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FIRST-ORDER SYSTEM LEAST SQUARES FOR VELOCITY-VORTICITY-PRESSURE FORM OF THE STOKES EQUATIONS, WITH APPLICATION TO LINEAR ELASTICITY*

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Abstract. In this paper, we study the least-squares method for the generalized Stokes equations (including linear elasticity) based on the velocity-vorticity-pressure formulation in $d = 2$ or 3 dimensions. The least-squares functional is defined in terms of the sum of the L^2- and H^{-1}-norms of the residual equations, which is similar to that in [7], but weighted appropriately by the Reynolds number (Poisson ratio). Our approach for establishing ellipticity of the functional does not use ADN theory, but is founded more on basic principles. We also analyze the case where the H^{-1}-norm in the functional is replaced by a discrete functional to make the computation feasible. We show that the resulting algebraic equations can be preconditioned by well-known techniques uniformly well in the Reynolds number (Poisson ratio).

Key words. least squares, Stokes, elasticity.

AMS subject classifications. 65F10, 65F30.

1. Introduction.

Recently, there has been substantial interest in the use of least-squares principles for numerical approximation of the incompressible Stokes and Navier-Stokes equations, especially those based on vorticity (more precisely, velocity-vorticity-pressure); for example, see [6, 14, 15, 16, 21]. Its attractions include accurate approximation to meaningful physical quantities, formulation of a well-posed minimization principle, elimination of the need for artificial stablization techniques, and freedom in the choice of finite element spaces (which are not subject to the LBB condition). The computational results provided in these papers indicate that such methods have great promise. However, they do not yield optimally accurate approximations for the case of Dirichlet boundary conditions (see the analysis in [7]). In recent work by Bochev and Gunzburger [7], the ADN approach (see [2]) was extended to the vorticity formulation of the Stokes equations with rigorous error analysis. The least-squares functional is defined to be the sum of squares of the norms of the residual of each equation, where the norms are determined by the indices assigned to each equation by the ADN theory (see [1]). To be specific, consider the three-dimensional stationary Stokes equations with Dirichlet boundary conditions. Then ADN theory was used in [7] to show that the least-squares functional $\|\mathbf{f}-(\nu\nabla\times\boldsymbol{\omega}+\nabla p)\|_q^2+\|\nabla\times\mathbf{u}-\boldsymbol{\omega}\|_{q+1}^2+\|\nabla\cdot\mathbf{u}\|_{q+1}^2$ is equivalent to the sum of squared norms of each variable, $\|\mathbf{u}\|_{q+2}^2+\|\boldsymbol{\omega}\|_{q+1}^2+\|p\|_{q+1}^2$, for all $q \in R$ and $\mathbf{f} = \mathbf{0}$. In particular, they consider the above functional with $q = 0$, then replace the H^1-norms by mesh-dependent L^2-norms, $h^{-2}\|\cdot\|_0^2$ (see also [2]). This mesh-dependent least-squares

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approach yields optimally accurate approximations for each variable with respect to approximation subspaces. However, it is not clear that an optimal solution algorithm for the resulting discrete equations can be developed at this stage of research, albeit the matrix is symmetric and positive definite.

In this paper, we consider a least-squares functional similar to that in [7] with $q = -1$, but weighted appropriately by the Reynolds number, Re. This is designed for the vorticity formulation of the pressure-perturbed variant of the generalized Stokes equation (which includes linear elasticity, where the Reynolds number becomes an expression in the Poisson ratio) with Dirichlet boundary conditions in two and three dimensions. Instead of applying ADN theory, we directly establish ellipticity and continuity of the functional in a product norm involving Re and the L^2- and H^1-norms. The H^{-1}-norm in the functional is further replaced by the discrete H^{-1}-norm to make the computation feasible, following the discrete H^{-1} least-squares approach proposed by Bramble, Lazarov, and Pasciak [3] for scalar second-order elliptic equations. Such discrete H^{-1} functionals are shown to be uniformly equivalent to the Sobolev norms weighted by the Reynolds number. This property enables us to show that standard finite element discretization error estimates are optimal with respect to the order of approximation as well as the required regularity of the solution, and that they are uniform in the Reynolds number (Poisson ratio). Moreover, the resulting discrete equations can be preconditioned by multigrid associated with velocity and by diagonal matrices associated with vorticity and pressure uniformly well with respect to the Reynolds number (Poisson ratio), the mesh size, and the number of levels.

The general theory presented here for Stokes–elasticity equations is obtained by a fairly simple extension of the theory for standard Stokes equations first obtained in [10]. Similar results for Stokes are obtained in a later report by Bramble and Pasciak [4].

The paper is organized as follows. Section 2.1 introduces the (generalized) Stokes equations, its vorticity formulation, and some preliminary results. We introduce the least-squares functional weighted appropriately by ν for the vorticity system, then establish its ellipticity and continuity in Section 2.2. Section 3 discusses the finite element approximation and Section 4 considers the discrete H^{-1}-norm least-squares functional and solution method for the resulting system of linear equations.

