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ETNA
Kent State University
etna@mcs.kent.edu

NONCOMMUTATIVE EXTENSIONS OF RAMANUJAN'S ${}_1\psi_1$ SUMMATION*

MICHAEL SCHLOSSER†

Abstract. Using functional equations, we derive noncommutative extensions of Ramanujan's ${}_1\psi_1$ summation.

Key words. noncommutative basic hypergeometric series, Ramanujan's ${}_1\psi_1$ summation

AMS subject classifications. 33D15, 33D99

1. Introduction. Hypergeometric series with noncommutative parameters and argument, in the special case involving square matrices, have been the subject of recent study, see e.g. the papers by Duval and Ovsienko [4], Grünbaum [6], Tirao [16], and some of the references mentioned therein. Of course, this subject is also closely related to the theory of orthogonal matrix polynomials which was initiated by Krein [12] and has experienced a steady development, see e.g. Durán and López-Rodríguez [3].

Very recently, Tirao [16] considered a particular type of a matrix valued hypergeometric function (which, in our terminology, belongs to noncommutative hypergeometric series of "type I"). He showed, in particular, that the matrix valued hypergeometric function satisfies the matrix valued hypergeometric differential equation, and conversely that any solution of the latter is a matrix valued hypergeometric function.

In [14], the present author investigated hypergeometric and basic hypergeometric series involving noncommutative parameters and argument (short: *noncommutative hypergeometric series*, and *noncommutative basic* or *Q-hypergeometric series*) over a unital ring R (or, when considering nonterminating series, over a unital Banach algebra R) from a different, nevertheless completely elementary, point of view. These investigations were exclusively devoted to the derivation of summation formulae (which quite surprisingly even exist in the noncommutative case), aiming to build up a theory of explicit identities analogous to the rich theory of identities for hypergeometric and basic hypergeometric series in the classical, commutative case (cf. [15] and [5]). Two closely related types of noncommmutative series, of "type I" and "type II", were considered in [14]. Most of the summations obtained there concern terminating series and were proved by induction. An exception are the noncommutative extensions of the nonterminating q-binomial theorem [14, Th. 7.2] which were established using functional equations. Aside from the latter and some conjectured Q-Gauß summations, no other explicit summations for nonterminating noncommutative basic hypergeometric series were given. Furthermore, noncommutative *bilateral* basic hypergeometric series were not even considered.

In this paper, we define noncommutative bilateral basic hypergeometric series of type I and type II (over an abstract unital Banach algebra R) and prove, using functional equations, noncommutative extensions of Ramanujan's ${}_1\psi_1$ summation. These generalize the noncommutative Q-binomial theorem of [14, Th. 7.2]. Our proof of the ${}_1\psi_1$ sum here is similar to Andrews and Askey's [1] proof in the classical commutative case. Ramanujan's ${}_1\psi_1$ summation (displayed in (2.4)) is one of the fundamental identities in q-series. It is thus just natural to look for different extensions, including noncommutative ones.

This paper is organized as follows. In Section 2, we review some standard notations for

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†Fakultät für Mathematik, Universität Wien, Nordbergstraße 15, A-1090 Wien, Austria (michael.schlosser@univie.ac.at), http://www.mat.univie.ac.at/~schlosse.

ETNA
Kent State University
etna@mcs.kent.edu

basic hypergeometric series and then explain the notation we utilize in the noncommutative case. Section 3, is devoted to the derivation of noncommutative $_1\psi_1$ summations.

We stress again, as in [14], that by "noncommutative" we do not mean "q-commutative" or "quasi-commutative" (i.e., where the variables satisfy a relation like $yx = qxy$; such series are considered e.g. in [11] and [17]) but that the parameters in our series are elements of some noncommutative unital ring (or unital Banach algebra).

2. Preliminaries.

2.1. Classical (commutative) basic hypergeometric series.

For convenience, we recall some standard notations for basic hypergeometric series (cf. [5]). When considering the noncommutative extensions in Subsection 2.2 and in Section 3, the reader may find it useful to compare with the classical, commutative case.

