tilde{\mathbf{i}}b + b\tilde{\mathbf{i}}$ of degree 0 and $\nu = (\tilde{\mathbf{i}} \otimes \tilde{\mathbf{i}})B_2 - \tilde{\mathbf{i}} - \tilde{v}B_1$ of degree -1. Then equations (7.5.1), (7.5.2) imply that $\lambda_m = 0$, $\nu_m = 0$ for $m < n$. The identity $\lambda b - b\lambda = 0$ implies that

$$\lambda_n d = \lambda_n b_1 - \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\lambda_n = 0.$$

The n-th component of the identity

$$\nu B_1 = (\tilde{\mathbf{i}} \otimes \tilde{\mathbf{i}})B_2 B_1 - \tilde{\mathbf{i}}B_1 = -(\tilde{\mathbf{i}} \otimes \tilde{\mathbf{i}})(1 \otimes B_1 + B_1 \otimes 1)B_2 - \lambda = -(\tilde{\mathbf{i}} \otimes \lambda)B_2 + (\lambda \otimes \tilde{\mathbf{i}})B_2 - \lambda$$

gives

$$\begin{aligned} \nu_n d = \nu_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\nu_n &= -(\mathbf{i}_0^{\mathcal{C}} \otimes \lambda_n)b_2 + (\lambda_n \otimes \mathbf{i}_0^{\mathcal{C}})b_2 - \lambda_n \\ &= -\lambda_n(\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 + \lambda_n(1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2 - \lambda_n = -\lambda_n u'. \end{aligned}$$

Here the chain map

$$u' = ({}_{X_0}\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 - (1 \otimes {}_{X_n}\mathbf{i}_0^{\mathcal{C}})b_2 + 1 : s\mathcal{C}(X_0, X_n) \to s\mathcal{C}(X_0, X_n)$$

is homotopic to -1 by Lemma 7.4. Hence, the map

$$u = \operatorname{Hom}(N, u') : \operatorname{Hom}^\bullet(N, s\mathcal{C}(X_0, X_n)) \to \operatorname{Hom}^\bullet(N, s\mathcal{C}(X_0, X_n)), \quad \lambda_n \mapsto \lambda_n u'$$

is also homotopic to -1 for each complex of $\Bbbk$-modules N, in particular, for $N = s\mathcal{C}(X_0, X_1) \otimes_\Bbbk \dots \otimes_\Bbbk s\mathcal{C}(X_{n-1}, X_n)$. Therefore, the complex $\operatorname{Cone}(u)$ is contractible by Lemma Appendix B.1. Since $-\lambda_n d = 0$ and $\nu_n d + \lambda_n u = 0$, the element

$$(\nu_n, \lambda_n) \in \operatorname{Hom}_\Bbbk^{-1}(N, s\mathcal{C}(X_0, X_n)) \oplus \operatorname{Hom}_\Bbbk^0(N, s\mathcal{C}(X_0, X_n)) = \operatorname{Cone}^{-1}(u)$$

is a cycle. Due to acyclicity of $\operatorname{Cone}(u)$ this element is a boundary of some element

$$(v_n, \mathbf{i}_n^{\mathcal{C}}) \in \operatorname{Hom}_\Bbbk^{-2}(N, s\mathcal{C}(X_0, X_n)) \oplus \operatorname{Hom}_\Bbbk^{-1}(N, s\mathcal{C}(X_0, X_n)) = \operatorname{Cone}^{-2}(u),$$

that is, $v_n d + \mathbf{i}_n^{\mathcal{C}} u = \nu_n$ and $-\mathbf{i}_n^{\mathcal{C}} d = \lambda_n$. These equations can be rewritten as follows:

$$-\mathbf{i}_n^{\mathcal{C}} b_1 - \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\mathbf{i}_n^{\mathcal{C}} = (\tilde{\mathbf{i}}b)_n + (b\tilde{\mathbf{i}})_n,$$

$$\begin{aligned} v_n b_1 - \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})v_n - (\mathbf{i}_0^{\mathcal{C}} \otimes \mathbf{i}_n^{\mathcal{C}})b_2 - (\mathbf{i}_n^{\mathcal{C}} \otimes \mathbf{i}_0^{\mathcal{C}})b_2 + \mathbf{i}_n^{\mathcal{C}} \\ = [(\tilde{\mathbf{i}} \otimes \tilde{\mathbf{i}})B_2]_n - (\tilde{v}b)_n + (b\tilde{v})_n. \end{aligned}$$

In other words, $(1,1)$-coderivations with components $(\mathbf{i}_0^{\mathcal{C}}, \dots, \mathbf{i}_{n-1}^{\mathcal{C}}, \mathbf{i}_n^{\mathcal{C}}, 0, \dots)$, $(v_0, \dots, v_{n-1}, v_n, 0, \dots)$ satisfy equations (7.5.1), (7.5.2) for $m \leqslant n$. The construction of $\mathbf{i}^{\mathcal{C}}$, v goes on inductively. □

7.6 Definition. A *unit transformation* of an A_∞-category $\mathcal{C}$ is a natural A_∞-transformation $\mathbf{i}^{\mathcal{C}} : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ such that $(\mathbf{i}^{\mathcal{C}} \otimes \mathbf{i}^{\mathcal{C}})B_2 \equiv \mathbf{i}^{\mathcal{C}}$, and for each pair X, Y of objects of $\mathcal{C}$ the chain maps $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2$, $({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 : s\mathcal{C}(X, Y) \to s\mathcal{C}(X, Y)$ are homotopy invertible.

We have shown in Proposition 7.5 that an A_∞-category $\mathcal{C}$ is unital if and only if it has a unit transformation. Similar (although not identical to our) definitions of units and unital A_∞-categories are proposed by Kontsevich and Soibelman [**KS02**] and Lefèvre-Hasegawa [**LH02**].

7.7 Proposition. *Let $\mathcal{A}$, $\mathcal{C}$ be A_∞-categories. If $\mathcal{C}$ is unital, then $A_\infty(\mathcal{A},\mathcal{C})$ is unital as well.*

Proof. We claim that

$$(1\otimes \mathbf{i}^{\mathcal{C}})M : (1\otimes \mathrm{id}_{\mathcal{C}})M \to (1\otimes \mathrm{id}_{\mathcal{C}})M : A_\infty(\mathcal{A},\mathcal{C}) \to A_\infty(\mathcal{A},\mathcal{C})$$

is a unit of $A_\infty(\mathcal{A},\mathcal{C})$. Indeed, $(1\otimes \mathrm{id}_{\mathcal{C}})M = \mathrm{id}_{A_\infty(\mathcal{A},\mathcal{C})}$, and there is a 3-morphism $v : (\mathbf{i}^{\mathcal{C}}\otimes \mathbf{i}^{\mathcal{C}})B_2 \to \mathbf{i}^{\mathcal{C}} : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$. Hence, $(\mathbf{i}^{\mathcal{C}}\otimes \mathbf{i}^{\mathcal{C}})B_2 = \mathbf{i}^{\mathcal{C}} + vB_1$, and by (6.3.1)

$$\begin{aligned}
&[(1\otimes \mathbf{i}^{\mathcal{C}})M\otimes(1\otimes \mathbf{i}^{\mathcal{C}})M]\tilde{B}_2 = [1\otimes(\mathbf{i}^{\mathcal{C}}\otimes \mathbf{i}^{\mathcal{C}})B_2]M\\
&\qquad = (1\otimes \mathbf{i}^{\mathcal{C}})M + (1\otimes vB_1)M = (1\otimes \mathbf{i}^{\mathcal{C}})M + (1\otimes v)M\tilde{B}_1 \equiv (1\otimes \mathbf{i}^{\mathcal{C}})M.
\end{aligned}$$

Let $f : \mathcal{A} \to \mathcal{C}$ be an A_∞-functor. The 0-th component of $(1\otimes \mathbf{i}^{\mathcal{C}})M$ is ${}_f[(1\otimes \mathbf{i}^{\mathcal{C}})M]_0 : \Bbbk \to sA_\infty(\mathcal{A},\mathcal{C})(f,f)$, $1\mapsto f\mathbf{i}^{\mathcal{C}}$. It remains to prove that for each pair of A_∞-functors $f,g : \mathcal{A}\to\mathcal{C}$ the maps

$$\begin{aligned}
&\big(1\otimes {}_g[(1\otimes \mathbf{i}^{\mathcal{C}})M]_0\big)B_2 : sA_\infty(\mathcal{A},\mathcal{C})(f,g) \to sA_\infty(\mathcal{A},\mathcal{C})(f,g), \quad r\mapsto (r\otimes g\mathbf{i}^{\mathcal{C}})B_2,\\
&\big({}_f[(1\otimes \mathbf{i}^{\mathcal{C}})M]_0\otimes 1\big)B_2 : sA_\infty(\mathcal{A},\mathcal{C})(f,g) \to sA_\infty(\mathcal{A},\mathcal{C})(f,g), \quad r\mapsto r(f\mathbf{i}^{\mathcal{C}}\otimes 1)B_2,
\end{aligned}$$

are homotopy invertible.

Let us define a decreasing filtration of the complex $(sA_\infty(\mathcal{A},\mathcal{C})(f,g),B_1)$. For $n\in\mathbb{Z}_{\geqslant 0}$, we set

$$\begin{aligned}
\Phi_n &= \{r\in sA_\infty(\mathcal{A},\mathcal{C})(f,g) \mid \forall l<n \quad r_l = 0\}\\
&= \{r\in sA_\infty(\mathcal{A},\mathcal{C})(f,g) \mid \forall l<n \quad (T^l s\mathcal{A})r = 0\}.
\end{aligned}$$

Clearly, Φ_n is stable under $B_1 = [\_,b]$, and we have

$$sA_\infty(\mathcal{A},\mathcal{C})(f,g) = \Phi_0 \supset \Phi_1 \supset \cdots \supset \Phi_n \supset \Phi_{n+1} \supset \dots \quad .$$

Due to (5.1.3) and (3.0.1) the chain maps $a = (1\otimes g\mathbf{i}^{\mathcal{C}})B_2$, $c = (f\mathbf{i}^{\mathcal{C}}\otimes 1)B_2$ preserve the subcomplex Φ_n. By Definition 2.6 $sA_\infty(\mathcal{A},\mathcal{C})(f,g) = \prod_{n=0}^\infty V_n$, where V_n is the $\Bbbk$-module (2.7.1) of n-th components r_n of (f,g)-coderivations r. The filtration consists of $\Bbbk$-submodules $\Phi_n = 0\times\cdots\times 0\times\prod_{m=n}^\infty V_m$.

The graded complex associated with this filtration is $\oplus_{n=0}^\infty V_n$, and the differential $d : V_n \to V_n$ induced by B_1 is given by formula (2.7.2). The associated endomorphisms $\operatorname{gr} a$, $\operatorname{gr} c$ of $\oplus_{n=0}^\infty V_n$ are given by the formulas

$$\begin{aligned}
(r_n)\operatorname{gr} a &= (r_n\otimes {}_g\mathbf{i}_0^{\mathcal{C}})b_2 = \prod_{X_0,\dots,X_n\in\operatorname{Ob}\mathcal{A}} ({}_{X_0,\dots,X_n}r_n \otimes {}_{X_n g}\mathbf{i}_0^{\mathcal{C}})b_2,\\
(r_n)\operatorname{gr} c &= r_n({}_f\mathbf{i}_0^{\mathcal{C}}\otimes 1)b_2 = \prod_{X_0,\dots,X_n\in\operatorname{Ob}\mathcal{A}} {}_{X_0,\dots,X_n}r_n({}_{X_0 f}\mathbf{i}_0^{\mathcal{C}}\otimes 1)b_2,
\end{aligned}$$

$r_n \in V_n$, as formulas (3.0.1), (5.1.3) show. Due to Lemma 7.4 for each pair X, Y of objects of $\mathcal{C}$ the chain maps $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2$, $-({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 : s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y)$ are homotopic to the identity map, that is, $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2 = 1 + hd + dh$, $({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 = -1 + h'd + dh'$ for some $\Bbbk$-linear maps $h, h' : s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y)$ of degree -1. Let us choose such homotopies ${}_{X_0,X_n}h$, ${}_{X_0,X_n}h' : s\mathcal{C}(X_0f, X_ng) \to s\mathcal{C}(X_0f, X_ng)$ for each pair X_0, X_n of objects of $\mathcal{A}$. Denote by $H, H' : \prod_{n=0}^{\infty} V_n \to \prod_{n=0}^{\infty} V_n$ the diagonal maps ${}_{X_0,\dots,X_n}r_n \mapsto {}_{X_0,\dots,X_n}r_n\,{}_{X_0,X_n}h$, ${}_{X_0,\dots,X_n}r_n \mapsto {}_{X_0,\dots,X_n}r_n\,{}_{X_0,X_n}h'$. Then $\operatorname{gr} a = 1 + Hd + dH$, $\operatorname{gr} c = -1 + H'd + dH'$. The chain maps $a - HB_1 - B_1H$, $c - H'B_1 - B_1H'$, being restricted to maps $\oplus_{m=0}^{\infty} V_m \to \prod_{m=0}^{\infty} V_m$ give upper triangular $\mathbb{N} \times \mathbb{N}$ matrices which, in turn, determine the whole map. Thus, $a - HB_1 - B_1H = 1 + N$, $c - H'B_1 - B_1H' = -1 + N'$, where the $\mathbb{N} \times \mathbb{N}$ matrices N, N' are strictly upper triangular. Therefore, $1 + N$ and $-1 + N'$ are invertible (since their inverse maps $\sum_{i=0}^{\infty}(-N)^i$ and $-\sum_{i=0}^{\infty}(N')^i$ make sense). Hence, $a = (1 \otimes g\mathbf{i}^{\mathcal{C}})B_2$ and $c = (f\mathbf{i}^{\mathcal{C}} \otimes 1)B_2$ are invertible in $\mathcal{K}$. □