2. Formulations of Least-Squares Functionals. In this section, we describe the weighted least-squares functional for the vorticity formulation and show its ellipticity and continuity in the appropriate Hilbert spaces. In Subsection 2.1, we start by defining the (generalized) Stokes equation and its vorticity formulation; we next give some notation for Sobolev spaces, the divergence and curl related Hilbert spaces, and their norms; we then include some preliminary results of functional analysis. In Subsection 2.3, we introduce a least-squares functional weighted appropriately by the Reynolds number, then directly show its ellipticity and continuity.

2.1. The Stokes Equation and Its Vorticity Formulation. Let Ω be a bounded open domain in $\Re^d$ ($d = 2$ or 3) with Lipschitz boundary $\partial\Omega$. The pressure-perturbed form of the generalized stationary Stokes equation in dimensionless variables may be written as

$$\text{(2.1)} \qquad \left\{ \begin{array}{rcll} -\nu\Delta\,\mathbf{u} + \nabla p & = & \mathbf{f}, & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} + \delta\, p & = & 0, & \text{in} \quad \Omega, \end{array} \right.$$

where: the symbols Δ, ∇, and $\nabla\cdot$ stand for the Laplacian, gradient, and divergence operators, respectively; $\mathbf{f}$ is a given vector function; ν is the reciprocal of the Reynolds

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number Re; $\mathbf{f}$ is a given vector function; and δ is some nonnegative constant ($\delta = 0$ for Stokes and $\delta = 1$ for linear elasticity with $\nu = \frac{\mu}{\lambda+\mu}$, where μ and λ are the (positive) Lamé constants). For more details on linear elasticity, see [10]. We consider the (generalized) Stokes equations (2.1) together with the Dirichlet velocity boundary condition

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \tag{2.2}$$

and the mean pressure condition

$$\int_\Omega p\, dx = 0. \tag{2.3}$$

Let $\mathbf{curl} \equiv \nabla\times$ denote the curl operator. (Here and henceforth, we use notation for the case $d = 3$ and consider the special case $d = 2$ in the natural way by identifying $\Re^2$ with the (x_1, x_2)-plane in $\Re^3$. Thus, if $\mathbf{u}$ is two dimensional, then the curl of $\mathbf{u}$ means the scalar function

$$\nabla\times \mathbf{u} = \partial_1 u_2 - \partial_2 u_1$$

where u_1 and u_2 are the components of $\mathbf{u}$.) It can be easily checked that

$$\nabla\times (\nabla\times \mathbf{u}) = -\Delta\, \mathbf{u} + \nabla\, (\nabla\cdot\, \mathbf{u}). \tag{2.4}$$

(For $d = 2$, relation (2.4) is interpreted as

$$\nabla^\perp (\nabla\times \mathbf{u}) = -\Delta\, \mathbf{u} + \nabla\, (\nabla\cdot\, \mathbf{u}),$$

where $\nabla^\perp$ is the formal adjoint of $\nabla\times$ defined by

$$\nabla^\perp q = \begin{pmatrix} \partial_2 q \\ -\partial_1 q \end{pmatrix}.)$$

Introducing the vorticity variable

$$\boldsymbol{\omega} = \nabla\times \mathbf{u},$$

using the identity (2.4), and remembering the "continuity" condition $\nabla\cdot\mathbf{u} + \delta\, p = 0$, then the generalized Stokes equation (2.1) may be rewritten in vorticity form as follows:

$$\left\{ \begin{array}{rcll} \nu\nabla\times\boldsymbol{\omega} + (1+\nu\delta)\nabla p & = & \mathbf{f}, & \text{in } \Omega, \\ \nabla\times\mathbf{u} - \boldsymbol{\omega} & = & \mathbf{0}, & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} + \delta\, p & = & 0, & \text{in } \Omega. \end{array} \right. \tag{2.5}$$

Next, we establish notation. We use the standard notation and definition for the Sobolev space $H^s(\Omega)^d$ for $s \geq 0$; the standard associated inner product and norm are denoted by $(\cdot,\,\cdot)_{s,\,\Omega}$ and $\|\cdot\|_{s,\,\Omega}$, respectively. (We supress the subscript d because dependence of the vector norms on dimension will be clear by context. We will omit the measure Ω from the inner product and norm designation when there is no risk of confusion.) For $s = 0$, $H^s(\Omega)^d$ coincides with $L^2(\Omega)^d$. In this case, the norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\,\cdot)$, respectively. As usual, for $s > 0$,