Let q be a complex number such that $0 < |q| < 1$. Define the *q-shifted factorial* for all integers k (including infinity) by

$$(a;q)_k := \prod_{j=1}^{k}(1 - aq^j).$$

We write

$$ {}_r\phi_{r-1}\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_{r-1} \end{matrix}; q, z\right] := \sum_{k=0}^{\infty} \frac{(a_1;q)_k(a_2;q)_k \dots (a_r;q)_k}{(q,q)_k(b_1;q)_k \dots (b_{r-1};q)_k} z^k, \tag{2.1}$$

to denote the (unilateral) *basic hypergeometric $_r\phi_{r-1}$ series*. Further, we write

$$ {}_r\psi_r\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; q, z\right] := \sum_{k=-\infty}^{\infty} \frac{(a_1;q)_k \dots (a_r;q)_k}{(b_1;q)_k \dots (b_r;q)_k} z^k, \tag{2.2}$$

to denote the *bilateral basic hypergeometric $_r\psi_r$ series*.

In (2.1) and (2.2), $a_1, \dots, a_r$ are called the *upper parameters*, $b_1, \dots, b_r$ the *lower parameters*, z is the *argument*, and q the *base* of the series. The bilateral $_r\psi_r$ series in (2.2) reduces to a unilateral $_r\phi_{r-1}$ series if one of the lower parameters, say b_r, equals q (or more generally, an integral power of q).

The basic hypergeometric $_r\phi_{r-1}$ series terminates if one of the upper parameters, say a_r, is of the form q^{-n}, for a nonnegative integer n. If the basic hypergeometric series does not terminate then it converges by the ratio test when $|z| < 1$. Similarly, the bilateral basic hypergeometric series converges when $|z| < 1$ and $|b_1 \dots b_r/a_1 \dots a_r z| < 1$.

We recall two important summations. One of them is the (nonterminating) q-binomial theorem,

$$ {}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix}; q, z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}, \tag{2.3}$$

where $|z| < 1$ (cf. [5, Appendix (II.3)]). It was discovered independently by several mathematicians, including Cauchy, Gauß, and Heine. A bilateral extension of (2.3) was found by the legendary Indian mathematician Ramanujan (see Hardy [8]),

$$ {}_1\psi_1\left[\begin{matrix} a \\ b \end{matrix}; q, z\right] = \frac{(q;q)_\infty(b/a;q)_\infty(az;q)_\infty(q/az;q)_\infty}{(b;q)_\infty(q/a;q)_\infty(z;q)_\infty(b/az;q)_\infty}, \tag{2.4}$$

where $|z| < 1$ and $|b/az| < 1$ (cf. [5, Appendix (II.29)]). Unfortunately, Ramanujan (who very rarely gave any proofs) did not provide a proof of the above bilateral summation. The first proof of (2.4) was given by Hahn [7, $\kappa = 0$ in Eq. (4.7)]. Other proofs were given by Jackson [10], Ismail [9], Andrews and Askey [1], the author [13, Sec. 3], and others. Some immediate applications of Ramanujan's summation formula to arithmetic number theory are considered in [2, Sec. 10.6].

2.2. Noncommutative basic hypergeometric series. Most of the following definitions are taken from [14]. However, the definitions for noncommutative *bilateral* basic hypergeometric series in (2.8) and (2.9) (although obvious) are new.

Let R be a unital ring (i.e., a ring with a multiplicative identity). When considering infinite series and infinite products of elements of R we shall further assume that R is a Banach algebra (with some norm $\|\cdot\|$). The elements of R will be denoted by capital letters $A, B, \ldots$. In general these elements do not commute with each other; however, we may sometimes specify certain commutation relations explicitly. We denote the identity by I and the zero element by O. Whenever a multiplicative inverse element exists for any $A \in R$, we denote it by A^{-1}. (Since R is a unital ring, we have $AA^{-1} = A^{-1}A = I$.) On the other hand, as we shall implicitly assume that all the expressions which appear are well defined, whenever we write A^{-1} we assume its existence. For instance, in (2.6) and (2.7) we assume that $I - B_iQ^j$ is invertible for all $1 \le i \le r$, $0 \le j < k$.