7.8 Corollary. *Let $f, g : \mathcal{A} \to \mathcal{C}$ be A_∞-functors. If $\mathcal{C}$ is unital, then*

$$(1 \otimes g\mathbf{i}^{\mathcal{C}})B_2 \sim 1 : sA_\infty(\mathcal{A},\mathcal{C})(f,g) \to sA_\infty(\mathcal{A},\mathcal{C})(f,g), \text{ and}$$
$$(f\mathbf{i}^{\mathcal{C}} \otimes 1)B_2 \sim -1 : sA_\infty(\mathcal{A},\mathcal{C})(f,g) \to sA_\infty(\mathcal{A},\mathcal{C})(f,g).$$

Proof. Follows from Proposition 7.7 and Lemma 7.4. □

7.9 Corollary. *Let $r : f \to g : \mathcal{A} \to \mathcal{C}$ be a natural A_∞-transformation. If $\mathcal{C}$ is unital, then*

$$(r \otimes g\mathbf{i}^{\mathcal{C}})B_2 \equiv r, \qquad (f\mathbf{i}^{\mathcal{C}} \otimes r)B_2 \equiv r.$$

Proof. By Corollary 7.8 there are homotopies $h, h' : sA_\infty(\mathcal{A},\mathcal{C})(f,g) \to sA_\infty(\mathcal{A},\mathcal{C})(f,g)$, which give

$$(r \otimes g\mathbf{i}^{\mathcal{C}})B_2 = r(1 \otimes g\mathbf{i}^{\mathcal{C}})B_2 = r + rB_1h + rhB_1 = r + (rh)B_1 \equiv r,$$
$$(f\mathbf{i}^{\mathcal{C}} \otimes r)B_2 = -r(f\mathbf{i}^{\mathcal{C}} \otimes 1)B_2 = r + rB_1h' + rh'B_1 = r + (rh')B_1 \equiv r.$$

□

7.10 Corollary. *The unit transformation of a unital category is determined uniquely up to equivalence.*

Indeed, take $f = \mathrm{id}_{\mathcal{C}}$ and notice that any two unit transformations $\mathbf{i}^{\mathcal{C}}$ and ${}'\mathbf{i}^{\mathcal{C}}$ of $\mathcal{C}$ satisfy ${}'\mathbf{i}^{\mathcal{C}} \equiv {}'\mathbf{i}^{\mathcal{C}} \cdot \mathbf{i}^{\mathcal{C}} \equiv \mathbf{i}^{\mathcal{C}}$.

7.11 Corollary. *The full $\mathcal{K}$-2-subcategory $\mathcal{K}^uA_\infty$ of non-2-unital $\mathcal{K}$-2-category $\mathcal{K}A_\infty$, whose objects are unital A_∞-categories and the other data are as in $\mathcal{K}A_\infty$, is a 1-2-unital $\mathcal{K}$-2-category. The unit 2-endomorphism of an A_∞-functor $f : \mathcal{A} \to \mathcal{C}$ is the homotopy class of the chain map*

$$1_f : \Bbbk \to (A_\infty(\mathcal{A},\mathcal{C})(f,f), m_1), \qquad 1 \mapsto (f\mathbf{i}^{\mathcal{C}})s^{-1}.$$

Proof. The composition

$$A_\infty(\mathcal{A},\mathcal{C})(f,g) \xrightarrow{1\otimes 1_g} A_\infty(\mathcal{A},\mathcal{C})(f,g) \otimes A_\infty(\mathcal{A},\mathcal{C})(g,g) \xrightarrow{m_2} A_\infty(\mathcal{A},\mathcal{C})(f,g),$$
$$rs^{-1} \mapsto rs^{-1} \otimes (g\mathbf{i}^{\mathcal{C}})s^{-1} \mapsto (rs^{-1} \otimes (g\mathbf{i}^{\mathcal{C}})s^{-1})(s \otimes s)B_2s^{-1} = (r \otimes g\mathbf{i}^{\mathcal{C}})B_2s^{-1},$$

is homotopic to the identity map by Corollary 7.8. Similarly, the composition

$$
\begin{aligned}
A_\infty(\mathcal{A},\mathcal{C})(f,g) &\xrightarrow{1_f\otimes 1} A_\infty(\mathcal{A},\mathcal{C})(f,f)\otimes A_\infty(\mathcal{A},\mathcal{C})(f,g) \xrightarrow{m_2} A_\infty(\mathcal{A},\mathcal{C})(f,g),\\
rs^{-1} &\mapsto (f\mathbf{i}^{\mathcal{C}})s^{-1}\otimes rs^{-1} \mapsto (-)^{r-1}(f\mathbf{i}^{\mathcal{C}}\otimes r)B_2s^{-1} = -r(f\mathbf{i}^{\mathcal{C}}\otimes 1)B_2s^{-1},
\end{aligned}
$$

is homotopic to the identity map. □

7.12 Corollary. *The full 2-subcategory ${}^uA_\infty$ of non-2-unital 2-category A_∞, which consists of unital A_∞-categories, all A_∞-functors between them, and equivalence classes of all natural A_∞-transformations is a (usual 1-2-unital) 2-category. The unit 2-endomorphism of an A_∞-functor $f:\mathcal{A}\to\mathcal{C}$ is $f\mathbf{i}^{\mathcal{C}}$. In this 2-category the notions of an isomorphism between A_∞-functors, an equivalence between A_∞-categories, etc. make sense. For instance, $r:f\to g:\mathcal{A}\to\mathcal{B}$ is an* isomorphism *if there is a natural A_∞-transformation $p:g\to f$, such that $(r\otimes p)B_2\equiv f\mathbf{i}^{\mathcal{B}}$ and $(p\otimes r)B_2\equiv g\mathbf{i}^{\mathcal{B}}$. An A_∞-functor $f:\mathcal{A}\to\mathcal{B}$ is an* equivalence *if there exists an A_∞-functor $g:\mathcal{B}\to\mathcal{A}$ and isomorphisms $\mathrm{id}_{\mathcal{A}}\to fg$ and $\mathrm{id}_{\mathcal{B}}\to gf$.*

Proof. Follows from Corollary 7.9 or 7.11. □

7.13. Invertible transformations. Let $\mathcal{B}$, $\mathcal{C}$ be A_∞-categories, and let $f,g:\operatorname{Ob}\mathcal{C}\to\operatorname{Ob}\mathcal{B}$ be maps. Assume that $\mathcal{B}$ is unital and that for each object X of $\mathcal{C}$ there are $\Bbbk$-linear maps

$$
\begin{aligned}
{}_Xr_0 &: \Bbbk\to(s\mathcal{B})^{-1}(Xf,Xg), &\quad {}_Xp_0 &: \Bbbk\to(s\mathcal{B})^{-1}(Xg,Xf),\\
{}_Xw_0 &: \Bbbk\to(s\mathcal{B})^{-2}(Xf,Xf), &\quad {}_Xv_0 &: \Bbbk\to(s\mathcal{B})^{-2}(Xg,Xg),
\end{aligned}
$$

such that

$$
\begin{gathered}
{}_Xr_0b_1=0,\qquad {}_Xp_0b_1=0,\\
({}_Xr_0\otimes{}_Xp_0)b_2-{}_{Xf}\mathbf{i}_0^{\mathcal{B}}={}_Xw_0b_1,\\
({}_Xp_0\otimes{}_Xr_0)b_2-{}_{Xg}\mathbf{i}_0^{\mathcal{B}}={}_Xv_0b_1.
\end{gathered}
\tag{7.13.1}
$$

7.14 Lemma. *Let the above assumptions hold. Then for all objects X of $\mathcal{C}$ and Y of $\mathcal{B}$ the chain maps*

$$
\begin{aligned}
&(r_0\otimes 1)b_2: s\mathcal{B}(Xg,Y)\to s\mathcal{B}(Xf,Y) \text{ and } (p_0\otimes 1)b_2: s\mathcal{B}(Xf,Y)\to s\mathcal{B}(Xg,Y),\\
&(1\otimes r_0)b_2: s\mathcal{B}(Y,Xf)\to s\mathcal{B}(Y,Xg) \ \text{ and } (1\otimes p_0)b_2: s\mathcal{B}(Y,Xg)\to s\mathcal{B}(Y,Xf)
\end{aligned}
$$

are homotopy inverse to each other.

Proof. We have

$$
\begin{aligned}
&(r_0\otimes 1)b_2(p_0\otimes 1)b_2=(p_0\otimes r_0\otimes 1)(1\otimes b_2)b_2\\
&=-(p_0\otimes r_0\otimes 1)[(b_2\otimes 1)b_2+b_3b_1+(1\otimes 1\otimes b_1)b_3]\\
&=-({}_{Xg}\mathbf{i}_0^{\mathcal{B}}\otimes 1)b_2-(v_0b_1\otimes 1)b_2-(p_0\otimes r_0\otimes 1)b_3b_1-b_1(p_0\otimes r_0\otimes 1)b_3\\
&\sim 1+b_1(v_0\otimes 1)b_2+(v_0\otimes 1)b_2b_1\sim 1: s\mathcal{B}(Xg,Y)\to s\mathcal{B}(Xg,Y).
\end{aligned}
\tag{7.14.1}
$$

For symmetry reasons also

$$(p_0\otimes 1)b_2(r_0\otimes 1)b_2\sim 1: s\mathcal{B}(Xf,Y)\to s\mathcal{B}(Xf,Y).$$

Therefore, $(r_0 \otimes 1)b_2$ and $(p_0 \otimes 1)b_2$ are homotopy inverse to each other.

Similarly, $(1 \otimes r_0)b_2$ and $(1 \otimes p_0)b_2$ are homotopy inverse to each other. □

7.15 Proposition. *Let $r : f \to g : \mathcal{C} \to \mathcal{B}$ be a natural A_∞-transformation, where $\mathcal{B}$ is unital, and let p_0, v_0, w_0 be as in Section 7.13 so that equations* (7.13.1) *hold. Then p_0, w_0 extend to a natural A_∞-transformation $p : g \to f : \mathcal{C} \to \mathcal{B}$, a 3-morphism w, and there is a 3-morphism t as follows:*

$$w : (r \otimes p)B_2 \to f\mathbf{i}^{\mathcal{B}} : f \to f : \mathcal{C} \to \mathcal{B}, \tag{7.15.1}$$

$$t : (p \otimes r)B_2 \to g\mathbf{i}^{\mathcal{B}} : g \to g : \mathcal{C} \to \mathcal{B}. \tag{7.15.2}$$

In particular, r is invertible and $p = r^{-1}$ in A_∞.

Proof. Let us drop equation (7.15.2) and prove the existence of p and w, satisfying (7.15.1). We have to satisfy the equations

$$pb + bp = 0, \tag{7.15.3}$$

$$(r \otimes p)B_2 - f\mathbf{i}^{\mathcal{B}} = [w, b]. \tag{7.15.4}$$

Let us construct the components of p and w by induction. Given a positive n, assume that we have already found components p_m, w_m of the sought p, w for $m < n$, such that the equations

$$(pb)_m + (bp)_m = 0 : s\mathcal{C}(X_0, X_1) \otimes \cdots \otimes s\mathcal{C}(X_{m-1}, X_m) \to s\mathcal{B}(X_0 g, X_m f), \tag{7.15.5}$$

$$\begin{aligned}[][(r \otimes p)B_2]_m - (f\mathbf{i}^{\mathcal{B}})_m = (wb)_m - (bw)_m :& \\ s\mathcal{C}(X_0, X_1) \otimes \cdots \otimes s\mathcal{C}(X_{m-1}, X_m) &\to s\mathcal{B}(X_0 f, X_m f)\end{aligned} \tag{7.15.6}$$

are satisfied for all $m < n$. Introduce a (g, f)-coderivation $\tilde{p} : Ts\mathcal{C} \to Ts\mathcal{B}$ of degree -1 by its components $(p_0, p_1, \dots, p_{n-1}, 0, 0, \dots)$ and an (f, f)-coderivation $\tilde{w} : Ts\mathcal{C} \to Ts\mathcal{B}$ of degree -2 by its components $(w_0, w_1, \dots, w_{n-1}, 0, 0, \dots)$. Define a (g, f)-coderivation $\lambda = \tilde{p}b + b\tilde{p}$ of degree 0 and an (f, f)-coderivation $\nu = (r \otimes \tilde{p})B_2 - f\mathbf{i}^{\mathcal{B}} - [\tilde{w}, b]$ of degree -1. Then equations (7.15.3), (7.15.4) imply that $\lambda_m = 0$, $\nu_m = 0$ for $m < n$. The identity $\lambda b - b\lambda = 0$ implies that

$$\lambda_n d = \lambda_n b_1 - \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\lambda_n = 0.$$

