Electronic Transactions on Numerical Analysis.
Volume 24, pp. 88-93, 2006.

ISSN 1068-9613.

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GENERALIZED WEIGHTED SOBOLEV SPACES AND APPLICATIONS TO SOBOLEV ORTHOGONAL POLYNOMIALS: A SURVEY*

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Abstract. In this paper we present a definition of Sobolev spaces with respect to general measures, prove some useful technical results, some of them generalizations of classical results with Lebesgue measure and find general conditions under which these spaces are complete. These results have important consequences in Approximation Theory. We also find conditions under which the evaluation operator is bounded.

Key words. Sobolev spaces, weights, orthogonal polynomials

AMS subject classifications. 41A10, 46E35, 46G10

1. Introduction. Weighted Sobolev spaces are an interesting topic in many fields of Mathematics. In the classical books [7], [8], we can find the point of view of Partial Differential Equations. (See also [20] and [6]). We are interested in the relationship between this topic and Approximation Theory in general, and Sobolev Orthogonal Polynomials in particular.

The specific problems we want to solve are the following:

1) Given a Sobolev scalar product with general measures in $\mathbb{R}$, find hypotheses on the measures, as general as possible, so that we can define a Sobolev space whose elements are functions.

2) If a Sobolev scalar product with general measures in $\mathbb{R}$ is well defined for polynomials, what is the completion, $P^{k,2}$, of the space of polynomials with respect to the norm associated to that scalar product? This problem has been studied in some particular cases (see e.g. [4], [3], [5]), but at this moment no general theory has been built.

Our study has as application an answer to the question of finding the most general conditions under which the multiplication operator, $Mf(x) = x\,f(x)$, is bounded in the space $P^{k,2}$. We know by a theorem in [10] that, the zeroes of the Sobolev orthogonal polynomials are contained in the disk $\{z : |z| \le \|M\|\}$. The location of these zeroes allows to prove results on the asymptotic behaviour of Sobolev orthogonal polynomials (see [9]). In the second part of this paper, [15], and in [17] and [1], we answer the question stated also in [9] about general conditions for M to be bounded.

The completeness that we study now is one of the central questions in the theory of weighted Sobolev spaces, together with the density of C^∞ functions. In particular, when all the measures are finite, have compact support and are such that $C_c^\infty(\mathbb{R})$ is dense in a Sobolev space that is complete, then the closure of the polynomials is the whole Sobolev space. This is deduced from Bernstein's proof of Weierstrass' theorem, where the polynomials he builds

*Received November 30, 2004. Accepted for publication February 28, 2005. Recommended by J. Arvesú.

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approximate uniformly up to the k-th derivative any function in $C^k([a,b])$ (see e.g. [2], p.113).

In the paper we also prove some inequalities which generalize classical results about Sobolev spaces with respect to Lebesgue measure (see Theorem 3.2).

What we present here is an abridged version of the paper [14], where the complete proofs of the results may be found, together with the corresponding lemmas and related results.

In the first part of the article we obtain a good definition of Sobolev space with respect to very general measures. We allow the measures to be almost independent of each other. The main result that we present in the paper is Theorem 3.1. It states very general conditions on the measures under which this Sobolev space is complete.

2. Definitions and previous results. The main concepts that we need to understand the statement of our results are contained in the following definitions. The first one is a class of weights that will be the absolutely continuous part of our measures.

DEFINITION 2.1. *We say that a weight w belongs to $B_p(J)$, if and only if,*

$$\begin{aligned} w^{-1} &\in L^{1/(p-1)}_{loc}(J)\,, \quad && \text{for } 1 \le p < \infty\,, \\ w^{-1} &\in L^{1}_{loc}(J)\,, && \text{for } p = \infty\,. \end{aligned}$$

This class contains the classical A_p weights appearing in Harmonic Analysis, but is larger. We consider vectorial measures $\mu = (\mu_0, \dots, \mu_k)$ in the definition of our Sobolev space and make for each one the decomposition $d\mu_j = d(\mu_j)_s + w_j dx$, where $(\mu_j)_s$ is singular with respect to the Lebesgue measure and w_j is a Lebesgue measurable function.