An important special case is when R is the ring of $n \times n$ square matrices (our notation is certainly suggestive with respect to this interpretation), or, more generally, one may view R as a space of some abstract operators.

Let $\mathbb{Z}$ be the set of integers. For $l, m \in \mathbb{Z} \cup \{\pm\infty\}$ we define the noncommutative product as follows:

$$\prod_{j=l}^{m} A_j = \begin{cases} 1 & m = l-1 \\ A_l A_{l+1} \ldots A_m & m \ge l \\ A_{l-1}^{-1} A_{l-2}^{-1} \ldots A_{m+1}^{-1} & m < l-1 \end{cases}.$$

Note that

$$\prod_{j=l}^{m} A_j = \prod_{j=m+1}^{l-1} A_{m+l-j}^{-1}, \tag{2.5}$$

for all $l, m \in \mathbb{Z} \cup \{\pm\infty\}$.

Throughout this paper, Q will be a parameter which commutes with any of the other parameters appearing in the series. (For instance, a central element such as $Q = qI$, a scalar multiple of the unit element in R, for $qI \in R$, trivially satisfies this requirement.)

Let $k \in \mathbb{Z} \cup \{\infty\}$. We define the generalized *noncommutative Q-shifted factorial of type I* by

$$\left\lceil \begin{matrix} A_1, A_2, \ldots, A_r \\ B_1, B_2, \ldots, B_r \end{matrix} ; Q, Z \right\rfloor_k := \prod_{j=1}^{k} \left[\left(\prod_{i=1}^{r} (I - B_iQ^{k-j})^{-1}(I - A_iQ^{k-j}) \right) Z \right]. \tag{2.6}$$

Similarly, we define the generalized *noncommutative Q-shifted factorial of type II* by

$$\left\lfloor \begin{matrix} A_1, A_2, \ldots, A_r \\ B_1, B_2, \ldots, B_r \end{matrix} ; Q, Z \right\rceil_k := \prod_{j=1}^{k} \left[\left(\prod_{i=1}^{r} (I - B_iQ^{j-1})^{-1}(I - A_iQ^{j-1}) \right) Z \right]. \tag{2.7}$$

Note the unusual usage of brackets ("floors" and "ceilings" are intermixed) on the left-hand sides of (2.6) and (2.7) which is intended to suggest that the products involve noncommuting factors in a prescribed order. In both cases, the product, read from left to right, starts

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Kent State University
etna@mcs.kent.edu

with a denominator factor. The brackets in the form "$\lceil - \rfloor$" are intended to denote that the factors are *falling*, while in "$\lfloor - \rceil$" that they are *rising*.

We define the *noncommutative basic hypergeometric series of type I* by

$$ {}_r\phi_{r-1}\left\lceil \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_{r-1} \end{matrix} ; Q, Z \right\rfloor := \sum_{k \ge 0} \left\lceil \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_{r-1}, Q \end{matrix} ; Q, Z \right\rfloor_k , $$

and the *noncommutative basic hypergeometric series of type II* by

$$ {}_r\phi_{r-1}\left\lfloor \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_{r-1} \end{matrix} ; Q, Z \right\rceil := \sum_{k \ge 0} \left\lfloor \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_{r-1}, Q \end{matrix} ; Q, Z \right\rceil_k . $$

Further, we define the *noncommutative bilateral basic hypergeometric series of type I* by

$$ {}_r\psi_r\left\lceil \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_r \end{matrix} ; Q, Z \right\rfloor := \sum_{k=-\infty}^{\infty} \left\lceil \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_r \end{matrix} ; Q, Z \right\rfloor_k , \tag{2.8} $$

and the *noncommutative bilateral basic hypergeometric series of type II* by

$$ {}_r\psi_r\left\lfloor \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_r \end{matrix} ; Q, Z \right\rceil := \sum_{k=-\infty}^{\infty} \left\lfloor \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_r \end{matrix} ; Q, Z \right\rceil_k . \tag{2.9} $$