The identity

$$\begin{aligned}[][\nu, b] = \nu B_1 = (r \otimes \tilde{p})B_2 B_1 - (f\mathbf{i}^{\mathcal{B}})B_1 - \tilde{w}B_1 B_1 &= -(r \otimes \tilde{p})(1 \otimes B_1 + B_1 \otimes 1)B_2 \\ &= -(r \otimes \tilde{p}B_1)B_2 = -(r \otimes \lambda)B_2\end{aligned}$$

implies that

$$\nu_n b_1 + \sum_{q+1+t=n} (1^{\otimes q} \otimes b_1 \otimes 1^{\otimes t})\nu_n = -(r_0 \otimes \lambda_n)b_2,$$

that is, $\nu_n d = -\lambda_n u$. Here the map $u = \operatorname{Hom}(N, (r_0 \otimes 1)b_2)$ for $N = s\mathcal{C}(X_0, X_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} s\mathcal{C}(X_{n-1}, X_n)$ is homotopy invertible, and the complex $\operatorname{Cone}(u)$ is contractible by Lemma Appendix B.1. Hence, the cycle

$$(\nu_n, \lambda_n) \in \operatorname{Hom}^{-1}_{\Bbbk}(N, s\mathcal{B}(X_0 f, X_n f)) \oplus \operatorname{Hom}^0_{\Bbbk}(N, s\mathcal{B}(X_0 g, X_n f)) = \operatorname{Cone}^{-1}(u)$$

is the boundary of some element

$$(w_n, p_n) \in \mathrm{Hom}_{\Bbbk}^{-2}(N, s\mathcal{B}(X_0 f, X_n f)) \oplus \mathrm{Hom}_{\Bbbk}^{-1}(N, s\mathcal{B}(X_0 g, X_n f)) = \mathrm{Cone}^{-2}(u),$$

that is, $w_n d + p_n u = \nu_n$ and $-p_n d = \lambda_n$. In other words, equations (7.15.5), (7.15.6) are satisfied for $m = n$, and we prove the statement by induction.

For similar reasons using Lemma 7.14 there exists a natural A_∞-transformation $q : g \to f : \mathcal{C} \to \mathcal{B}$ with $q_0 = p_0$ and a 3-morphism

$$v : (q \otimes r)B_2 \to g\mathbf{i}^{\mathcal{B}} : g \to g : \mathcal{C} \to \mathcal{B}$$

with given v_0. Since r has a left inverse and a right inverse, it is invertible in A_∞ and p is equivalent to q. Hence, $(p \otimes r)B_2$ is equivalent to $(q \otimes r)B_2$, and there exists t of (7.15.2). $\square$

8. Unital A_∞-functors

8.1 Definition. Let $\mathcal{A}$, $\mathcal{B}$ be unital A_∞-categories. An A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is called *unital* if for all objects X of $\mathcal{A}$ we have ${}_X\mathbf{i}_0^{\mathcal{A}} f_1 - {}_{Xf}\mathbf{i}_0^{\mathcal{B}} \in \mathrm{Im}\, b_1$.

For instance, an A_∞-homomorphism $f : \mathcal{A} \to \mathcal{B}$ of A_∞-algebras is unital if the cycles $\mathbf{i}_0^{\mathcal{A}} f_1$, $\mathbf{i}_0^{\mathcal{B}} \in (s\mathcal{B})^{-1}$ are cohomologous in $(s\mathcal{B}, b_1)$. We may say that a unital A_∞-functor (or A_∞-homomorphism) preserves the cohomology classes of unit elements.

8.2 Proposition. *Let $\mathcal{A}$, $\mathcal{B}$ be unital A_∞-categories. An A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is unital if and only if $\mathbf{i}^{\mathcal{A}} f \equiv f\mathbf{i}^{\mathcal{B}}$.*

Proof. If $\mathbf{i}^{\mathcal{A}} f = f\mathbf{i}^{\mathcal{B}} + vB_1$, then ${}_X\mathbf{i}_0^{\mathcal{A}} f_1 = {}_X(\mathbf{i}^{\mathcal{A}} f)_0 = {}_X(f\mathbf{i}^{\mathcal{B}} + vB_1)_0 = {}_{Xf}\mathbf{i}_0^{\mathcal{B}} + {}_X v_0 b_1$, hence, f is unital.

Assume now that f is unital. We want to find a 3-morphism

$$v : \mathbf{i}^{\mathcal{A}} f \to f\mathbf{i}^{\mathcal{B}} : f \to f : \mathcal{A} \to \mathcal{B},$$

that is, an (f, f)-coderivation v of degree -2 such that

$$vB_1 = \mathbf{i}^{\mathcal{A}} f - f\mathbf{i}^{\mathcal{B}}. \tag{8.2.1}$$

We subject it to an additional condition described below. Consider 3-morphisms

$$x : (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}})B_2 \to \mathbf{i}^{\mathcal{A}} : \mathrm{id}_{\mathcal{A}} \to \mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A},$$
$$y : (\mathbf{i}^{\mathcal{B}} \otimes \mathbf{i}^{\mathcal{B}})B_2 \to \mathbf{i}^{\mathcal{B}} : \mathrm{id}_{\mathcal{B}} \to \mathrm{id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B},$$

so that

$$xB_1 = (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}})B_2 - \mathbf{i}^{\mathcal{A}}, \qquad yB_1 = (\mathbf{i}^{\mathcal{B}} \otimes \mathbf{i}^{\mathcal{B}})B_2 - \mathbf{i}^{\mathcal{B}}.$$

The following equations between (f, f)-coderivations of degree -2 are due to (7.1.2), (7.1.3):

$$(xf)B_1 = (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}})B_2 f - \mathbf{i}^{\mathcal{A}} f = (\mathbf{i}^{\mathcal{A}} f \otimes \mathbf{i}^{\mathcal{A}} f)B_2 + (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}} \mid f)M_{20}B_1 - \mathbf{i}^{\mathcal{A}} f,$$
$$(fy)B_1 = f(\mathbf{i}^{\mathcal{B}} \otimes \mathbf{i}^{\mathcal{B}})B_2 - f\mathbf{i}^{\mathcal{B}} = (f\mathbf{i}^{\mathcal{B}} \otimes f\mathbf{i}^{\mathcal{B}})B_2 - f\mathbf{i}^{\mathcal{B}}.$$

Combining them with (8.2.1) we find that

$$\begin{aligned}
&(xf)B_1 - (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}} \mid f)M_{20}B_1 + vB_1 \\
&= (\mathbf{i}^{\mathcal{A}} f \otimes \mathbf{i}^{\mathcal{A}} f)B_2 - f\mathbf{i}^{\mathcal{B}} \\
&= (\mathbf{i}^{\mathcal{A}} f \otimes vB_1)B_2 + (\mathbf{i}^{\mathcal{A}} f \otimes f\mathbf{i}^{\mathcal{B}})B_2 - f\mathbf{i}^{\mathcal{B}} \\
&= -(\mathbf{i}^{\mathcal{A}} f \otimes v)B_2B_1 + (vB_1 \otimes f\mathbf{i}^{\mathcal{B}})B_2 + (f\mathbf{i}^{\mathcal{B}} \otimes f\mathbf{i}^{\mathcal{B}})B_2 - f\mathbf{i}^{\mathcal{B}} \\
&= -(\mathbf{i}^{\mathcal{A}} f \otimes v)B_2B_1 + (v \otimes f\mathbf{i}^{\mathcal{B}})B_2B_1 + (fy)B_1.
\end{aligned}$$

Now we may formulate the problem: we are looking for v as above and an (f, f)-coderivation w of degree -3, such that

$$wB_1 = xf - (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}} \mid f)M_{20} + v + (\mathbf{i}^{\mathcal{A}} f \otimes v)B_2 - (v \otimes f\mathbf{i}^{\mathcal{B}})B_2 - fy,$$

in other terms, a 4-morphism

$$\begin{aligned}
w : xf - (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}} \mid f)M_{20} + v \to fy - (\mathbf{i}^{\mathcal{A}} f \otimes v)B_2 + (v \otimes f\mathbf{i}^{\mathcal{B}})B_2 : \\
(\mathbf{i}^{\mathcal{A}} f \otimes \mathbf{i}^{\mathcal{A}} f)B_2 \to f\mathbf{i}^{\mathcal{B}} : f \to f : \mathcal{A} \to \mathcal{B}.
\end{aligned}$$

Using the chain map

$$u = (1\otimes f\mathbf{i}^{\mathcal{B}})B_2 - 1 - (\mathbf{i}^{\mathcal{A}} f \otimes 1)B_2 : (sA_\infty(\mathcal{A}, \mathcal{B})(f, f), B_1) \to (sA_\infty(\mathcal{A}, \mathcal{B})(f, f), B_1),$$

we may rewrite our system of equations as follows:

$$\begin{aligned}
-vB_1 &= f\mathbf{i}^{\mathcal{B}} - \mathbf{i}^{\mathcal{A}} f, \\
wB_1 + vu &= xf - (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}} \mid f)M_{20} - fy.
\end{aligned} \tag{8.2.2}$$

In other words, we look for an element

$$(w, v) \in [sA_\infty(\mathcal{A}, \mathcal{B})(f, f)]^{-3} \oplus [sA_\infty(\mathcal{A}, \mathcal{B})(f, f)]^{-2} = \operatorname{Cone}^{-3}(u),$$

whose boundary is

$$\begin{aligned}
&(xf - (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}} \mid f)M_{20} - fy, f\mathbf{i}^{\mathcal{B}} - \mathbf{i}^{\mathcal{A}} f) \\
&\qquad \in [sA_\infty(\mathcal{A}, \mathcal{B})(f, f)]^{-2} \oplus [sA_\infty(\mathcal{A}, \mathcal{B})(f, f)]^{-1} = \operatorname{Cone}^{-2}(u).
\end{aligned}$$

Let us prove that u is homotopy invertible. Since

$${}_X\mathbf{i}_0^{\mathcal{A}} f_1 = {}_{Xf}\mathbf{i}_0^{\mathcal{B}} + {}_X z b_1 : \Bbbk \to (s\mathcal{B})^{-1}(Xf, Xf),$$

for some ${}_X z$, the cycles ${}_X r_0 = {}_X\mathbf{i}_0^{\mathcal{A}} f_1$ and ${}_X p_0 = {}_{Xf}\mathbf{i}_0^{\mathcal{B}}$ satisfy conditions (7.13.1) for $g = f : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, that is,

$$\begin{aligned}
({}_X\mathbf{i}_0^{\mathcal{A}} f_1 \otimes {}_{Xf}\mathbf{i}_0^{\mathcal{B}})b_2 - {}_{Xf}\mathbf{i}_0^{\mathcal{B}} &= ({}_{Xf}\mathbf{i}_0^{\mathcal{B}} \otimes {}_{Xf}\mathbf{i}_0^{\mathcal{B}})b_2 - {}_{Xf}\mathbf{i}_0^{\mathcal{B}} + ({}_X z \otimes {}_{Xf}\mathbf{i}_0^{\mathcal{B}})b_2b_1 \in \operatorname{Im} b_1, \\
({}_{Xf}\mathbf{i}_0^{\mathcal{B}} \otimes {}_X\mathbf{i}_0^{\mathcal{A}} f_1)b_2 - {}_{Xf}\mathbf{i}_0^{\mathcal{B}} &= ({}_{Xf}\mathbf{i}_0^{\mathcal{B}} \otimes {}_{Xf}\mathbf{i}_0^{\mathcal{B}})b_2 - {}_{Xf}\mathbf{i}_0^{\mathcal{B}} - ({}_{Xf}\mathbf{i}_0^{\mathcal{B}} \otimes {}_X z)b_2b_1 \in \operatorname{Im} b_1.
\end{aligned}$$

Hence, the natural A_∞-transformation $r = \mathbf{i}^{\mathcal{A}} f : f \to f : \mathcal{A} \to \mathcal{B}$ is invertible by Proposition 7.15. In detail, there exists a natural A_∞-transformation $p : f \to f : \mathcal{A} \to \mathcal{B}$ and 3-morphisms q, t such that

$$(\mathbf{i}^{\mathcal{A}} f \otimes p)B_2 - f\mathbf{i}^{\mathcal{B}} = qB_1, \qquad (p \otimes \mathbf{i}^{\mathcal{A}} f)B_2 - f\mathbf{i}^{\mathcal{B}} = tB_1. \tag{8.2.3}$$

These equations are interpreted as equations (7.13.1) for the following data. Let $\mathcal{C} = \mathbb{1}$ be a 1-object-0-morphisms A_∞-category, $\operatorname{Ob}\mathcal{C} = \{*\}$, $\mathcal{C}(*,*) = 0$. Consider a map $\operatorname{Ob}\mathcal{C} \to \operatorname{Ob} A_\infty(\mathcal{A},\mathcal{B})$, $* \mapsto f$, and elements $\mathbf{i}^{\mathcal{A}} f, p \in [sA_\infty(\mathcal{A},\mathcal{B})(f,f)]^{-1}$, $q, t \in [sA_\infty(\mathcal{A},\mathcal{B})(f,f)]^{-2}$. Equations (7.13.1) for these data are precisely (8.2.3), since ${}_f\mathbf{i}_0^{A_\infty(\mathcal{A},\mathcal{B})} = f\mathbf{i}^{\mathcal{B}}$. By Lemma 7.14 we deduce that

$$(\mathbf{i}^{\mathcal{A}} f \otimes 1)B_2 : (sA_\infty(\mathcal{A},\mathcal{B})(f,f), B_1) \to (sA_\infty(\mathcal{A},\mathcal{B})(f,f), B_1),$$

is homotopy invertible. Since the map $(1 \otimes f\mathbf{i}^{\mathcal{B}})B_2 - 1$ is homotopic to 0 by Corollary 7.8, we deduce that u is homotopy invertible. Therefore, $\operatorname{Cone}(u)$ is contractible by Lemma Appendix B.1.