$H_0^s(\Omega)$ will denote the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_s$ and $H^{-s}(\Omega)$ will denote its dual with norm defined by

$$\|\varphi\|_{-s} = \sup_{0\neq\phi\in H_0^s(\Omega)} \frac{<\varphi,\,\phi>}{\|\phi\|_s},$$

where $<\cdot,\cdot>$ denotes the duality pairing. Define the product spaces $H_0^s(\Omega)^d = \prod_{i=1}^d H_0^s(\Omega)$ and $H^{-1}(\Omega)^d = \prod_{i=1}^d H^{-1}(\Omega)$ with standard product norms. Let

$$H(\mathrm{div};\,\Omega) = \{\mathbf{v}\in L^2(\Omega)^d \,:\, \nabla\cdot\mathbf{v}\in L^2(\Omega)\}$$

and

$$H(\mathbf{curl};\,\Omega) = \{\mathbf{v}\in L^2(\Omega)^d \,:\, \nabla\times\mathbf{v}\in L^2(\Omega)^{2d-3}\},$$

which are Hilbert spaces under the respective norms

$$\|\mathbf{v}\|_{H(\mathrm{div};\,\Omega)} \equiv \left(\|\mathbf{v}\|^2 + \|\nabla\cdot\,\mathbf{v}\|^2\right)^{\frac{1}{2}}$$

and

$$\|\mathbf{v}\|_{H(\mathbf{curl};\,\Omega)} \equiv \left(\|\mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2\right)^{\frac{1}{2}}.$$

Define their subspaces

$$H_0(\mathrm{div};\,\Omega) = \{\mathbf{v}\in H(\mathrm{div};\,\Omega) \,:\, \mathbf{v}\cdot\mathbf{n}=0 \text{ on } \partial\Omega\}$$

and

$$H_0(\mathbf{curl};\,\Omega) = \{\mathbf{v}\in H(\mathbf{curl};\,\Omega) \,:\, \gamma_\tau\mathbf{v}=0 \text{ on } \partial\Omega\},$$

where $\gamma_\tau\mathbf{v} = \mathbf{v}\cdot\boldsymbol{\tau}$ for $d=2$ and $\gamma_\tau\mathbf{v} = \mathbf{v}\times\mathbf{n}$ for $d=3$, and $\mathbf{n}$ and $\boldsymbol{\tau}$ denote the respective unit vectors normal and tangent to the boundary. Finally, define the subspace $L_0^2(\Omega)^d$ of $L^2(\Omega)^d$ by

$$L_0^2(\Omega)^d = \{\mathbf{v}\in L^2(\Omega)^d \,:\, \int_\Omega v_i\,dx = 0 \text{ for } i=1,...,d\}.$$

Here and henceforth, we will use C with or without subscripts to denote a generic positive constant, possibly different at different occurrences, which is independent of the parameter ν and other parameters introduced in this paper, but may depend on the domain Ω. The next lemma is an immediate consequence of a general result of functional analysis due to Nečas [20] (see also [13]).

LEMMA 2.1. *For any $p\in L_0^2(\Omega)$, there exists a positive constant C such that*

$$\|p\| \le C\,\|\nabla\,p\|_{-1}. \tag{2.6}$$

A result analogous to Green's formula also follows:

$$(\nabla\times\mathbf{z},\ \boldsymbol{\phi}) = (\mathbf{z},\ \nabla\times\boldsymbol{\phi}) - \int_{\partial\Omega} \boldsymbol{\phi}\cdot(\mathbf{z}\times\mathbf{n})\,ds \tag{2.7}$$

for $\mathbf{z}\in H(\mathbf{curl}\,;\,\Omega)$ and $\boldsymbol{\phi}\in H^1(\Omega)^d$.

Finally, we will summarize results of Lemma 2.5 and Remark 2.7 in Chapter I of [13] that we will need in subsequent sections.

LEMMA 2.2. *For any $\mathbf{v}\in H_0(\mathrm{div};\,\Omega)\cap H_0(\mathbf{curl};\,\Omega)$, there exists a positive constant C such that*

$$\|\mathbf{v}\|_1 \le C\,\left(\|\nabla\cdot\,\mathbf{v}\| + \|\nabla\times\mathbf{v}\|\right). \tag{2.8}$$

2.2. Least-Squares Functional. Our least-squares functional is defined by the weighted sum of the L^2- and H^{-1}-norms of the residual equations of system (2.5):

$$G(\mathbf{u},\,\boldsymbol{\omega},\,p;\,\mathbf{f}) = \|\mathbf{f}-(\nu\nabla\times\boldsymbol{\omega}+(1+\nu\delta)\nabla\,p)\|_{-1}^2+\nu^2\,\|\nabla\times\mathbf{u}-\boldsymbol{\omega}\|^2+\nu^2\,\|\nabla\cdot\mathbf{u}+\delta\,p\|^2. \tag{2.9}$$

(A similar functional without the weights of the Reynolds parameter ν for the Stokes equations was considered by Bochev and Gunzburger in [7].) The least-squares problem we consider is to minimize this quadratic functional over $\mathbf{V} \equiv H_0^1(\Omega)^d \times L^2(\Omega)^{2d-3} \times L_0^2(\Omega)$: find $(\mathbf{u},\,\boldsymbol{\omega},\,p) \in \mathbf{V}$ such that

$$G(\mathbf{u},\,\boldsymbol{\omega},\,p;\,\mathbf{f}) = \inf_{(\mathbf{v},\,\boldsymbol{\sigma},\,q)\in\mathbf{V}} G(\mathbf{v},\,\boldsymbol{\sigma},\,q;\,\mathbf{f}). \tag{2.10}$$

Next, we use an approach that departs from the established ADN theory (cf. [7]) to show ellipticity of the functional.