We also refer to the respective series as *(noncommutative) Q-hypergeometric series*. In each case (of type I and type II), the ${}_r\phi_{r-1}$ series terminates if one of the upper parameters A_i is of the form Q^{-n}. If the ${}_r\phi_{r-1}$ series does not terminate, then (implicitly assuming that R is a unital Banach algebra with some norm $\|\cdot\|$) it converges when $\|Z\| < 1$. Similarly, the ${}_r\psi_r$ series converges in R when $\|Z\| < 1$ and $\|Z^{-1}\prod_{i=1}^r A_{r+1-i}^{-1} B_{r+1-i}\| < 1$.

We also consider reversed (or "transposed") versions of generalized noncommutative Q-shifted factorials and noncommutative bilateral basic hypergeometric series of type I and II. These are defined as follows (compare with (2.6), (2.7), (2.8) and (2.9)):

$$ {}^{\sim}\left\lceil \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_r \end{matrix} ; Q, Z \right\rfloor_k := \prod_{j=1}^{k} \left(Z \prod_{i=1}^{r} (I - A_i Q^{j-1})(I - B_i Q^{j-1})^{-1} \right) , $$

$$ {}^{\sim}\left\lfloor \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_r \end{matrix} ; Q, Z \right\rceil_k := \prod_{j=1}^{k} \left(Z \prod_{i=1}^{r} (I - A_i Q^{k-j})(I - B_i Q^{k-j})^{-1} \right) , $$

$$ {}_r\psi_r^{\sim}\left\lceil \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_r \end{matrix} ; Q, Z \right\rfloor := \sum_{k=-\infty}^{\infty} {}^{\sim}\left\lceil \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_r \end{matrix} ; Q, Z \right\rfloor_k , $$

and

$$ {}_r\psi_r^{\sim}\left\lfloor \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_r \end{matrix} ; Q, Z \right\rceil := \sum_{k=-\infty}^{\infty} {}^{\sim}\left\lfloor \begin{matrix} A_1, A_2, \dots, A_r \\ B_1, B_2, \dots, B_r \end{matrix} ; Q, Z \right\rceil_k . $$

Of course, reversed versions of the unilateral noncommutative basic hypergeometric series are defined analogously.

ETNA
Kent State University
etna@mcs.kent.edu

3. Noncommutative ${}_1\psi_1$ summations. In [14, Th. 7.2], the following two noncommutative extensions of the nonterminating q-binomial theorem (which generalize [5, II.3]) were given.

PROPOSITION 3.1. *Let A and Z be noncommutative parameters of some unital Banach algebra, and suppose that Q commutes with both A and Z. Further, assume that $\|Z\| < 1$. Then we have the following summation for a noncommutative basic hypergeometric series of type I.*

$$
{}_1\phi_0\left\lceil \begin{matrix} A \\ - \end{matrix} ; Q, Z \right\rfloor = \left\lfloor \begin{matrix} AZ \\ Z \end{matrix} ; Q, I \right\rceil_\infty . \tag{3.1}
$$

Further, we we have the following summation for a noncommutative basic hypergeometric series of type II.

$$
{}_1\phi_0\left\lfloor \begin{matrix} A \\ - \end{matrix} ; Q, Z \right\rceil = {}^{\sim}\left\lfloor \begin{matrix} AZ \\ Z \end{matrix} ; Q, I \right\rceil_\infty . \tag{3.2}
$$

Here we extend Proposition 3.1 to summations for bilateral series. Our proof is similar to that of the classical result given in [1] (see also [2, p. 502, first proof of Th. 10.5.1]), but also similar to the proof of Proposition 3.1 given in [14].