To prove the existence of (w,v) satisfying (8.2.2) it suffices to show that $(xf - (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}} \mid f)M_{20} - fy, f\mathbf{i}^{\mathcal{B}} - \mathbf{i}^{\mathcal{A}} f) \in \operatorname{Cone}^{-2}(u)$ is a cycle. And indeed,

$$\begin{aligned}
&[xf - (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}} \mid f)M_{20} - fy]B_1 + (f\mathbf{i}^{\mathcal{B}} - \mathbf{i}^{\mathcal{A}} f)u \\
&= (xB_1)f - (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}} \mid f)M_{20}B_1 - f(yB_1) \\
&\qquad + [(f\mathbf{i}^{\mathcal{B}} - \mathbf{i}^{\mathcal{A}} f) \otimes f\mathbf{i}^{\mathcal{B}}]B_2 - f\mathbf{i}^{\mathcal{B}} + \mathbf{i}^{\mathcal{A}} f + [\mathbf{i}^{\mathcal{A}} f \otimes (f\mathbf{i}^{\mathcal{B}} - \mathbf{i}^{\mathcal{A}} f)]B_2 \\
&= (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}})B_2 f - \mathbf{i}^{\mathcal{A}} f - (\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}} \mid f)M_{20}B_1 - f(\mathbf{i}^{\mathcal{B}} \otimes \mathbf{i}^{\mathcal{B}})B_2 + f\mathbf{i}^{\mathcal{B}} \\
&\qquad + (f\mathbf{i}^{\mathcal{B}} \otimes f\mathbf{i}^{\mathcal{B}})B_2 - (\mathbf{i}^{\mathcal{A}} f \otimes f\mathbf{i}^{\mathcal{B}})B_2 - f\mathbf{i}^{\mathcal{B}} + \mathbf{i}^{\mathcal{A}} f + (\mathbf{i}^{\mathcal{A}} f \otimes f\mathbf{i}^{\mathcal{B}})B_2 \\
&\qquad - (\mathbf{i}^{\mathcal{A}} f \otimes \mathbf{i}^{\mathcal{A}} f)B_2 \\
&= -[(\mathbf{i}^{\mathcal{A}} \otimes \mathbf{i}^{\mathcal{A}})(1 \otimes B_1 + B_1 \otimes 1) \mid f]M_{20} = 0
\end{aligned}$$

due to (7.1.2) and (7.1.3). Clearly, $(f\mathbf{i}^{\mathcal{B}} - \mathbf{i}^{\mathcal{A}} f)B_1 = 0$, so the proposition is proven. □

Clearly, the composition of unital functors is unital. If $\mathcal{B}$, $\mathcal{C}$ are unital A_∞-categories, $r : f \to g : \mathcal{B} \to \mathcal{C}$ is an isomorphism of A_∞-functors and f is unital, then g is unital as well. Indeed, distributivity law in A_∞ implies

$$\begin{tikzcd}[column sep=large] \mathcal{B} \arrow[r, shift left=5, "f"] \arrow[r, "g" description] \arrow[r, shift right=5, "g"'] \arrow[r, phantom, shift left=2.5, "r\Downarrow"] \arrow[r, phantom, shift right=2.5, "\mathbf{i}^{\mathcal{B}} g\Downarrow"] & \mathcal{C} \end{tikzcd} = \left(\begin{tikzcd} \mathcal{B} \arrow[r, shift left=3, "\mathrm{id}_{\mathcal{B}}"] \arrow[r, shift right=3, "\mathrm{id}_{\mathcal{B}}"'] \arrow[r, phantom, "\mathbf{i}^{\mathcal{B}}\Downarrow"] & \mathcal{B} \arrow[r, shift left=3, "f"] \arrow[r, shift right=3, "g"'] \arrow[r, phantom, "r\Downarrow"] & \mathcal{C} \end{tikzcd} \right) = \begin{tikzcd}[column sep=huge] \mathcal{B} \arrow[r, shift left=5, "f"] \arrow[r, "f" description] \arrow[r, shift right=5, "g"'] \arrow[r, phantom, shift left=2.5, "\mathbf{i}^{\mathcal{B}} f\Downarrow\equiv f\mathbf{i}^{\mathcal{C}}"] \arrow[r, phantom, shift right=2.5, "r\Downarrow"] & \mathcal{C} \end{tikzcd}$$

$$= \left(\begin{tikzcd} \mathcal{B} \arrow[r, shift left=3, "f"] \arrow[r, shift right=3, "g"'] \arrow[r, phantom, "r\Downarrow"] & \mathcal{C} \arrow[r, shift left=3, "\mathrm{id}_{\mathcal{C}}"] \arrow[r, shift right=3, "\mathrm{id}_{\mathcal{C}}"'] \arrow[r, phantom, "\mathbf{i}^{\mathcal{C}}\Downarrow"] & \mathcal{C} \end{tikzcd} \right) = \begin{tikzcd}[column sep=large] \mathcal{B} \arrow[r, shift left=5, "f"] \arrow[r, "g" description] \arrow[r, shift right=5, "g"'] \arrow[r, phantom, shift left=2.5, "r\Downarrow"] \arrow[r, phantom, shift right=2.5, "g\mathbf{i}^{\mathcal{C}}\Downarrow"] & \mathcal{C}, \end{tikzcd}$$

or $$r \cdot (\mathbf{i}^{\mathcal{B}} g) = \mathbf{i}^{\mathcal{B}} \circ_h r = (\mathbf{i}^{\mathcal{B}} f) \cdot r = (f\mathbf{i}^{\mathcal{C}}) \cdot r = r \circ_h \mathbf{i}^{\mathcal{C}} = r \cdot (g\mathbf{i}^{\mathcal{C}}), \tag{8.2.4}$$

where $\cdot$ and $\circ_h$ denote the vertical and the horizontal compositions of 2-morphisms, hence, $\mathbf{i}^{\mathcal{B}} g \equiv g\mathbf{i}^{\mathcal{C}}$.

8.3 Definition. The 2-category A_∞^u is a 2-subcategory of ${}^u A_\infty$, whose class of objects consists of all unital A_∞-categories, 1-morphisms are all unital A_∞-functors, and 2-morphisms are equivalence classes of all natural A_∞-transformations between such functors.

8.4 Proposition. *Let $\mathcal{A}$ be an A_∞-category. The strict 2-functor $A_\infty(\mathcal{A},\_)$ maps a unital A_∞-category $\mathcal{C}$ to the unital A_∞-category $A_\infty(\mathcal{A},\mathcal{C})$, and a unital functor to a unital functor. Its restrictions $A_\infty(\mathcal{A},\_) : {}^uA_\infty \to {}^uA_\infty$, $A_\infty(\mathcal{A},\_) : A_\infty^u \to A_\infty^u$ preserve 1-units and 2-units, thus, they are 2-functors in the usual sense.*

Proof. Proposition 7.7 shows that $A_\infty(\mathcal{A},\mathcal{C})$ is unital, if $\mathcal{C}$ is unital. If $g : \mathcal{B} \to \mathcal{C}$ is a unital A_∞-functor between unital A_∞-categories $\mathcal{B}$ and $\mathcal{C}$, then $\mathbf{i}^{\mathcal{B}}g \equiv g\mathbf{i}^{\mathcal{C}}$ implies

$$(1\otimes \mathbf{i}^{\mathcal{B}})M(1\otimes g)M = (1\otimes \mathbf{i}^{\mathcal{B}}g)M \equiv (1\otimes g\mathbf{i}^{\mathcal{C}})M = (1\otimes g)M(1\otimes \mathbf{i}^{\mathcal{C}})M, \quad (8.4.1)$$

hence, $(1\otimes g)M$ is unital. The fact, that $A_\infty(\mathcal{A},\_)$ preserves 1-units and 2-units is already proven in Proposition 7.7. □

8.5. Categories modulo homotopy. $\mathcal{K}$-categories form a 2-category $\mathcal{K}$-$\mathcal{C}at$. We consider also non-unital $\mathcal{K}$-categories. They form a 2-category $\mathcal{K}$-$\mathcal{C}at^{nu}$ without 2-units (but with 1-units – identity functors).

8.6 Proposition. *There is a strict 2-functor $\mathsf{k} : A_\infty \to \mathcal{K}\text{-}\mathcal{C}at^{nu}$ of non-2-unital 2-categories, which assigns to an A_∞-category $\mathcal{C}$ the $\mathcal{K}$-category $\mathsf{k}\mathcal{C}$ with the same class of objects $\operatorname{Ob}\mathsf{k}\mathcal{C} = \operatorname{Ob}\mathcal{C}$, the same graded $\Bbbk$-module of morphisms $\mathsf{k}\mathcal{C}(X,Y) = \mathcal{C}(X,Y)$, equipped with the differential m_1. Composition in $\mathsf{k}\mathcal{C}$ is given by (the homotopy equivalence class of) $m_2 : \mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$. To an A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is assigned $\mathsf{k}f : \mathsf{k}\mathcal{A} \to \mathsf{k}\mathcal{B}$ such that $\operatorname{Ob}\mathsf{k}f = f : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, and for each pair of objects X, Y of $\mathcal{A}$ we have $\mathsf{k}f = sf_1s^{-1} : \mathcal{A}(X,Y) \to \mathcal{B}(Xf,Yf)$. To a natural A_∞-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}$ is assigned $\mathsf{k}r = r_0s^{-1} : \mathsf{k}f \to \mathsf{k}g$, that is, for each object X of $\mathcal{A}$ the component ${}_X\mathsf{k}r$ is the homotopy equivalence class of chain map ${}_Xr_0s^{-1} : \Bbbk \to \mathcal{B}(Xf,Xg)$. Unital A_∞-categories and unital A_∞-functors are mapped by k to unital $\mathcal{K}$-categories and unital $\mathcal{K}$-functors. The restriction $\mathsf{k} : A_\infty^u \to \mathcal{K}\text{-}\mathcal{C}at$ is a 2-functor, which preserves 1-units and 2-units.*

Proof. The identity (7.1.1) shows that m_2 is associative in $\mathcal{K}$. The identity

$$(\mathsf{k}f\otimes\mathsf{k}f)m_2 + (s\otimes s)f_2s^{-1}m_1 + (1\otimes m_1 + m_1\otimes 1)(s\otimes s)f_2s^{-1} + m_2\mathsf{k}f = 0$$

shows that $\mathsf{k}f$ preserves the multiplication in $\mathcal{K}$.

The map ${}_X\mathsf{k}r$ is a chain map since ${}_Xr_0s^{-1}m_1 = {}_Xr_0b_1s^{-1} = 0$. If $r \equiv p : f \to g : \mathcal{A} \to \mathcal{B}$, then ${}_Xr_0 = {}_Xp_0 + {}_Xv_0b_1$ for some ${}_Xv_0 \in (s\mathcal{B})^{-2}(Xf,Xg)$, therefore, ${}_Xr_0s^{-1} = {}_Xp_0s^{-1} + ({}_Xv_0s^{-1})m_1$ and chain maps ${}_Xr_0s^{-1}$ and ${}_Xp_0s^{-1}$ are homotopic to each other, that is, $\mathsf{k}r = \mathsf{k}p$. The identity

$$\begin{aligned} 0 &= s[(f_1\otimes{}_Yr_0)b_2 + ({}_Xr_0\otimes g_1)b_2 + r_1b_1 + b_1r_1]s^{-1} \\ &= (\mathsf{k}f\otimes{}_Y\mathsf{k}r)m_2 - ({}_X\mathsf{k}r\otimes\mathsf{k}g)m_2 + sr_1s^{-1}m_1 + m_1sr_1s^{-1} : \mathcal{A}(X,Y) \to \mathcal{B}(Xf,Yg) \end{aligned}$$

shows that the following diagram commutes in $\mathcal{K}$ for all objects X, Y of $\mathcal{A}$

$$\begin{tikzcd} \mathcal{A}(X,Y) \arrow[r, "\mathsf{k}f"] \arrow[d, "\mathsf{k}g"'] & \mathcal{B}(Xf,Yf) \arrow[d, "(1\otimes{}_Y\mathsf{k}r)m_2"] \\ \mathcal{B}(Xg,Yg) \arrow[r, "({}_X\mathsf{k}r\otimes 1)m_2"] & \mathcal{B}(Xf,Yg) \end{tikzcd}$$

Thus $\mathsf{k}r$ is, indeed, a $\mathcal{K}$-natural transformation.