THEOREM 2.1. *For any $(\mathbf{u},\,\boldsymbol{\omega},\,p) \in \mathbf{V}$, there exist positive constants C_1 and C_2 independent of ν such that*

$$C_1\,\left(\nu^2\|\mathbf{u}\|_1^2+\nu^2\|\boldsymbol{\omega}\|^2+(1+\nu\delta)^2\|p\|^2\right) \le G(\mathbf{u},\,\boldsymbol{\omega},\,p;\,\mathbf{0}) \tag{2.11}$$

and

$$G(\mathbf{u},\,\boldsymbol{\omega},\,p;\,\mathbf{0}) \le C_2\,\left(\nu^2\|\mathbf{u}\|_1^2+\nu^2\|\boldsymbol{\omega}\|^2+(1+\nu\delta)^2\|p\|^2\right). \tag{2.12}$$

Proof. Upper bound (2.12) is straightforward from the triangle and Cauchy-Schwarz inequalities. We proceed to show the validity of (2.11) for $(\mathbf{u},\,\boldsymbol{\omega},\,p) \in H_0^1(\Omega)^d \times H(\mathbf{curl};\,\Omega) \times (L_0^2(\Omega) \cap H^1(\Omega))$. It will then follow for $(\mathbf{u},\,\boldsymbol{\omega},\,p) \in \mathbf{V}$ by continuity (c.f. [9]). Now from (2.7) and the Cauchy-Schwarz inequality, for any $\boldsymbol{\phi} \in H_0^1(\Omega)^d$ we have

$$\begin{aligned}
&\frac{1+\nu\delta}{\nu}(\nabla\,p,\,\boldsymbol{\phi})\\
=\;& \frac{1}{\nu}(\nu\nabla\times\boldsymbol{\omega}+(1+\nu\delta)\nabla\,p,\,\boldsymbol{\phi})+(\nabla\times\mathbf{u}-\boldsymbol{\omega},\,\nabla\times\boldsymbol{\phi})-(\nabla\times\mathbf{u},\,\nabla\times\boldsymbol{\phi})\\
\le\;& C\left(\frac{1}{\nu}\|\nu\nabla\times\boldsymbol{\omega}+(1+\nu\delta)\nabla\,p\|_{-1}+\|\nabla\times\mathbf{u}-\boldsymbol{\omega}\|+\|\nabla\times\mathbf{u}\|\right)\|\boldsymbol{\phi}\|_1,
\end{aligned}$$

which, together with Lemma 2.1, implies that

$$\begin{aligned}
\frac{1+\nu\delta}{\nu}\|p\| \;&\le\; C\,\frac{1+\nu\delta}{\nu}\,\|\nabla\,p\|_{-1}\\
&\le\; C\left(\frac{1}{\nu}\|\nu\nabla\times\boldsymbol{\omega}+(1+\nu\delta)\nabla\,p\|_{-1}+\|\nabla\times\mathbf{u}-\boldsymbol{\omega}\|+\|\nabla\times\mathbf{u}\|\right).
\end{aligned} \tag{2.13}$$

By (2.7), the Cauchy-Schwarz and triangle inequalities, Lemma 2.2, and (2.13), we have that

$$\begin{aligned}
&\|\nabla\times\mathbf{u}\|^2\\
=\;& (\nabla\times\mathbf{u}-\boldsymbol{\omega},\,\nabla\times\mathbf{u})+\frac{1}{\nu}(\nu\nabla\times\boldsymbol{\omega}+(1+\nu\delta)\nabla\,p,\,\mathbf{u})\\
&\quad+\frac{1+\nu\delta}{\nu}(p,\,\nabla\cdot\mathbf{u}+\delta\,p)-\frac{\delta(1+\nu\delta)}{\nu}(p,\,p)
\end{aligned}$$