THEOREM 3.2. *Let A, B and Z be noncommutative parameters of some unital Banach algebra, suppose that Q and B both commute with any of the other parameters. Further, assume that $\|Z\| < 1$ and $\|BZ^{-1}A^{-1}\| < 1$. Then we have the following summation for a noncommutative bilateral basic hypergeometric series of type I.*

$$
\begin{aligned}
&{}_1\psi_1\left\lceil \begin{matrix} A \\ B \end{matrix} ; Q, Z \right\rfloor = \\
&\left\lfloor \begin{matrix} Q, BZ^{-1}A^{-1}Z \\ B, BZ^{-1}A^{-1} \end{matrix} ; Q, I \right\rceil_\infty \left\lceil \begin{matrix} Z^{-1}A^{-1}Q \\ Z^{-1}A^{-1}ZQ \end{matrix} ; Q, I \right\rfloor_\infty \left\lfloor \begin{matrix} AZ \\ Z \end{matrix} ; Q, I \right\rceil_\infty .
\end{aligned} \tag{3.3}
$$

Further, we have the following summation for a noncommutative bilateral basic hypergeometric series of type II.

$$
\begin{aligned}
&{}_1\psi_1\left\lfloor \begin{matrix} A \\ B \end{matrix} ; Q, Z \right\rceil = \\
&{}^{\sim}\left\lfloor \begin{matrix} AZ \\ Z \end{matrix} ; Q, I \right\rceil_\infty \left\lfloor \begin{matrix} Z^{-1}A^{-1}Q, Q \\ B, A^{-1}Q \end{matrix} ; Q, I \right\rceil_\infty {}^{\sim}\left\lfloor \begin{matrix} BA^{-1} \\ BZ^{-1}A^{-1} \end{matrix} ; Q, I \right\rceil_\infty .
\end{aligned} \tag{3.4}
$$

Clearly, Theorem 3.2 reduces to Proposition 3.1 when $B = Q$.

Proof of Theorem 3.2. We prove (3.3) using (3.1), leaving the proof of (3.4) using (3.2), which is similar, to the reader.

Let $f(A, B, Z)$ denote the series on the left-hand side of (3.3). We make use of the two simple identities

$$
Z = AZQ^k + (I - AQ^k)Z, \tag{3.5a}
$$

$$
I = BQ^k + (I - BQ^k), \tag{3.5b}
$$

to obtain two functional equations for f. We also make use of the simple relation

$$
f(A, B, Z) = f(AQ, BQ, Z)(I - B)^{-1}(I - A)Z, \tag{3.6}
$$

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Kent State University
etna@mcs.kent.edu

obtained by shifting the summation index in f by one.

First, (3.5a) gives

$$Zf(A,B,Z) = AZf(A,B,ZQ) + f(AQ,B,Z)(I-A)Z, \tag{3.7}$$

while (3.5b) gives

$$f(A,BQ,Z) = Bf(A,BQ,ZQ) + (I-B)f(A,B,Z). \tag{3.8}$$

Combining (3.8), (3.7), and (3.6), one readily deduces

$$\begin{aligned} f(A,BQ,Z) &= (I-B)f(A,B,Z) + BZ^{-1}A^{-1}Zf(A,BQ,Z) \\ &\quad - BZ^{-1}A^{-1}f(AQ,BQ,Z)(I-A)Z \\ &= (I-B)f(A,B,Z) + BZ^{-1}A^{-1}Zf(A,BQ,Z) \\ &\quad - BZ^{-1}A^{-1}f(A,B,Z)(I-B), \end{aligned}$$

or equivalently

$$(I - BZ^{-1}A^{-1}Z)f(A,BQ,Z) = (I-B)(I-BZ^{-1}A^{-1})f(A,B,Z),$$

thus

$$f(A,B,Z) = (I-BZ^{-1}A^{-1})^{-1}(I-BZ^{-1}A^{-1}Z)(I-B)^{-1}f(A,BQ,Z). \tag{3.9}$$