DEFINITION 2.2. *Let us consider $1 \le p \le \infty$ and a vectorial measure $\mu = (\mu_0, \dots, \mu_k)$. For $0 \le j \le k$, we define the open set*

$$\Omega_j := \{x \in \mathbb{R} \ : \ \exists \text{ an open neighbourhood } V \text{ of } x \text{ with } w_j \in B_p(V)\}\,.$$

Observe that we always have $w_j \in B_p(\Omega_j)$, for any $0 \le j \le k$. In fact, Ω_j is the largest open set U with $w_j \in B_p(U)$. Obviously, Ω_j depends on p and μ, although p and μ do not appear explicitly in the symbol Ω_j. It is easy to check that if $f^{(j)} \in L^p(\Omega_j, w_j)$ with $0 \le j \le k$, then, $f^{(j)} \in L^1_{loc}(\Omega_j)$, and therefore $f^{(j-1)} \in AC_{loc}(\Omega_j)$ if $1 \le j \le k$. The notation AC_{loc} refers to the class of locally absolutely continuous functions.

We denote by $\Omega^{(j)}$ the set of "good" points at the level j for the vectorial weight $(w_0, \dots, w_k)$. These are in essence the points x for which there exists a weight w_i with $j < i \le k$ that is, in a neighbourhood of x, in the class B_p.

Let us present now the class of measures that we use and the definition of Sobolev space.

DEFINITION 2.3. *We say that the vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ is p-admissible if $(\mu_j)_s(\mathbb{R} \setminus \Omega^{(j)}) = 0$, for $1 \le j < k$, and $(\mu_k)_s \equiv 0$.*

Remarks.

1. The hypothesis of p-admissibility is natural. It would not be reasonable to consider Dirac deltas in μ_j in the points where $f^{(j)}$ is not continuous.

2. Observe that there is not any restriction on supp$(\mu_0)_s$.

3. Every absolutely continuous measure is p-admissible.

DEFINITION 2.4. *Let us consider $1 \le p \le \infty$, an open set $\Omega \subseteq \mathbb{R}$ and a p-admissible vectorial measure $\mu = (\mu_0, \dots, \mu_k)$ in $\overline{\Omega}$. We define the Sobolev space $W^{k,p}(\overline{\Omega}, \mu)$ as the space of equivalence classes of*

$$\begin{aligned} V^{k,p}(\overline{\Omega}, \mu) := \Big\{ f : \overline{\Omega} \to \mathbb{C} \ / \ & f^{(j)} \in AC_{loc}(\Omega^{(j)}) \text{ for } j = 0, 1, \dots, k-1 \text{ and} \\ & \|f^{(j)}\|_{L^p(\overline{\Omega}, \mu_j)} < \infty \ \text{ for } j = 0, 1, \dots, k \Big\}\,, \end{aligned}$$

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with respect to the seminorms

$$\|f\|_{W^{k,p}(\overline{\Omega},\mu)} := \Big(\sum_{j=0}^{k} \|f^{(j)}\|^p_{L^p(\overline{\Omega},\mu_j)}\Big)^{1/p}, \qquad \textit{for } 1 \le p < \infty\,,$$

$$\|f\|_{W^{k,\infty}(\overline{\Omega},\mu)} := \max_{0\le j\le k} \|f^{(j)}\|_{L^\infty(\overline{\Omega},\mu_j)}\,.$$

Here

$$\|g\|_{L^\infty(\Omega,\mu_j)} := \max\Big\{ \operatorname*{ess\,sup}_{x\in\Omega} |g(x)w_j(x)|, \sup_{x\in \operatorname{supp}(\mu_j)_s} |g(x)|\Big\}\,,$$

where ess sup *refers to Lebesgue measure, and we assume the usual convention* $\sup \emptyset = -\infty$.

Before we state our theorems, let us recall a classical result that will be generalized in our Theorem 3.2.