$$\begin{aligned}
&\leq \|\nabla\times \mathbf{u}-\boldsymbol{\omega}\|\,\|\nabla\times \mathbf{u}\| + \frac{1}{\nu}\|\nu\nabla\times\boldsymbol{\omega}+(1+\nu\delta)\nabla p\|_{-1}\,\|\mathbf{u}\|_1 \\
&\qquad + \frac{1+\nu\delta}{\nu}\|p\|\,\|\nabla\cdot\mathbf{u}+\delta\, p\| \\
&\leq \frac{C}{\nu}\|\nu\nabla\times\boldsymbol{\omega}+(1+\nu\delta)\nabla p\|_{-1}\,(\|\nabla\times\mathbf{u}\|+\|\nabla\cdot\mathbf{u}+\delta\, p\|+\delta\|p\|) \\
&\qquad + \|\nabla\times\mathbf{u}-\boldsymbol{\omega}\|\,\|\nabla\times\mathbf{u}\| + \frac{1+\nu\delta}{\nu}\|p\|\,\|\nabla\cdot\mathbf{u}+\delta\, p\| \\
&\leq \|\nabla\times\mathbf{u}\|\left(\|\nabla\times\mathbf{u}-\boldsymbol{\omega}\|+\frac{C}{\nu}\|\nu\nabla\times\boldsymbol{\omega}+(1+\nu\delta)\nabla p\|_{-1}+\|\nabla\cdot\mathbf{u}+\delta\, p\|\right) \\
&\qquad + \frac{C}{\nu^2}G(\mathbf{u},\,\boldsymbol{\omega},\,p;\,\mathbf{0}) \\
&\leq \frac{1}{2}\,\|\nabla\times\mathbf{u}\|^2+\frac{C}{\nu^2}G(\mathbf{u},\,\boldsymbol{\omega},\,p;\,\mathbf{0}).
\end{aligned}$$

Hence,

$$\|\nabla\times\mathbf{u}\|^2 \leq \frac{C}{\nu^2}G(\mathbf{u},\,\boldsymbol{\omega},\,p;\,\mathbf{0}). \tag{2.14}$$

But (2.14), (2.13), the bounds

$$\|\boldsymbol{\omega}\| \leq \|\nabla\times\mathbf{u}-\boldsymbol{\omega}\|+\|\nabla\times\mathbf{u}\| \quad \text{and} \quad \|\nabla\cdot\mathbf{u}\| \leq \|\nabla\cdot\mathbf{u}+\delta\, p\|+\delta\,\|p\|,$$

and Lemma 2.2 imply (2.11). This completes the proof of the theorem. □

3. Finite Element Approximations. We approximate the minimum of the least-squares functional $G(\mathbf{u},\,\boldsymbol{\omega},\,p;\mathbf{f})$ in (2.9) using a Rayleigh-Ritz type finite element method. Assuming that the domain Ω is a polyhedron, let $\mathcal{T}_h$ be a partition of the Ω into finite elements, i.e., $\Omega = \cup_{K\in\mathcal{T}_h}K$ with $h = \max\{\mathrm{diam}(K) : K \in \mathcal{T}_h\}$. Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform, i.e., it is regular and satisfies the inverse assumption (see [12]). Let $\mathbf{V}_h = \mathbf{U}_h \times \mathbf{W}_h \times P_h$ be a finite-dimensional subspace of $\mathbf{V}$ with the following properties: for any $(\mathbf{u},\,\boldsymbol{\omega},\,p) \in \left(H^{r+1}(\Omega)^d \times H^r(\Omega)^{2d-2}\right) \cap \mathbf{V}$,

$$\inf_{\mathbf{v}\in\mathbf{U}_h}(\|\mathbf{u}-\mathbf{v}\|+h\,\|\mathbf{u}-\mathbf{v}\|_1) \;\leq\; Ch^{r+1}\|\mathbf{u}\|_{r+1}, \tag{3.1}$$

$$\inf_{\boldsymbol{\sigma}\in\mathbf{W}_h}(\|\boldsymbol{\omega}-\boldsymbol{\sigma}\|+h\,\|\boldsymbol{\omega}-\boldsymbol{\sigma}\|_1) \;\leq\; Ch^{r}\|\boldsymbol{\omega}\|_{r}, \tag{3.2}$$

$$\inf_{q\in P_h}(\|p-q\|+h\,\|p-q\|_1) \;\leq\; Ch^{r}\|p\|_{r}, \tag{3.3}$$

where r is an integer with $r \geq 0$ for (3.1) and $r \geq 1$ for (3.2)–(3.3). It is well-known that (3.1)–(3.3) holds for typical finite element spaces consisting of continuous piecewise polynomials with respect to quasi-uniform triangulations (cf. [12]). Note that these assumptions mandate the use of finite element subspaces of $H^1(\Omega)$. These restrictions stem from the hypotheses required for Lemma 4.1 in the next section.