Iteration of (3.9) gives

$$f(A,B,Z) = \prod_{j=0}^{\infty}\left[(I-BZ^{-1}A^{-1}Q^j)^{-1}(I-BZ^{-1}A^{-1}ZQ^j)(I-BQ^j)^{-1}\right] f(A,O,Z). \tag{3.10}$$

We still need to compute $f(A,O,Z)$. It is not easy to do this directly but we know the value of $f(A,Q,Z)$ (by Proposition 3.1). We set $B = Q$ in (3.10) which gives

$$f(A,Q,Z) = \prod_{j=0}^{\infty}\left[(I-Z^{-1}A^{-1}Q^{j+1})^{-1}(I-Z^{-1}A^{-1}ZQ^{j+1})(I-Q^{j+1})^{-1}\right] f(A,O,Z),$$

thus we obtain

$$f(A,O,Z) = \left[\prod_{j=0}^{\infty}(I-Q^{j+1})\right]\left[\begin{matrix} Z^{-1}A^{-1}Q \\ Z^{-1}A^{-1}ZQ \end{matrix};Q,I\right]_{\infty} f(A,Q,Z). \tag{3.11}$$

Combination of (3.10), (3.11) and (3.1) establishes the result. ◻

In the ${}_1\psi_1$ summations of Theorem 3.2, the lower parameter B commutes with both A and Z while A does not commute with Z. In the next theorem the roles of A and B are interchanged. Here A commutes with both B and Z while B does not commute with Z.

THEOREM 3.3. *Let A, B and Z be noncommutative parameters of some Banach algebra, suppose that Q and A both commute with any of the other parameters. Further, assume*

ETNA
Kent State University
etna@mcs.kent.edu

that $\|Z\| < 1$ and $\|BZ^{-1}A^{-1}\| < 1$. Then we have the following summation for a noncommutative bilateral basic hypergeometric series of type I.

$$(3.12)\quad {}_1\psi_1\left[\begin{matrix}A\\B\end{matrix};Q,Z\right] = Z^{-1}\left[\begin{matrix}Q, ZBZ^{-1}A^{-1}\\A^{-1}Q, Z\end{matrix};Q,I\right]_\infty \left[\begin{matrix}AZ\\ZBZ^{-1}\end{matrix};Q,I\right]_\infty \left[\begin{matrix}Z^{-1}A^{-1}Q\\BZ^{-1}A^{-1}\end{matrix};Q,I\right]_\infty Z.$$

Further, we have the following summation for a noncommutative bilateral basic hypergeometric series of type II.

$$(3.13)\quad {}_1\psi_1\left\lfloor\begin{matrix}A\\B\end{matrix};Q,Z\right\rfloor = Z^{-1}\,{}^{\sim}\!\left\lfloor\begin{matrix}Z^{-1}A^{-1}Q\\BZ^{-1}A^{-1}\end{matrix};Q,I\right\rfloor_\infty \left\lfloor\begin{matrix}AZ, Q\\A^{-1}Q, B\end{matrix};Q,I\right\rfloor_\infty {}^{\sim}\!\left\lfloor\begin{matrix}A^{-1}B\\Z\end{matrix};Q,I\right\rfloor_\infty Z.$$

Proof. We indicate the derivation of (3.12) from (3.3). (The derivation of (3.13) from (3.4) is analogous.) The sum on the left-hand side of (3.3) remains unchanged if the summation index, say k, is replaced by $-k$. Using (2.5), we compute

$$\begin{aligned}
\left[\begin{matrix}A\\B\end{matrix};Q,Z\right]_{-k} &= \prod_{j=1}^{-k}\left[(I-BQ^{-k-j})^{-1}(I-AQ^{-k-j})Z\right]\\
&= \prod_{j=1-k}^{0}\left[(I-BQ^{-1+j})^{-1}(I-AQ^{-1+j})Z\right]^{-1}\\
&= \prod_{j=1}^{k}\left[Z^{-1}(I-AQ^{-1-k+j})^{-1}(I-BQ^{-1-k+j})\right]\\
&= \prod_{j=1}^{k}\left[Z^{-1}A^{-1}(I-A^{-1}Q^{1+k-j})^{-1}(I-B^{-1}Q^{1+k-j})B\right]\\
&= Z^{-1}A^{-1}\left[\begin{matrix}B^{-1}Q\\A^{-1}Q\end{matrix};Q,BZ^{-1}A^{-1}\right]_k AZ.
\end{aligned}$$