One checks easily that the composition of functors is preserved, and the both compositions of 1-morphisms and 2-morphisms are preserved. The vertical composition of 2-morphisms is preserved due to the property

$$_X\mathsf{k}[(r \otimes p)B_2] = {}_X[(r \otimes p)B_2]_0 s^{-1} = ({}_X r_0 \otimes {}_X p_0)b_2 s^{-1} = ({}_X\mathsf{k}r \otimes {}_X\mathsf{k}p)m_2.$$

Let $\mathcal{B}$ be a unital category. Then $\mathsf{k}\mathcal{B}$ is a unital $\mathcal{K}$-category. Indeed, for each object X of $\mathcal{B}$ consider the corresponding element $1_X = {}_X\mathbf{i}_0^{\mathcal{B}} s^{-1} = {}_X\mathsf{k}\mathbf{i}^{\mathcal{B}} : \Bbbk \to \mathcal{B}^0(X,X)$. Then for each pair X, Y of objects of $\mathcal{C}$ the following equations hold in $\mathcal{K}$

$$\begin{aligned}(1 \otimes 1_Y)m_2 &= s(1 \otimes 1_Y s)b_2 s^{-1} = s(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{B}})b_2 s^{-1} = ss^{-1} \\ &= 1 : \mathcal{B}(X,Y) \to \mathcal{B}(X,Y), \\ (1_X \otimes 1)m_2 &= -s(1_X s \otimes 1)b_2 s^{-1} = -s({}_X\mathbf{i}_0^{\mathcal{B}} \otimes 1)b_2 s^{-1} \\ &= ss^{-1} = 1 : \mathcal{B}(X,Y) \to \mathcal{B}(X,Y).\end{aligned}$$

That is, 1_X is the unit endomorphism of X.

If $f : \mathcal{A} \to \mathcal{C}$ is unital then, applying k to the equivalence $\mathbf{i}^{\mathcal{A}} f \equiv f\mathbf{i}^{\mathcal{C}}$, we find that $(1_{\mathrm{id}_{\mathsf{k}\mathcal{A}}})\mathsf{k}f = (\mathsf{k}f)1_{\mathrm{id}_{\mathsf{k}\mathcal{C}}} = 1_{\mathsf{k}f}$, that is, $\mathsf{k}f$ is unital (it maps units into units). □

8.7 Lemma (Cancellation). *Let $\phi : \mathcal{C} \to \mathcal{B}$ be an A_∞-functor, such that for all objects X, Y of $\mathcal{C}$ the chain map $\phi_1 : (s\mathcal{C}(X,Y), b_1) \to (s\mathcal{B}(X\phi, Y\phi), b_1)$ is invertible in $\mathcal{K}$. Let $f, g : \mathcal{A} \to \mathcal{C}$ be A_∞-functors. Let $y : f\phi \to g\phi : \mathcal{A} \to \mathcal{B}$ be a natural A_∞-transformation. Then there is a unique up to equivalence natural A_∞-transformation $t : f \to g : \mathcal{A} \to \mathcal{C}$ such that $y \equiv t\phi$.*

Proof. First we prove the existence. We are looking for a 2-morphism $t : f \to g : \mathcal{A} \to \mathcal{C}$ and a 3-morphism $v : y \to t\phi : f\phi \to g\phi : \mathcal{A} \to \mathcal{B}$. We have to satisfy the equations

$$tb + bt = 0, \qquad y - t\phi = vb - bv.$$

Let us construct the components of t and v by induction. Given a non-negative integer n, assume that we have already found components t_m, v_m of the sought t, v for $m < n$, such that the equations

$$(tb)_m + (bt)_m = 0 : s\mathcal{A}(X_0, X_1) \otimes \cdots \otimes s\mathcal{A}(X_{m-1}, X_m) \to s\mathcal{C}(X_0 f, X_m g), \tag{8.7.1}$$

$$y_m - (t\phi)_m = (vb - bv)_m : s\mathcal{A}(X_0, X_1) \otimes \cdots \otimes s\mathcal{A}(X_{m-1}, X_m) \to s\mathcal{B}(X_0 f\phi, X_m g\phi), \tag{8.7.2}$$

are satisfied for all $m < n$. Introduce an (f,g)-coderivation $\tilde{t} : Ts\mathcal{A} \to Ts\mathcal{C}$ of degree -1 by its components $(t_0, \dots, t_{n-1}, 0, 0, \dots)$ and an $(f\phi, g\phi)$-coderivation $\tilde{v} : Ts\mathcal{A} \to Ts\mathcal{B}$ of degree -2 by its components $(v_0, \dots, v_{n-1}, 0, 0, \dots)$. Define an (f,g)-coderivation $\lambda = \tilde{t}b + b\tilde{t}$ of degree 0 and an $(f\phi, g\phi)$-coderivation $\nu = y - \tilde{t}\phi - \tilde{v}b + b\tilde{v}$ of degree -1. Then equations (8.7.1), (8.7.2) imply that $\lambda_m = 0$, $\nu_m = 0$ for $m < n$. The identity $\lambda b - b\lambda = 0$ implies that

$$\lambda_n d = \lambda_n b_1 - \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta})\lambda_n = 0. \tag{8.7.3}$$

The identity

$$\nu b + b\nu = -\tilde{t}\phi b - b\tilde{t}\phi = -\lambda\phi$$

implies that

$$\nu_n d = \nu_n b_1 + \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta})\nu_n = -\lambda_n \phi_1. \tag{8.7.4}$$

Denote $N = s\mathcal{A}(X_0, X_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} s\mathcal{A}(X_{n-1}, X_n)$, and consider the chain map

$$u = \operatorname{Hom}(N, \phi_1) : \operatorname{Hom}^\bullet(N, s\mathcal{C}(X_0 f, X_n g)) \to \operatorname{Hom}^\bullet(N, s\mathcal{C}(X_0 f\phi, X_n g\phi)).$$

Since ϕ_1 is homotopy invertible, the map u is homotopy invertible as well. Therefore, the complex $\operatorname{Cone}(u)$ is acyclic. Moreover, it is contractible by Lemma Appendix B.1. Equations (8.7.3) and (8.7.4) in the form $-\lambda_n d = 0$, $\nu_n d + \lambda_n \phi_1 = 0$ imply that

$$(\nu_n, \lambda_n) \in \operatorname{Hom}^{-1}_{\Bbbk}(N, s\mathcal{B}(X_0 f\phi, X_n g\phi)) \oplus \operatorname{Hom}^0_{\Bbbk}(N, s\mathcal{C}(X_0 f, X_n g)) = \operatorname{Cone}^{-1}(u)$$

is a boundary of some element

$$(v_n, t_n) \in \operatorname{Hom}^{-2}_{\Bbbk}(N, s\mathcal{B}(X_0 f\phi, X_n g\phi)) \oplus \operatorname{Hom}^{-1}_{\Bbbk}(N, s\mathcal{C}(X_0 f, X_n g)) = \operatorname{Cone}^{-2}(u),$$

that is, $v_n d + t_n u = \nu_n$ and $-t_n d = \lambda_n$. In other words, equations (8.7.1), (8.7.2) are satisfied for $m = n$, and we prove the existence of t with the required properties by induction.

Now we prove the uniqueness of t. Assume that we have 2-morphisms $t, t' : f \to g : \mathcal{A} \to \mathcal{C}$ and 3-morphisms $v : y \to t\phi : f\phi \to g\phi : \mathcal{A} \to \mathcal{B}$, $v' : y \to t'\phi : f\phi \to g\phi : \mathcal{A} \to \mathcal{B}$. We look for a 3-morphism w and a 4-morphism x:

$$w : t \to t' : f \to g : \mathcal{A} \to \mathcal{C},$$
$$x : v' \to v + w\phi : y \to t'\phi : f\phi \to g\phi : \mathcal{A} \to \mathcal{B}.$$

They have to satisfy equations

$$t - t' = wb - bw, \qquad v' - v - w\phi = xb + bx.$$

Let us construct the components of w and x by induction. Given a non-negative integer n, assume that we have already found components w_m and x_m of the sought w, x for $m < n$, such that the equations

$$t_m - t'_m = (wb - bw)_m : s\mathcal{A}(X_0, X_1)$$
$$\otimes \cdots \otimes s\mathcal{A}(X_{m-1}, X_m) \to s\mathcal{C}(X_0 f, X_m g), \tag{8.7.5}$$

$$v'_m - v_m - (w\phi)_m = (xb + bx)_m : s\mathcal{A}(X_0, X_1)$$
$$\otimes \cdots \otimes s\mathcal{A}(X_{m-1}, X_m) \to s\mathcal{B}(X_0 f\phi, X_m g\phi), \tag{8.7.6}$$

are satisfied for all $m < n$. Introduce an (f, g)-coderivation $\tilde{w} : Ts\mathcal{A} \to Ts\mathcal{C}$ of degree -2 by its components $(w_0, \dots, w_{n-1}, 0, 0, \dots)$ and an $(f\phi, g\phi)$-coderivation $\tilde{x} : Ts\mathcal{A} \to Ts\mathcal{B}$ of degree -3 by its components $(x_0, \dots, x_{n-1}, 0, 0, \dots)$. Define an

(f,g)-coderivation $\lambda = t' - t + \tilde{w}b + b\tilde{w}$ of degree -1 and an $(f\phi, g\phi)$-coderivation $\nu = v' - v - \tilde{w}\phi - \tilde{x}b - b\tilde{x}$ of degree -2. Then equations (8.7.5), (8.7.6) imply that $\lambda_m = 0$, $\nu_m = 0$ for $m < n$. The identity $\lambda b + b\lambda = 0$ implies that

$$\lambda_n d = \lambda_n b_1 + \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta})\lambda_n = 0.$$

The identity

$$\nu b - b\nu = v'B_1 - vB_1 - \tilde{w}\phi b + b\tilde{w}\phi = y - t'\phi - y + t\phi - \tilde{w}b\phi + b\tilde{w}\phi = -\lambda\phi$$

implies that

$$\nu_n d = \nu_n b_1 - \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta})\nu_n = -\lambda_n\phi_1.$$

Hence,

$$(\nu_n, \lambda_n) \in \mathrm{Hom}_{\Bbbk}^{-2}(N, s\mathcal{B}(X_0 f\phi, X_n g\phi)) \oplus \mathrm{Hom}_{\Bbbk}^{-1}(N, s\mathcal{C}(X_0 f, X_n g)) = \mathrm{Cone}^{-2}(u)$$

is a cycle, therefore, it is a boundary of an element

$$(x_n, w_n) \in \mathrm{Hom}_{\Bbbk}^{-3}(N, s\mathcal{B}(X_0 f\phi, X_n g\phi)) \oplus \mathrm{Hom}_{\Bbbk}^{-2}(N, s\mathcal{C}(X_0 f, X_n g)) = \mathrm{Cone}^{-3}(u),$$

that is, $x_n d + w_n\phi_1 = \nu_n$ and $-w_n d = \lambda_n$. In other words, equations (8.7.5), (8.7.6), are satisfied for $m = n$, and we prove the uniqueness of t, using induction. □

A version of the following theorem is proved by Fukaya [**Fuk**, Theorem 8.6] with a different notion of unitality and under the additional assumption that the $\Bbbk$-modules $\mathcal{B}(W,Z)$, $\mathcal{C}(X,Y)$ are free.

8.8 Theorem. *Let $\mathcal{C}$ be an A_∞-category and let $\mathcal{B}$ be a unital A_∞-category. Let $\phi : \mathcal{C} \to \mathcal{B}$ be an A_∞-functor such that for all objects X, Y of $\mathcal{C}$ the chain map $\phi_1 : (s\mathcal{C}(X,Y), b_1) \to (s\mathcal{B}(X\phi, Y\phi), b_1)$ is invertible in $\mathcal{K}$. Let $h : \mathrm{Ob}\,\mathcal{B} \to \mathrm{Ob}\,\mathcal{C}$ be a mapping. Assume that for each object U of $\mathcal{B}$ the $\Bbbk$-linear maps*

$$_Ur_0 : \Bbbk \to (s\mathcal{B})^{-1}(U, Uh\phi), \qquad _Up_0 : \Bbbk \to (s\mathcal{B})^{-1}(Uh\phi, U),$$
$$_Uw_0 : \Bbbk \to (s\mathcal{B})^{-2}(U, Uh\phi), \qquad _Uv_0 : \Bbbk \to (s\mathcal{B})^{-2}(Uh\phi, U)$$

are given such that

$$\begin{gathered} _Ur_0 b_1 = 0, \qquad _Up_0 b_1 = 0, \\ (_Ur_0 \otimes {_Up_0})b_2 - {_U\mathbf{i}_0^{\mathcal{B}}} = {_Uw_0} b_1, \\ (_Up_0 \otimes {_Ur_0})b_2 - {_{Uh\phi}\mathbf{i}_0^{\mathcal{B}}} = {_Uv_0} b_1. \end{gathered} \tag{8.8.1}$$

Then there is an A_∞-functor $\psi : \mathcal{B} \to \mathcal{C}$ such that $\mathrm{Ob}\,\psi = h$, there are natural A_∞-transformations $r : \mathrm{id}_{\mathcal{B}} \to \psi\phi$, $p : \psi\phi \to \mathrm{id}_{\mathcal{B}}$ such that their 0-th components are the given $_Ur_0$, $_Up_0$. Moreover, r and p are inverse to each other in the sense that

$$(r \otimes p)B_2 \equiv \mathbf{i}^{\mathcal{B}}, \qquad (p \otimes r)B_2 \equiv \psi\phi\mathbf{i}^{\mathcal{B}}.$$

There exist unique up to equivalence natural A_∞-transformations $t : \mathrm{id}_{\mathcal{C}} \to \phi\psi$, $q : \phi\psi \to \mathrm{id}_{\mathcal{C}}$ such that $t\phi \equiv \phi r : \phi \to \phi\psi\phi$ and $q\phi \equiv \phi p : \phi\psi\phi \to \phi$.