The finite element approximation to (2.9) becomes: find $(\mathbf{u}_h,\,\boldsymbol{\omega}_h,\,p_h) \in \mathbf{V}_h$ that satisfies

$$G(\mathbf{u}_h,\,\boldsymbol{\omega}_h,\,p_h;\,\mathbf{f}) = \inf_{(\mathbf{v},\,\boldsymbol{\sigma},\,q)\in\mathbf{V}_h} G(\mathbf{v},\,\boldsymbol{\sigma},\,q;\,\mathbf{f}). \tag{3.4}$$

Denote the norm that is equivalent to the norm induced by the functional as

$$|||(\mathbf{u},\,\boldsymbol{\omega},\,p)|||_{\mathbf{V}} \equiv \left(\nu^2\|\mathbf{u}\|_1^2+\nu^2\|\boldsymbol{\omega}\|^2+(1+\nu\delta)^2\|p\|^2\right)^{\frac{1}{2}}.$$

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THEOREM 3.1. *Assume that* $(\mathbf{u},\, \boldsymbol{\omega},\, p) \in \left(H^{r+1}(\Omega)^d \times H^r(\Omega)^{2d-2}\right) \cap \mathbf{V}$ *is the solution of problem* (2.10)*. Then*

$$|||(\mathbf{u},\, \boldsymbol{\omega},\, p) - (\mathbf{u}_h,\, \boldsymbol{\omega}_h,\, p_h)|||_{\mathbf{V}} \le C\, h^r d_r(\mathbf{u},\, \boldsymbol{\omega},\, p), \tag{3.5}$$

where C *depends only on the domain* Ω *and the ratio of the constants* C_2 *and* C_1 *in* Theorem 2.1 *and where*

$$d_r(\mathbf{u},\, \boldsymbol{\omega},\, p) = \left(\nu^2 \|\mathbf{u}\|^2_{r+1} + \nu^2 \|\boldsymbol{\omega}\|^2_r + (1+\nu\delta)^2 \|p\|^2_r\right)^{\frac{1}{2}}. \tag{3.6}$$

Proof. It is easy to see that the error $(\mathbf{u} - \mathbf{u}_h,\, \boldsymbol{\omega} - \boldsymbol{\omega}_h,\, p - p_h)$ is orthogonal to $\mathbf{V}_h$ with respect to the inner product corresponding to the functional $G(\mathbf{u},\, \boldsymbol{\omega},\, p;\, \mathbf{0})$. Bound (3.5) now follows from Theorem 2.1 and approximation properties (3.1)–(3.3). □

REMARK 3.1. *The above result indicates that the finite element approximation is optimal, both with respect to the order of approximation and the required regularity of the solution* (*see* [3])*. More specifically, bound* (3.5) *holds with*

$$d_r(\mathbf{u},\, \boldsymbol{\omega},\, p) = \left(\nu^2 \|\mathbf{u}\|^2_{r+1} + \|p\|^2_r\right)^{\frac{1}{2}}$$

since $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ *and* $\|\nabla\times\mathbf{u}\|_r \le C\, \|\mathbf{u}\|_{r+1}$.

4. Solution Method and Discrete H^{-1} Functional. Theorem 3.1 indicates that the finite element approximation based on functional G is also optimal with respect to the required regularity of the solution. Notice that the functional involves the H^{-1} norm, which in turn requires solution of a boundary value problem for its evaluation. There are two existing approaches to make the method computationally feasible: the mesh-dependent least-squares scheme proposed by Aziz, Kellogg, and Stephens [2] (see also [7]) and the discrete H^{-1}-norm scheme proposed by Bramble, Lazarov, and Pasciak [3]. As mentioned in the introduction, it is not clear that a fast solution algorithm for the resulting discrete equations from the mesh-dependent least-squares method can be developed at this stage of research. In this paper, we will therefore adopt the discrete H^{-1}-norm approach. Following [3], the H^{-1}-norm in the functional is replaced by a discrete norm. This discrete H^{-1} functional is computable and can be uniformly preconditioned by well-known techniques.

To this end, let $A : H^{-1}(\Omega)^d \longrightarrow H^1_0(\Omega)^d$ denote the solution operator for the Poisson problem

$$\left\{ \begin{array}{rcll} -\Delta\, \boldsymbol{\phi} & = & \mathbf{v}, & \text{in} \quad \Omega, \\ \boldsymbol{\phi} & = & 0, & \text{on} \quad \partial\Omega, \end{array} \right. \tag{4.1}$$

i.e., $A\mathbf{v} = \boldsymbol{\phi}$ for a given $\mathbf{v} \in H^{-1}(\Omega)^d$ is the solution to (4.1). It is well-known that $\sqrt{(A\cdot,\, \cdot)}$ defines a norm that is equivalent to the $H^{-1}(\Omega)^d$ norm. Let $A_h : L^2(\Omega)^d \longrightarrow \mathbf{U}_h$ be defined by $A_h\boldsymbol{\varphi} = \boldsymbol{\phi}$, where $\boldsymbol{\phi}$ is the unique solution in $\mathbf{U}_h$ satisfying

$$\int_\Omega \nabla\, \boldsymbol{\phi} \cdot \nabla\, \boldsymbol{\psi}\, dx = (\boldsymbol{\varphi},\, \boldsymbol{\psi}), \quad \forall\, \boldsymbol{\psi} \in \mathbf{U}_h.$$