Thus, by performing the simultaneous replacements $A \mapsto B^{-1}Q$, $B \mapsto A^{-1}Q$, $Z \mapsto BZ^{-1}A^{-1}$, in (3.3), we obtain (3.12). □

We complete this paper with four more ${}_1\psi_1$ summations, immediately obtained from corresponding summations in Theorems 3.2 and 3.3 by "reversing all products" (cf. [14, Subsec. 8.2]) on each side of the respective identities.

THEOREM 3.4. *Let A, B and Z be noncommutative parameters of some unital Banach algebra, suppose that Q and B both commute with any of the other parameters. Further, assume that $\|Z\| < 1$ and $\|A^{-1}Z^{-1}B\| < 1$. Then we have the following summation for a reversed noncommutative bilateral basic hypergeometric series of type I.*

$${}_1\psi_1^{\sim}\left[\begin{matrix}A\\B\end{matrix};Q,Z\right] = {}^{\sim}\!\left[\begin{matrix}ZA\\Z\end{matrix};Q,I\right]_\infty {}^{\sim}\!\left[\begin{matrix}A^{-1}Z^{-1}Q\\ZA^{-1}Z^{-1}Q\end{matrix};Q,I\right]_\infty {}^{\sim}\!\left[\begin{matrix}ZA^{-1}Z^{-1}B, Q\\A^{-1}Z^{-1}B, B\end{matrix};Q,I\right]_\infty .$$

ETNA
Kent State University
etna@mcs.kent.edu

Further, we have the following summation for a noncommutative bilateral basic hypergeometric series of type II.

$$
{}_1\psi_1^{\sim}\left\lfloor \begin{matrix} A \\ B \end{matrix}; Q, Z \right\rfloor =
$$
$$
\left\lfloor \begin{matrix} A^{-1}B \\ A^{-1}Z^{-1}B \end{matrix}; Q, I \right\rfloor_\infty^{\sim} \left\lfloor \begin{matrix} Q, A^{-1}Z^{-1}Q \\ A^{-1}Q, B \end{matrix}; Q, I \right\rfloor_\infty \left\lfloor \begin{matrix} ZA \\ Z \end{matrix}; Q, I \right\rfloor_\infty .
$$

THEOREM 3.5. *Let A, B and Z be noncommutative parameters of some Banach algebra, suppose that Q and A both commute with any of the other parameters. Further, assume that $\|Z\| < 1$ and $\|A^{-1}Z^{-1}B\| < 1$. Then we have the following summation for a noncommutative bilateral basic hypergeometric series of type I.*

$$
{}_1\psi_1^{\sim}\left\lceil \begin{matrix} A \\ B \end{matrix}; Q, Z \right\rceil =
$$
$$
Z^{\sim}\left\lfloor \begin{matrix} A^{-1}Z^{-1}Q \\ A^{-1}Z^{-1}B \end{matrix}; Q, I \right\rfloor_\infty^{\sim} \left\lceil \begin{matrix} ZA \\ Z^{-1}BZ \end{matrix}; Q, I \right\rceil_\infty^{\sim} \left\lfloor \begin{matrix} A^{-1}Z^{-1}BZ, Q \\ Z, A^{-1}Q \end{matrix}; Q, I \right\rfloor_\infty Z^{-1}.
$$

Further, we have the following summation for a noncommutative bilateral basic hypergeometric series of type II.