Finally, $\mathcal{C}$ is unital with the unit

$$\mathbf{i}^{\mathcal{C}} = (t \otimes q)B_2 : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C},$$

ϕ and ψ are unital A_∞-equivalences, quasi-inverse to each other via mutually inverse isomorphisms r and p, t and q (in particular, $(q \otimes t)B_2 \equiv \phi\psi\mathbf{i}^{\mathcal{C}}$).

Proof. We have to satisfy the equations

$$\psi b = b\psi, \qquad rb + br = 0.$$

We already know the map $\operatorname{Ob}\psi$ and the component r_0. Let us construct the remaining components of ψ and r by induction. Given a positive integer n, assume that we have already found components ψ_m, r_m of the sought ψ, r for $m < n$, such that the equations

$$(\psi b)_m + (b\psi)_m = 0 : s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0 h, X_m h), \tag{8.8.2}$$

$$(rb + br)_m = 0 : s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{B}(X_0, X_m h\phi) \tag{8.8.3}$$

are satisfied for all $m < n$. Introduce a cocategory homomorphism $\tilde{\psi} : Ts\mathcal{B} \to Ts\mathcal{C}$ of degree 0 by its components $(\psi_1, \dots, \psi_{n-1}, 0, 0, \dots)$ and a $(\mathrm{id}_{\mathcal{B}}, \tilde{\psi}\phi)$-coderivation $\tilde{r} : Ts\mathcal{B} \to Ts\mathcal{B}$ of degree -1 by its components $(r_0, r_1, \dots, r_{n-1}, 0, 0, \dots)$. Define a $(\tilde{\psi}, \tilde{\psi})$-coderivation $\lambda = \tilde{\psi}b - b\tilde{\psi}$ of degree 1 and a map $\nu = -\tilde{r}b - b\tilde{r} + (\tilde{r} \otimes \lambda\phi)\theta : Ts\mathcal{B} \to Ts\mathcal{B}$ of degree 0. The commutator $\tilde{r}b + b\tilde{r}$ has the following property:

$$(\tilde{r}b + b\tilde{r})\Delta = \Delta\big[1 \otimes (\tilde{r}b + b\tilde{r}) + (\tilde{r}b + b\tilde{r}) \otimes \tilde{\psi}\phi + \tilde{r} \otimes \lambda\phi\big].$$

By Proposition 3.1 the map $(\tilde{r} \otimes \lambda\phi)\theta$ has a similar property

$$(\tilde{r} \otimes \lambda\phi)\theta\Delta = \Delta\big[1 \otimes (\tilde{r} \otimes \lambda\phi)\theta + (\tilde{r} \otimes \lambda\phi)\theta \otimes \tilde{\psi}\phi + \tilde{r} \otimes \lambda\phi\big].$$

Taking the difference we find that ν is an $(\mathrm{id}_{\mathcal{B}}, \tilde{\psi}\phi)$-coderivation. Equations (8.8.2), (8.8.3) imply that $\lambda_m = 0$, $\nu_m = 0$ for $m < n$ (the image of $(\tilde{r} \otimes \lambda\phi)\theta$ is contained in $T^{\geqslant 2}s\mathcal{B}$).

The identity $\lambda b + b\lambda = 0$ implies that

$$\lambda_n d = \lambda_n b_1 + \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta})\lambda_n = 0. \tag{8.8.4}$$

The identity

$$\nu b - b\nu = (\tilde{r} \otimes \lambda\phi)\theta b - b(\tilde{r} \otimes \lambda\phi)\theta$$

implies that

$$\nu_n b_1 - \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta})\nu_n = -(r_0 \otimes \lambda_n\phi_1)b_2 = -\lambda_n\phi_1(r_0 \otimes 1)b_2. \tag{8.8.5}$$

Set $N = s\mathcal{B}(X_0, X_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} s\mathcal{B}(X_{n-1}, X_n)$, and introduce a chain map

$$u = \operatorname{Hom}(N, \phi_1(r_0 \otimes 1)b_2) : \operatorname{Hom}^\bullet(N, s\mathcal{C}(X_0 h, X_n h)) \to \operatorname{Hom}^\bullet(N, s\mathcal{B}(X_0, X_n h\phi)).$$

Since ϕ_1 and $(r_0 \otimes 1)b_2$ are homotopy invertible by Lemma 7.14, the map u is homotopy invertible as well. Therefore, the complex $\operatorname{Cone}(u)$ is contractible by

Lemma Appendix B.1. Equations (8.8.4) and (8.8.5) in the form $-\lambda_n d = 0$, $\nu_n d + \lambda_n u = 0$ imply that

$$(\nu_n, \lambda_n) \in \mathrm{Hom}^0_{\Bbbk}(N, s\mathcal{B}(X_0, X_n h\phi)) \oplus \mathrm{Hom}^1_{\Bbbk}(N, s\mathcal{C}(X_0 h, X_n h)) = \mathrm{Cone}^0(u)$$

is a cycle. Hence, it is a boundary of some element

$$(r_n, \psi_n) \in \mathrm{Hom}^{-1}_{\Bbbk}(N, s\mathcal{B}(X_0, X_n h\phi)) \oplus \mathrm{Hom}^0_{\Bbbk}(N, s\mathcal{C}(X_0 h, X_n h)) = \mathrm{Cone}^{-1}(u),$$

that is, $r_n d + \psi_n \phi_1 (r_0 \otimes 1) b_2 = \nu_n$ and $-\psi_n d = \lambda_n$. In other words, equations (8.8.2), (8.8.3) are satisfied for $m = n$, and we prove the existence of ψ and r by induction.

Since r_0 and p_0 are homotopy inverse to each other in the sense of (8.8.1), we find by Proposition 7.15 that there exists a natural A_∞-transformation $p : \psi\phi \to \mathrm{id}_{\mathcal{B}}$ such that r and p are inverse to each other.

The existence of t, q such that $t\phi \equiv \phi r$ and $q\phi \equiv \phi p$ follows by Lemma 8.7. Let us prove that $\mathbf{i}^{\mathcal{C}} = (t \otimes q) B_2$ is a unit of $\mathcal{C}$. Due to Lemma 7.14 the maps $(r_0 \otimes 1) b_2$, $(1 \otimes r_0) b_2$, $(p_0 \otimes 1) b_2$, $(1 \otimes p_0) b_2$ are homotopy invertible. Let f denote a homotopy inverse map of $\phi_1 : s\mathcal{C}(X, Y) \to s\mathcal{B}(X\phi, Y\phi)$. The identity $t\phi \equiv \phi r$ implies that ${}_X t_0 \phi_1 = {}_{X\phi} r_0 + \kappa b_1$. Hence,

$$({}_{X\phi} r_0 \otimes 1) b_2 \sim ({}_X t_0 \phi_1 \otimes 1) b_2 \sim f({}_X t_0 \otimes 1) b_2 \phi_1.$$

Therefore, $({}_X t_0 \otimes 1) b_2 \sim \phi_1 ({}_{X\phi} r_0 \otimes 1) b_2 f$ is homotopy invertible. Similarly,

$$(1 \otimes {}_{Y\phi} r_0) b_2 \sim (1 \otimes {}_Y t_0 \phi_1) b_2 \sim f(1 \otimes {}_Y t_0) b_2 \phi_1$$

implies that $(1 \otimes {}_Y t_0) b_2 \sim \phi_1 (1 \otimes {}_{Y\phi} r_0) b_2 f$ is homotopy invertible. Similarly, $({}_X q_0 \otimes 1) b_2$ and $(1 \otimes {}_Y q_0) b_2$ are homotopy invertible.

The computation made in (7.14.1) shows that the product of the above homotopy invertible maps

$$(q_0 \otimes 1) b_2 (t_0 \otimes 1) b_2 \sim -(t_0 \otimes q_0 \otimes 1)(b_2 \otimes 1) b_2 = -(\mathbf{i}^{\mathcal{C}}_0 \otimes 1) b_2$$

is the map we are studying. Similarly,

$$(1 \otimes t_0) b_2 (1 \otimes q_0) b_2 \sim (1 \otimes t_0 \otimes q_0)(1 \otimes b_2) b_2 = (1 \otimes \mathbf{i}^{\mathcal{C}}_0) b_2.$$

We conclude that both $(\mathbf{i}^{\mathcal{C}}_0 \otimes 1) b_2$ and $(1 \otimes \mathbf{i}^{\mathcal{C}}_0) b_2$ are homotopy invertible.

Let us prove that $(\mathbf{i}^{\mathcal{C}} \otimes \mathbf{i}^{\mathcal{C}}) B_2 \equiv \mathbf{i}^{\mathcal{C}}$. Due to Proposition 7.1 we have

$$\mathbf{i}^{\mathcal{C}} \phi = (t \otimes q) B_2 \phi \equiv (t\phi \otimes q\phi) B_2 \equiv (\phi r \otimes \phi p) B_2 = \phi (r \otimes p) B_2 \equiv \phi \mathbf{i}^{\mathcal{B}}. \qquad (8.8.6)$$

Using Proposition 7.1 again we get

$$(\mathbf{i}^{\mathcal{C}} \otimes \mathbf{i}^{\mathcal{C}}) B_2 \phi \equiv (\mathbf{i}^{\mathcal{C}} \phi \otimes \mathbf{i}^{\mathcal{C}} \phi) B_2 \equiv (\phi \mathbf{i}^{\mathcal{B}} \otimes \phi \mathbf{i}^{\mathcal{B}}) B_2 = \phi (\mathbf{i}^{\mathcal{B}} \otimes \mathbf{i}^{\mathcal{B}}) B_2 \equiv \phi \mathbf{i}^{\mathcal{B}} \equiv \mathbf{i}^{\mathcal{C}} \phi.$$

By Lemma 8.7 we deduce that $(\mathbf{i}^{\mathcal{C}} \otimes \mathbf{i}^{\mathcal{C}}) B_2 \equiv \mathbf{i}^{\mathcal{C}}$, therefore, $\mathbf{i}^{\mathcal{C}}$ is a unit of $\mathcal{C}$.

Let us prove that t and q are inverse to each other. By definition, $(t \otimes q) B_2 = \mathbf{i}^{\mathcal{C}}$. Due to Proposition 7.1

$$(q \otimes t) B_2 \phi \equiv (q\phi \otimes t\phi) B_2 \equiv (\phi p \otimes \phi r) B_2 = \phi (p \otimes r) B_2 \equiv \phi\psi\phi \mathbf{i}^{\mathcal{B}} \equiv \phi\psi \mathbf{i}^{\mathcal{C}} \phi.$$

By Lemma 8.7 $(q \otimes t) B_2 \equiv \phi\psi \mathbf{i}^{\mathcal{C}}$. Hence, t and q are inverse to each other, as well as r and p. Therefore, ϕ and ψ are equivalences, quasi-inverse to each other.

Relation (8.8.6) shows that ϕ is unital. Let us prove that ψ is unital. We know that $\psi\phi$ is isomorphic to the identity functor. Thus, $\psi\phi$ is unital by (8.2.4). Hence, $\mathbf{i}^{\mathcal{B}}\psi\phi \equiv \psi\phi\mathbf{i}^{\mathcal{B}} \equiv \psi\mathbf{i}^{\mathcal{C}}\phi$. By Lemma 8.7 we have $\mathbf{i}^{\mathcal{B}}\psi \equiv \psi\mathbf{i}^{\mathcal{C}}$, and ψ is unital. The theorem is proven. □

8.9 Corollary. *Let $\mathcal{C}$, $\mathcal{B}$ be unital A_∞-categories, and let $\phi : \mathcal{C} \to \mathcal{B}$ be an equivalence. Then ϕ is unital.*

Proof. Since ϕ is an equivalence, $\mathsf{k}\phi$ is an equivalence as well. Hence, ϕ_1 is invertible in $\mathcal{K}$. There exists an A_∞-functor $\psi : \mathcal{B} \to \mathcal{C}$ quasi-inverse to ϕ, and mutually inverse isomorphisms $r : \mathrm{id}_{\mathcal{B}} \to \psi\phi$, $p : \psi\phi \to \mathrm{id}_{\mathcal{B}}$. In particular, the assumptions of Theorem 8.8 are satisfied by ϕ, $\mathrm{Ob}\,\psi : \mathrm{Ob}\,\mathcal{B} \to \mathrm{Ob}\,\mathcal{C}$, r_0 and p_0. The theorem implies that ϕ is unital. □

8.10 Corollary. *Let $\mathcal{C}$ be an A_∞-algebra and let $\mathcal{B}$ be a unital A_∞-algebra (viewed as A_∞-categories with one object). Let $\phi : \mathcal{C} \to \mathcal{B}$ be an A_∞-homomorphism such that $\phi_1 : (s\mathcal{C}, b_1) \to (s\mathcal{B}, b_1)$ is homotopy invertible. Then $\mathcal{C}$ and ϕ are unital, and ϕ is an A_∞-equivalence.*

Existence of ϕ with the above property might be taken as an equivalence relation on the class of unital A_∞-algebras.