Assume that there is a preconditioner $B_h : L^2(\Omega)^d \longrightarrow \mathbf{U}_h$ that is symmetric with respect to the $L^2(\Omega)$ inner product and spectrally equivalent to A_h, i.e., there are positive constants C_1 and C_2, not depending on h, that satisfy

$$C_1\, (A_h\boldsymbol{\phi},\, \boldsymbol{\phi}) \le (B_h\boldsymbol{\phi},\, \boldsymbol{\phi}) \le C_2\, (A_h\boldsymbol{\phi},\, \boldsymbol{\phi}), \quad \forall\, \boldsymbol{\phi} \in \mathbf{U}_h. \tag{4.2}$$

Following [3], define $\tilde{A}_h = h^2 I + B_h$, where I denotes the identity operator on $\mathbf{U}_h$. In the remainder of this section, we analyze the least-squares approximation based on the functional

$$\begin{aligned} G^h(\mathbf{u},\,\boldsymbol{\omega},\,p;\,\mathbf{f}) \;&=\; \Big(\tilde{A}_h(\mathbf{f} - (\nu\nabla\times\boldsymbol{\omega} + (1+\nu\delta)\nabla\, p)),\, \mathbf{f} - (\nu\nabla\times\boldsymbol{\omega} + (1+\nu\delta)\nabla\, p)\Big) \\ &+\; \nu^2\,\|\nabla\times\mathbf{u} - \boldsymbol{\omega}\|^2 + \nu^2\,\|\nabla\cdot\mathbf{u} + \delta\, p\|^2. \end{aligned} \tag{4.3}$$

Define the norm corresponding to the functional G^h by

$$|||(\mathbf{u},\,\boldsymbol{\omega},\,p)|||_{\mathbf{V}_h} \equiv \sqrt{G^h(\mathbf{u},\,\boldsymbol{\omega},\,p;\,\mathbf{0})}.$$

Let $Q_h : L^2(\Omega)^d \longrightarrow \mathbf{U}_h$ denote the $L^2(\Omega)^d$ orthogonal projection operator onto $\mathbf{U}_h$. We assume that Q_h is bounded on $H^1(\Omega)^d$, i.e.,

$$\|Q_h\mathbf{v}\|_1 \le C\,\|\mathbf{v}\|_1, \quad \forall\ \mathbf{v} \in H^1(\Omega)^d. \tag{4.4}$$

REMARK 4.1. *The symmetry of B_h with respect to the inner product on $L^2(\Omega)^d$ implies that $B_h = B_hQ_h$. Similarly, $A_h = A_hQ_h$. Thus, (4.2) holds for any $\mathbf{v} \in L^2(\Omega)^d$.*

It is easy to check that assumption (3.1) for $r = 0$ and (4.4) imply that

$$\|(I - Q_h)\mathbf{v}\|_{-1} \le Ch\,\|\mathbf{v}\|, \quad \forall\ \mathbf{v} \in L^2(\Omega)^d, \tag{4.5}$$

and that (see [3])

$$\|Q_h\mathbf{v}\|_{-1}^2 \le C\,(A_h\mathbf{v},\,\mathbf{v}) \le C\,\|\mathbf{v}\|_{-1}^2, \quad \forall\ \mathbf{v} \in L^2(\Omega)^d. \tag{4.6}$$

LEMMA 4.1. *For any $(\mathbf{u},\,\boldsymbol{\omega},\,p) \in H_0^1(\Omega)^d \times H(\mathbf{curl};\,\Omega) \times (L_0^2(\Omega) \cap H^1(\Omega))$, there exist positive constants C_1 and C_2, independent of h and ν, such that*

$$\begin{aligned} &C_1\,\left(\nu^2\|\mathbf{u}\|_1^2 + \nu^2\|\boldsymbol{\omega}\|^2 + (1+\nu\delta)^2\|p\|^2\right) \le |||(\mathbf{u},\,\boldsymbol{\omega},\,p)|||_{\mathbf{V}_h}^2 \\ &\le C_2\left(\nu^2\|\mathbf{u}\|_1^2 + \nu^2h^2\|\nabla\times\boldsymbol{\omega}\|^2 + \nu^2\|\boldsymbol{\omega}\|^2 + h^2(1+\nu\delta)^2\|\nabla p\|^2 + (1+\nu\delta)^2\|p\|^2\right). \end{aligned} \tag{4.7}$$