$$
{}_1\psi_1^{\sim}\left\lfloor \begin{matrix} A \\ B \end{matrix}; Q, Z \right\rfloor =
$$
$$
Z \left\lfloor \begin{matrix} BA^{-1} \\ Z \end{matrix}; Q, I \right\rfloor_\infty^{\sim} \left\lfloor \begin{matrix} Q, ZA \\ B, A^{-1}Q \end{matrix}; Q, I \right\rfloor_\infty \left\lfloor \begin{matrix} A^{-1}Z^{-1}Q \\ A^{-1}Z^{-1}B \end{matrix}; Q, I \right\rfloor_\infty Z^{-1}.
$$

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REFERENCES

[1] G. E. ANDREWS AND R. ASKEY, *A simple proof of Ramanujan's summation of the* ${}_1\psi_1$, Aequationes Math., 18 (1978), pp. 333–337.
[2] G. E. ANDREWS, R. ASKEY, AND R. ROY, *Special Functions*, Encyclopedia of Mathematics And Its Applications 71, Cambridge University Press, Cambridge, 1999.
[3] A. J. DURÁN AND P. LÓPEZ-RODRÍGUEZ, *Orthogonal matrix polynomials*, in Loredo Lectures on Orthogonal Polynomials And Special Functions, Advances in the theory of special functions and applications, Nova Science Publishers, New York, 2004, pp. 13–44.
[4] C. DUVAL AND V. OVSIENKO, *Projectively equivariant quantization and symbol calculus: noncommutative hypergeometric functions*, Letters Math. Phys., 57 (2001), pp. 61–67.
[5] G. GASPER AND M. RAHMAN, *Basic Hypergeometric Series*, 2nd ed., Encyclopedia of Mathematics And Its Applications 96, Cambridge University Press, Cambridge, 2004.
[6] F. A. GRÜNBAUM, *Matrix valued Jacobi polynomials*, Bull. Sci. Math., 127 (2003), pp. 207–214.
[7] W. HAHN, *Beiträge zur Theorie der Heineschen Reihen. Die hypergeometrische q-Differenzengleichung. Das q-Analogon der Laplace-Transformation*, Math. Nachr., 2 (1949), pp. 340–379.
[8] G. H. HARDY, *Ramanujan*, Cambridge University Press, Cambridge, 1940, reprinted by Chelsea, New York, 1978.
[9] M. E. H. ISMAIL, *A simple proof of Ramanujan's* ${}_1\psi_1$ *sum*, Proc. Amer. Math. Soc., 63 (1977), pp. 185–186.
[10] M. JACKSON, *On Lerch's transcendant and the basic bilateral hypergeometric series* ${}_2\psi_2$, J. London Math. Soc., 25 (1950), pp. 189–196.

[11] T. H. KOORNWINDER, *Special functions and q-commuting variables*, in Special Functions, q-Series, And Related Topics, M. E. H. Ismail, D. R. Masson, and M. Rahman, eds., Fields Institute Communications, 14, Amer. Math. Soc., 1997, pp. 131–166.

[12] M. G. KREIN, *Fundamental aspects of the representation theory of Hermitian operators with deficiency index* (m, m), Ukrain. Mat. Ž., 1 (1949), pp. 3–66; Amer. Math. Soc. Transl., 97 (1970), pp. 75–143.

[13] M. SCHLOSSER, *Abel–Rothe type generalizations of Jacobi's triple product identity*, in Theory And Applications of Special Functions. A Volume Dedicated to Mizan Rahman, M. E. H. Ismail and E. Koelink, eds., Dev. Math., 13 (2005), pp. 382–400.

[14] M. SCHLOSSER, *Summation formulae for noncommutative hypergeometric series*, preprint.

[15] L. J. SLATER, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

[16] J. A. TIRAO, *The matrix-valued hypergeometric equation*, Proc. Nat. Acad. Sci., 100 (2003), pp. 8138–8141.

[17] A. Y. VOLKOV, *Noncommutative hypergeometry*, Comm. Math. Phys., 258 (2005), pp. 257–273.