8.11. Strictly unital A_∞-categories. A *strict unit* of an object X of an A_∞-category $\mathcal{A}$ is an element $1_X \in \mathcal{A}^0(X,X)$, such that $(f \otimes 1_X)m_2 = f$, $(1_X \otimes g)m_2 = g$, whenever these make sense, and $(\dots \otimes 1_X \otimes \dots)m_n = 0$ if $n \neq 2$ (see e.g. [**FOOO, Fuk, Kel01**]). We may write it as a map $1_X : \Bbbk \to \mathcal{A}(X,X)$, $1 \mapsto 1_X$. Assume that $\mathcal{A}$ has a strict unit for each object X. For example, a differential graded category $\mathcal{A}$ has strict units. Then we introduce a coderivation $\mathbf{i}^{\mathcal{A}} : \mathrm{id}_{\mathcal{A}} \to \mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$, whose components are $\mathbf{i}_0^{\mathcal{A}} : \Bbbk \to s\mathcal{A}(X,X)$, $1 \mapsto 1_X s = {}_X\mathbf{i}_0^{\mathcal{A}}$, and $\mathbf{i}_k^{\mathcal{A}} = 0$ for $k > 0$. The conditions on 1_X imply that $(1\otimes \mathbf{i}_0^{\mathcal{A}})b_2 = 1 : s\mathcal{A}(Y,X) \to s\mathcal{A}(Y,X)$ and $(\mathbf{i}_0^{\mathcal{A}} \otimes 1)b_2 = -1 : s\mathcal{A}(X,Z) \to s\mathcal{A}(X,Z)$. One deduces that $\mathbf{i}^{\mathcal{A}}$ is a natural A_∞-transformation. If an A_∞-category $\mathcal{A}$ has two such transformations – strict units $\mathbf{i}$ and $\mathbf{i}'$, then they must coincide because of the above equations. We call $\mathcal{A}$ *strictly unital* if it has a strict unit $\mathbf{i}^{\mathcal{A}}$. Naturally, a strictly unital A_∞-category is unital.

For any A_∞-functor $f : \mathcal{C} \to \mathcal{A}$ the natural A_∞-transformation $1_f s = f\mathbf{i}^{\mathcal{A}} : f \to f : \mathcal{C} \to \mathcal{A}$ has the components ${}_X(f\mathbf{i}^{\mathcal{A}})_0 = {}_{Xf}\mathbf{i}_0^{\mathcal{A}} : \Bbbk \to s\mathcal{A}(Xf, Xf)$ and $(f\mathbf{i}^{\mathcal{A}})_k = 0$ for $k > 0$. It is the unit 2-endomorphism of f.

If A_∞-category $\mathcal{B}$ is strictly unital, then so is $\mathcal{C} = A_\infty(\mathcal{A}, \mathcal{B})$ for an arbitrary A_∞-category $\mathcal{A}$. Indeed, for an arbitrary A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ there is a unit 2-endomorphism $1_f s = f\mathbf{i}^{\mathcal{B}} : f \to f$. We set $\mathbf{i}_0^{\mathcal{C}} : \Bbbk \to [sA_\infty(\mathcal{A},\mathcal{B})]^{-1}(f,f)$, $1 \mapsto 1_f s$, and $\mathbf{i}_k^{\mathcal{C}} = 0$ for $k > 0$. For any element $r \in \mathcal{C}(g,f)$ we have $(r \otimes 1_f s)B_2 = r$. For any element $p \in \mathcal{C}(f,h)$ we have $p(1_f s \otimes 1)B_2 = p((f\mathbf{i}^{\mathcal{B}})_0 \otimes 1)b_2 = -p$. We have also $\mathbf{i}^{\mathcal{C}} B_1 = 0$ and $(\dots \otimes \mathbf{i}^{\mathcal{C}} \otimes \dots)B_n = 0$ if $n > 2$, due to (5.1.3). Therefore, $\mathbf{i}^{\mathcal{C}}$ satisfies the required conditions.

Another approach to $\mathbf{i}^{\mathcal{C}}$ uses the A_∞-functor $M : TsA_\infty(\mathcal{A},\mathcal{B}) \otimes TsA_\infty(\mathcal{B},\mathcal{B}) \to TsA_\infty(\mathcal{A},\mathcal{B}) = \mathcal{C}$. We have $(1\otimes \mathrm{id}_{\mathcal{B}})M = \mathrm{id}_{\mathcal{C}}$ by (4.1.3), and the natural A_∞-transformations $(1\otimes \mathbf{i}^{\mathcal{B}})M$ and $\mathbf{i}^{\mathcal{C}}$ of $\mathrm{id}_{\mathcal{C}}$ coincide. Indeed, $[(1\otimes \mathbf{i}^{\mathcal{B}})M]_0 : \Bbbk \to (s\mathcal{C})^{-1}(f,f)$,

$1 \mapsto (f \mid \mathbf{i}^{\mathcal{B}})M_{01} = f\mathbf{i}^{\mathcal{B}} = {}_f\mathbf{i}_0^{\mathcal{C}}$. For all $n \geqslant 0$ we have $[(1 \otimes \mathbf{i}^{\mathcal{B}})M]_n : r^1 \otimes \cdots \otimes r^n \mapsto (r^1 \otimes \cdots \otimes r^n \otimes \mathbf{i}^{\mathcal{B}})M_{n1}$. By (4.1.4) the components

$$[(r^1 \otimes \cdots \otimes r^n \otimes \mathbf{i}^{\mathcal{B}})M_{n1}]_k = \sum_l (r^1 \otimes \cdots \otimes r^n)\theta_{kl}\mathbf{i}_l^{\mathcal{B}} = (r^1 \otimes \cdots \otimes r^n)\theta_{k0}\mathbf{i}_0^{\mathcal{B}}$$

vanish for $n > 0$.

8.12. Other examples of unital A_∞-categories. More examples of unital categories might be obtained via Theorem 8.8. An A_∞-category with a homotopy unit in the sense of Fukaya, Oh, Ohta and Ono [**FOOO**, Definition 20.1] clarified by Fukaya [**Fuk**, Definition 5.11] is also a unital category in our sense. Indeed, these authors enlarge given A_∞-category $\mathcal{C}$ to a strictly unital A_∞-category $\mathcal{B}$ by adding extra elements to $\mathcal{C}(X,X)$, so that the natural embedding $(\mathcal{C}(X,Y), m_1) \hookrightarrow (\mathcal{B}(X,Y), m_1)$ were a homotopy equivalence. Setting $r_0 = p_0 = \mathbf{i}_0^{\mathcal{B}}$ we view the above situation as a particular case of Theorem 8.8.

If an A_∞-functor $\phi : \mathcal{C} \to \mathcal{B}$ to a unital A_∞-category $\mathcal{B}$ is invertible, then $\mathcal{C}$ is unital. Indeed, since there exists an A_∞-functor $\psi : \mathcal{B} \to \mathcal{C}$ such that $\phi\psi = \mathrm{id}_{\mathcal{C}}$ and $\psi\phi = \mathrm{id}_{\mathcal{B}}$, then the map ϕ_1 is invertible with inverse ψ_1. The remaining data are $\operatorname{Ob}\psi : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$ and ${}_Xr_0 = {}_Xp_0 = {}_X\mathbf{i}_0^{\mathcal{B}} : \Bbbk \to s\mathcal{B}(X,X)$. Since $(\mathbf{i}^{\mathcal{B}} \otimes \mathbf{i}^{\mathcal{B}})B_2 \equiv \mathbf{i}^{\mathcal{B}}$ we have $({}_X\mathbf{i}_0^{\mathcal{B}} \otimes {}_X\mathbf{i}_0^{\mathcal{B}})b_2 - {}_X\mathbf{i}_0^{\mathcal{B}} \in \operatorname{Im} b_1$ and conditions (8.8.1) are satisfied. The data constructed in Theorem 8.8 will be precisely $\psi : \mathcal{B} \to \mathcal{C}$ and $r = p = \mathbf{i}^{\mathcal{B}}$. Since ϕ is unital by Theorem 8.8, we may choose $\mathbf{i}^{\mathcal{C}} = \phi\mathbf{i}^{\mathcal{B}}\psi$ as a unit of $\mathcal{C}$.

If a unital A_∞-category $\mathcal{C}$ is equivalent to a strictly unital A_∞-category $\mathcal{B}$ via an A_∞-functor $\phi : \mathcal{C} \to \mathcal{B}$, then $(1 \otimes \phi)M : A_\infty(\mathcal{A}, \mathcal{C}) \to A_\infty(\mathcal{A}, \mathcal{C})$ is also an equivalence for an arbitrary A_∞-category $\mathcal{A}$ as Proposition 8.4 shows. Thus, a unital A_∞-category $A_\infty(\mathcal{A}, \mathcal{C})$ is equivalent to a strictly unital A_∞-category $A_\infty(\mathcal{A}, \mathcal{B})$. In particular, if ϕ is invertible, then $(1 \otimes \phi)M$ is invertible as well.

8.13. Cohomology of A_∞-categories. Using a lax monoidal functor from $\mathcal{K}$ to some monoidal category we get another 2-functor, which can be composed with k. For instance, there is a cohomology functor $H^\bullet : \mathcal{K} \to \mathbb{Z}\text{-grad-}\Bbbk\text{-mod}$, which induces a 2-functor $H^\bullet : \mathcal{K}\text{-}\mathcal{C}at \to \mathbb{Z}\text{-grad-}\Bbbk\text{-}\mathcal{C}at$. In practice we will use the 0-th cohomology functor $H^0 : \mathcal{K} \to \Bbbk\text{-mod}$, the corresponding 2-functor $H^0 : \mathcal{K}\text{-}\mathcal{C}at \to \Bbbk\text{-}\mathcal{C}at$, and the composite 2-functor

$$A_\infty^u \xrightarrow{\mathsf{k}} \mathcal{K}\text{-}\mathcal{C}at \xrightarrow{H^0} \Bbbk\text{-}\mathcal{C}at,$$

which is also denoted by H^0. It takes a unital A_∞-category $\mathcal{C}$ into a $\Bbbk$-linear category $H^0(\mathcal{C})$ with the same class of objects $\operatorname{Ob} H^0(\mathcal{C}) = \operatorname{Ob}\mathcal{C}$. Its morphism space between objects X and Y is $H^0(\mathcal{C})(X,Y) = H^0(\mathcal{C}(X,Y), m_1)$, the 0-th cohomology with respect to the differential $m_1 = sb_1s^{-1}$. The composition in $H^0(\mathcal{C})$ is induced by m_2, and the units by $\mathbf{i}_0^{\mathcal{C}}s^{-1}$.

For example, the homotopy category $\mathsf{K}(\mathcal{A})$ of complexes of objects of an abelian category $\mathcal{A}$ is the 0-th cohomology $H^0(\mathsf{C}(\mathcal{A}))$ of the differential graded category of complexes $\mathsf{C}(\mathcal{A})$.

Appendix A. Enriched 2-categories

Recall that $\mathscr{U}$ is a fixed universe. Let $\mathcal{V} = (\mathcal{V}, \otimes, c, \mathbb{1})$ be a symmetric monoidal $\mathscr{U}$-category (that is, all $\mathcal{V}(X,Y)$ are $\mathscr{U}$-small sets). In this article we shall use $(\mathcal{V}, \otimes, c, \mathbb{1}) = (\Bbbk\text{-mod}, \otimes_\Bbbk, \sigma, \Bbbk)$, where σ is the permutation isomorphism, or $(\mathcal{V}, \otimes, c, \mathbb{1}) = (\mathcal{K}, \otimes_\Bbbk, c, \Bbbk)$, where $\mathcal{K}$ is the category of differential graded $\Bbbk$-modules, whose morphisms are chain maps modulo homotopy, and c is its standard symmetry. There is a notion of a category $\mathcal{C}$ enriched in $\mathcal{V}$ ($\mathcal{V}$-categories, $\mathcal{V}$-functors, $\mathcal{V}$-natural transformations), see Kelly [**Kel82**], summarized e.g. in [**KL01**]: for all objects X, Y of $\mathcal{C}$ $\mathcal{C}(X,Y)$ is an object of $\mathcal{V}$. Denote by $\mathcal{V}$-$\mathcal{C}at$ the category, whose objects are $\mathcal{V}$-categories and morphisms are $\mathcal{V}$-functors. Since $\mathcal{V}$ is symmetric, the category $\mathcal{V}$-$\mathcal{C}at$ is symmetric monoidal with the tensor product $\mathcal{A} \otimes \mathcal{B}$ of $\mathcal{V}$-categories $\mathcal{A}$, $\mathcal{B}$ defined via $\operatorname{Ob}(\mathcal{A} \otimes \mathcal{B}) = \operatorname{Ob}\mathcal{A} \times \operatorname{Ob}\mathcal{B}$, $\mathcal{A} \otimes \mathcal{B}(X \times Y, U \times V) = \mathcal{A}(X,U) \otimes \mathcal{B}(Y,V)$. Thus, we may consider the 1-category $\mathcal{V}$-$\mathcal{C}at$-$\mathcal{C}at$ of $\mathcal{V}$-$\mathcal{C}at$-categories and $\mathcal{V}$-$\mathcal{C}at$-functors. We may interpret it in the same way, as $\mathcal{C}at$-$\mathcal{C}at$ is interpreted as the category of 2-categories. So we say that objects of $\mathcal{V}$-$\mathcal{C}at$-$\mathcal{C}at$ are $\mathcal{V}$-2-categories $\mathfrak{A}$, as defined below. To restore the definition of a usual 2-category, it suffices to take $\mathcal{V} = (\mathscr{U}\text{-Sets}, \times, \varnothing)$.