Proof. By Remark 4.1 and (4.6), we have that

$$(\tilde{A}_h\boldsymbol{\phi},\,\boldsymbol{\phi}) \le C\,\left(h^2\|\boldsymbol{\phi}\|^2 + (A_h\boldsymbol{\phi},\,\boldsymbol{\phi})\right) \le C\,\left(h^2\|\boldsymbol{\phi}\|^2 + \|\boldsymbol{\phi}\|_{-1}^2\right), \quad \forall\ \boldsymbol{\phi} \in L^2(\Omega)^d,$$

which, together with the triangle inequality and Theorem 2.1, imply the upper bound in (4.7). To prove the first inequality in (4.7), by Theorem 2.1 it suffices to show that

$$\|\nu\nabla\times\boldsymbol{\omega} + (1+\nu\delta)\nabla\, p\|_{-1}^2 \le C\,\Big(\tilde{A}_h(\nu\nabla\times\boldsymbol{\omega} + (1+\nu\delta)\nabla\, p),\, \nu\nabla\times\boldsymbol{\omega} + (1+\nu\delta)\nabla\, p\Big)$$

for any $\boldsymbol{\omega} \in H(\mathbf{curl};\,\Omega)$ and any $p \in H^1(\Omega)$. From (4.5), (4.6), and Remark 4.1, for any $\boldsymbol{\phi} \in L^2(\Omega)^d$ we have

$$\begin{aligned} \|\boldsymbol{\phi}\|_{-1}^2 \;&\le\; 2\,\left(\|(I - Q_h)\boldsymbol{\phi}\|_{-1}^2 + \|Q_h\boldsymbol{\phi}\|_{-1}^2\right) \\ &\le\; C\,\left(h^2\|\boldsymbol{\phi}\|^2 + (A_h\boldsymbol{\phi},\,\boldsymbol{\phi})\right) \\ &\le\; C\,(\tilde{A}_h\boldsymbol{\phi},\,\boldsymbol{\phi}). \end{aligned}$$

This completes the proof of the lemma. ◻

REMARK 4.2. *If $\mathbf{W}_h \subset H(\mathbf{curl};\Omega)$ and $P_h \subset L_0^2(\Omega) \cap H^1(\Omega)$ satisfy an inverse inequality of the form*

$$\|\nabla\times\boldsymbol{\omega}\| \le C\,h^{-1}\|\boldsymbol{\omega}\| \quad and \quad \|\nabla\, p\| \le C\,h^{-1}\|p\|,$$

respectively, then the second inequality of (4.7) can be replaced by $\nu^2\|\mathbf{u}\|_1^2 + \nu^2\,\|\boldsymbol{\omega}\|^2 + (1+\nu\delta)^2\|p\|^2$ for any $\mathbf{u} \in H_0^1(\Omega)^d$, any $\boldsymbol{\omega} \in \mathbf{W}_h$, and any $p \in P_h$. It is well-known (cf. [12]) that the above inverse inequalities hold for typical finite element spaces consisting of continuous piecewise polynomials on quasi-uniform triangulations.

THEOREM 4.1. *Let $(\mathbf{u}_h,\,\boldsymbol{\omega}_h,\,p_h) \in \mathbf{V}_h$ be the unique minimizer of $G^h(\mathbf{u},\,\boldsymbol{\omega},\,p;\,\mathbf{f})$ over $\mathbf{V}_h$ and let $(\mathbf{u},\,\boldsymbol{\omega},\,p) \in \left(H^{r+1}(\Omega)^d \times H^r(\Omega)^d \times H^r(\Omega)\right) \cap \mathbf{V}$ be the solution of problem (2.10). Then*

$$\begin{aligned} &\nu\,\|\mathbf{u}-\mathbf{u}_h\|_1 + \nu\,\|\boldsymbol{\omega}-\boldsymbol{\omega}_h\| + (1+\nu\delta)\|p-p_h\| \\ \le\ & C\,h^r\left(\nu^2\|\mathbf{u}\|_{r+1}^2 + (1+\nu\delta)^2\|p\|_r^2\right)^{\frac{1}{2}}, \end{aligned} \tag{4.8}$$

where C is independent of the mesh size h and the Reynolds (Poisson) parameter ν.

Proof. It is easy to see that the error $(\mathbf{u}-\mathbf{u}_h,\,\boldsymbol{\omega}-\boldsymbol{\omega}_h,\,p-p_h)$ is orthogonal to $\mathbf{V}_h$ with respect to the inner product corresponding to the functional $G^h(\mathbf{u},\,\boldsymbol{\omega},\,p;\,\mathbf{0})$. Bound (4.8) now follows from Lemma 4.1 and approximation properties (3.1)–(3.3). □

For the finite element spaces $\mathbf{W}_h$ and P_h satisfying the inverse inequalities in Remark 4.2, the discrete H^{-1} functional $G^h(\mathbf{u},\,\boldsymbol{\omega},\,p;\,\mathbf{0})$ can be preconditioned by the functional $\nu^2\|\mathbf{u}\|_1^2 + \nu^2\,\|\boldsymbol{\omega}\|^2 + (1+\nu\delta)^2\|p\|^2$ that decouples velocity, vorticity, and pressure unknowns, because they are spectrally equivalent uniformly in the mesh size h and the Reynolds (Poisson) parameter ν (see Lemma 4.1 and Remark 4.2). We can use any effective elliptic preconditioners associated with velocity $\mathbf{u}$, including those of multigrid type, and simple preconditioners associated with vorticity $\boldsymbol{\omega}$ and pressure p, including those of diagonal matrix type.

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