Appendix A.1 Definition ($\mathcal{V}$-2-category). A 1-unital 2-unital $\mathcal{V}$-2-category $\mathfrak{A}$ consists of

- a class of objects $\operatorname{Ob}\mathfrak{A}$;
- for any pair of objects $\mathcal{A}, \mathcal{B} \in \operatorname{Ob}\mathfrak{A}$ a $\mathcal{V}$-category $\mathfrak{A}(\mathcal{A},\mathcal{B})$;
- for any object $\mathcal{A} \in \operatorname{Ob}\mathfrak{A}$ a $\mathcal{V}$-functor $\underline{\mathbb{1}} \to \mathfrak{A}(\mathcal{A},\mathcal{A})$, $1 \mapsto \mathrm{id}_\mathcal{A}$;
- for any triple $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ of objects of $\mathfrak{A}$ a $\mathcal{V}$-functor

$$\mathfrak{A}(\mathcal{A},\mathcal{B}) \otimes \mathfrak{A}(\mathcal{B},\mathcal{C}) \to \mathfrak{A}(\mathcal{A},\mathcal{C}), \qquad (f,g) \mapsto fg,$$

such that the following $\mathcal{V}$-functors are equal (modulo the associativity isomorphism in $\mathcal{V}$): $f\,\mathrm{id} = f = \mathrm{id}\,f$, $f(gh) = (fg)h$.

Here the unit $\mathcal{V}$-category $\underline{\mathbb{1}}$ has the set of objects $\operatorname{Ob}\underline{\mathbb{1}} = \{1\}$, and $\underline{\mathbb{1}}(1,1) = \mathbb{1}$ is the unit object of $\mathcal{V}$. The above definition has an equivalent unpacked form, namely, Definition Appendix A.3. We also need generalizations of the above $\mathcal{V}$-2-categories – 1-unital non-2-unital $\mathcal{V}$-2-categories, which contain unit 1-morphisms, but do not contain unit 2-morphisms. An expanded definition of the latter follows. It seems that it does not have a concise version.

Appendix A.2 Definition (Non-2-unital $\mathcal{V}$-2-category). A 1-unital non-2-unital $\mathcal{V}$-2-category $\mathfrak{A}$ consists of

- a class of objects $\operatorname{Ob}\mathfrak{A}$;
- a class of 1-morphisms $\mathfrak{A}(\mathcal{A},\mathcal{B})$ for any pair $\mathcal{A}$, $\mathcal{B}$ of objects of $\mathfrak{A}$;
- an object of 2-morphisms $\mathfrak{A}(\mathcal{A},\mathcal{B})(f,g) \in \operatorname{Ob}\mathcal{V}$ for any pair of 1-morphisms $f,g \in \mathfrak{A}(\mathcal{A},\mathcal{B})$;
- a strictly associative composition of 1-morphisms $\mathfrak{A}(\mathcal{A},\mathcal{B}) \times \mathfrak{A}(\mathcal{B},\mathcal{C}) \to \mathfrak{A}(\mathcal{A},\mathcal{C})$, $(f,g) \mapsto fg$;

- a strict two-sided unit 1-morphism $\mathrm{id}_\mathcal{A} \in \mathfrak{A}(\mathcal{A},\mathcal{A})$ for each object $\mathcal{A}$ of $\mathfrak{A}$;
- a right action of a 1-morphism $k : \mathcal{B} \to \mathcal{C}$ on 2-morphisms $\cdot k : \mathfrak{A}(\mathcal{A},\mathcal{B})(f,g) \to \mathfrak{A}(\mathcal{A},\mathcal{C})(fk,gk) \in \operatorname{Mor}\mathcal{V}$;
- a left action of a 1-morphism $e : \mathcal{D} \to \mathcal{A}$ on 2-morphisms $e\cdot : \mathfrak{A}(\mathcal{A},\mathcal{B})(f,g) \to \mathfrak{A}(\mathcal{D},\mathcal{B})(ef,eg) \in \operatorname{Mor}\mathcal{V}$;
- a vertical composition of 2-morphisms $m_2 : \mathfrak{A}(\mathcal{A},\mathcal{B})(f,g) \otimes \mathfrak{A}(\mathcal{A},\mathcal{B})(g,h) \to \mathfrak{A}(\mathcal{A},\mathcal{B})(f,h) \in \operatorname{Mor}\mathcal{V}$,

such that

- m_2 is associative (in monoidal category $\mathcal{V}$);
- the right and the left actions
  (a) commute with each other:
$$(e\cdot)(\cdot k) = (\cdot k)(e\cdot), \qquad \text{for } \mathcal{D} \xrightarrow{e} \mathcal{A} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{B} \xrightarrow{k} \mathcal{C},$$
  (b) are associative:
$$(\cdot k)(\cdot k') = \cdot(kk'), \qquad \text{for } \mathcal{A} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{B} \xrightarrow{k} \mathcal{C} \xrightarrow{k'} \mathcal{D},$$
$$(e'\cdot)(e\cdot) = (e'e)\cdot, \qquad \text{for } \mathcal{C} \xrightarrow{e'} \mathcal{D} \xrightarrow{e} \mathcal{A} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{B},$$
  (c) and unital: $(\cdot\,\mathrm{id}_\mathcal{B}) = \mathrm{id}$, $(\mathrm{id}_\mathcal{A}\,\cdot) = \mathrm{id}$;
- the right and the left actions of 1-morphisms on 2-morphisms preserve the vertical composition:

$$\begin{tikzcd}
\mathfrak{A}(\mathcal{A},\mathcal{B})(f,g)\otimes\mathfrak{A}(\mathcal{A},\mathcal{B})(g,h) \arrow[r, "m_2"] \arrow[d, "(\cdot k)\otimes(\cdot k)"'] \arrow[dr, phantom, "="] & \mathfrak{A}(\mathcal{A},\mathcal{B})(f,h) \arrow[d, "\cdot k"] \\
\mathfrak{A}(\mathcal{A},\mathcal{C})(fk,gk)\otimes\mathfrak{A}(\mathcal{A},\mathcal{C})(gk,hk) \arrow[r, "m_2"] & \mathfrak{A}(\mathcal{A},\mathcal{C})(fk,hk)
\end{tikzcd} \quad \text{for } \mathcal{A} \overset{f}{\underset{h}{\overset{g}{\to}}} \mathcal{B} \xrightarrow{k} \mathcal{C},$$

$$\begin{tikzcd}
\mathfrak{A}(\mathcal{A},\mathcal{B})(f,g)\otimes\mathfrak{A}(\mathcal{A},\mathcal{B})(g,h) \arrow[r, "m_2"] \arrow[d, "(e\cdot)\otimes(e\cdot)"'] \arrow[dr, phantom, "="] & \mathfrak{A}(\mathcal{A},\mathcal{B})(f,h) \arrow[d, "e\cdot"] \\
\mathfrak{A}(\mathcal{D},\mathcal{B})(ef,eg)\otimes\mathfrak{A}(\mathcal{D},\mathcal{B})(eg,eh) \arrow[r, "m_2"] & \mathfrak{A}(\mathcal{D},\mathcal{B})(ef,eh)
\end{tikzcd} \quad \text{for } \mathcal{D} \xrightarrow{e} \mathcal{A} \overset{f}{\underset{h}{\overset{g}{\to}}} \mathcal{B};$$

- the distributivity law holds:

$$\begin{tikzcd}
 & \mathfrak{A}(\mathcal{B},\mathcal{C})(h,k)\otimes\mathfrak{A}(\mathcal{A},\mathcal{B})(f,g) \arrow[d, "(f\cdot)\otimes(\cdot k)"] \\
\mathfrak{A}(\mathcal{A},\mathcal{B})(f,g)\otimes\mathfrak{A}(\mathcal{B},\mathcal{C})(h,k) \arrow[ur, "c"] \arrow[d, "(\cdot h)\otimes(g\cdot)"'] \arrow[dr, phantom, "="] & \mathfrak{A}(\mathcal{A},\mathcal{C})(fh,fk)\otimes\mathfrak{A}(\mathcal{A},\mathcal{C})(fk,gk) \arrow[d, "m_2"] \\
\mathfrak{A}(\mathcal{A},\mathcal{C})(fh,gh)\otimes\mathfrak{A}(\mathcal{A},\mathcal{C})(gh,gk) \arrow[r, "m_2"] & \mathfrak{A}(\mathcal{A},\mathcal{C})(fh,gk)
\end{tikzcd}$$

$$\text{for } \mathcal{A} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{B} \underset{k}{\overset{h}{\rightrightarrows}} \mathcal{C}.$$

The following definition is equivalent to Definition Appendix A.1.

Appendix A.3 Definition (2-unital $\mathcal{V}$-2-category). A 1-unital 2-unital $\mathcal{V}$-2-category $\mathfrak{A}$ consists of the same data as in Definition Appendix A.2 plus a morphism $1_f : \mathbb{1} \to \mathfrak{A}(\mathcal{A},\mathcal{B})(f,f)$ for any 1-morphism f, which is a two-sided unit with respect to m_2, such that homomorphisms $\cdot k$, $e\cdot$ preserve the units 1_-.

Appendix B. Contractibility

One can avoid using the following lemma in this article. However, it might be used in order to replace inductive constructions with recurrent formulas.

Appendix B.1 Lemma. *Let a chain map $u : A \to C$ be homotopically invertible. Then* $\operatorname{Cone}(u)$ *is contractible.*

Proof. The homotopy category $\mathcal{K} = \mathsf{K}(\Bbbk\text{-mod})$ is triangulated and it has a distinguished triangle $A \xrightarrow{u} C \xrightarrow{p} \operatorname{Cone}(u) \xrightarrow{q} A[1] \xrightarrow{u[1]}$ (e.g. [**Gri87**, Corollaire 5.13]). Since $up = 0$, $qu[1] = 0$ (e.g. [**Gri87**, Proposition 2.8]), and u is invertible in $\mathcal{K}$, we deduce that $p = 0$ and $q = 0$ in $\mathcal{K}$. Since $\mathcal{K}(\operatorname{Cone}(u), \_)$ is a homological functor (e.g. [**Gri87**, Proposition 2.10]), we have $\mathcal{K}(\operatorname{Cone}(u), \operatorname{Cone}(u)) = 0$, that is, $\operatorname{Cone}(u) \simeq 0$ in $\mathcal{K}$. □

Let us construct an explicit homotopy between $\mathrm{id}_{\operatorname{Cone}(u)}$ and $0_{\operatorname{Cone}(u)}$. There exists a chain map $v : C \to A$ homotopically inverse to u. That is, there are maps $h' : A \to A$, $h'' : C \to C$ of degree -1 such that $uv = 1 + h'd^A + d^A h' : A \to A$, $vu = 1 + h''d^C + d^C h'' : C \to C$. Using the notation at the end of Section 1 we define a map $h : \operatorname{Cone}(u) \to \operatorname{Cone}(u)$ of degree -1 by the formula $(c,a)h = (-ch'', cv + ah')$, $(c,a) \in C^k \oplus A^{k+1} = \operatorname{Cone}^k(u)$. Let us compute the boundary of h:

$$\begin{aligned}(c,a)(hd + dh) &= (-ch'', cv + ah')d + (cd^C + au, -ad^A)h \\ &= (-ch''d^C + cvu + ah'u, -cvd^A - ah'd^A) \\ &\quad + (-cd^C h'' - auh'', cd^C v + auv - ad^A h') \\ &= (c + ah'u - auh'', a).\end{aligned}$$

Hence, $hd + dh = 1 - f : \operatorname{Cone}(u) \to \operatorname{Cone}(u)$, where the map $f : \operatorname{Cone}(u) \to \operatorname{Cone}(u)$ is defined via $(c,a)f = (auh'' - ah'u, 0)$. We conclude that f is a chain map homotopic to the identity map. A sequence of equivalences $\mathrm{id}_{\operatorname{Cone}(u)} \sim f = \mathrm{id}_{\operatorname{Cone}(u)} f \sim f^2 = 0$ proves that $\operatorname{Cone}(u)$ is contractible. It gives also an explicit homotopy – the map $\underline{h} = h + hf : \operatorname{Cone}(u) \to \operatorname{Cone}(u)$ of degree -1, which satisfies $\mathrm{id}_{\operatorname{Cone}(u)} = \underline{h}d + d\underline{h}$. This homotopy might be used to replace inductive constructions in this paper with recurrent formulas.

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