Muckenhoupt inequality. ([12], [11]) *Let us consider,* $1 \le p < \infty$ *and* μ_0, μ_1 *measures in* $(a, b]$ *with* $w_1 := d\mu_1/dx$. *Then there exists a positive constant c such that*

$$\Big\| \int_x^b g(t)\,dt \Big\|_{L^p((a,b],\mu_0)} \le c\,\|g\|_{L^p((a,b],\mu_1)}$$

for any measurable function g in $(a, b]$, *if and only if*

$$\Lambda_p(\mu_0, \mu_1) := \sup_{a<r<b} \mu_0((a, r])\|w_1^{-1}\|_{L^{1/(p-1)}([r,b))} < \infty\,.$$

3. Completeness of the Sobolev spaces. And now, here is our main theorem in the paper. In it and in Theorem 3.2 we consider special classes that we call $\mathcal{C}$ and $\mathcal{C}_0$. The conditions $(\overline{\Omega}, \mu) \in \mathcal{C}_0$ and $(\overline{\Omega}, \mu) \in \mathcal{C}$ are not very restrictive. The first one consists, roughly speaking, in considering measures μ such that $\|\cdot\|_{W^{k,p}(M_n,\mu)}$ is a norm for some sequence of compact sets $\{M_n\}$ growing to Ω. As to the class $\mathcal{C}$, it is a slight modification of $\mathcal{C}_0$, in which we consider measures $\mu = (\mu_0, \dots, \mu_k)$ such that by adding a minimal amount of deltas to μ_0 we obtain a measure in the class $\mathcal{C}_0$.

THEOREM 3.1. *Let us consider* $1 \le p \le \infty$, *an open set* $\Omega \subseteq \mathbb{R}$ *and a p-admissible vectorial measure* $\mu = (\mu_0, \dots, \mu_k)$ *in* $\overline{\Omega}$ *with* $(\overline{\Omega}, \mu) \in \mathcal{C}$. *Then the Sobolev space* $W^{k,p}(\overline{\Omega}, \mu)$ *is complete.*

The main ingredient of the proof of this result is Theorem 3.2. It allows us to control the L^∞ norm (in appropriate sets) of a function and its derivatives in terms of its Sobolev norm. It is also useful by its applications in the papers [15], [16], [17], [18], [1], [19] and [13]. Furthermore, it is important by itself, since it answers to the following main question: when the evaluation functional of f (or $f^{(j)}$) in a point is a bounded operator in $W^{k,p}(\overline{\Omega}, \mu)$?

THEOREM 3.2. *Let us consider* $1 \le p \le \infty$, *an open set* $\Omega \subseteq \mathbb{R}$ *and a p-admissible vectorial measure* μ *in* $\overline{\Omega}$. *If* K_j *is a finite union of compact intervals contained in* $\Omega^{(j)}$, *for* $0 \le j < k$, *then:*

(a) *If* $(\overline{\Omega}, \mu) \in \mathcal{C}_0$ *there exists a positive constant* $c_1 = c_1(K_0, \dots, K_{k-1})$ *such that*

$$c_1 \sum_{j=0}^{k-1} \|g^{(j)}\|_{L^\infty(K_j)} \le \|g\|_{W^{k,p}(\overline{\Omega},\mu)},$$

for all $g \in V^{k,p}(\overline{\Omega}, \mu)$.

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(b) *If* $\left(\overline{\Omega}, \mu\right) \in \mathcal{C}$ *there exists a positive constant* $c_2 = c_2(K_0, \ldots, K_{k-1})$ *such that for every* $g \in V^{k,p}(\overline{\Omega}, \mu)$*, there exists* $g_0 \in V^{k,p}(\overline{\Omega}, \mu)$*, independent of* $K_0, \ldots, K_{k-1}$ *and* c_2*, with*

$$\|g_0 - g\|_{W^{k,p}(\overline{\Omega},\mu)} = 0\,,$$

$$c_2 \sum_{j=0}^{k-1} \|g_0^{(j)}\|_{L^\infty(K_j)} \le \|g_0\|_{W^{k,p}(\overline{\Omega},\mu)} = \|g\|_{W^{k,p}(\overline{\Omega},\mu)}.$$

Furthermore, if g_0, f_0 *are these representatives of* g, f *respectively, we have for the same constant* c_2

$$c_2 \sum_{j=0}^{k-1} \|g_0^{(j)} - f_0^{(j)}\|_{L^\infty(K_j)} \le \|g - f\|_{W^{k,p}(\overline{\Omega},\mu)}.$$

This theorem has the following corollary, that we use in the proof of Theorem 3.1:

COROLLARY 3.3. *Let us consider,* $1 \le p \le \infty$*, an open set* $\Omega \subseteq \mathbb{R}$ *and a p-admissible vectorial measure* μ *in* $\overline{\Omega}$*. If* K_j *is a finite union of compact intervals contained in* $\Omega^{(j)}$*, for* $0 \le j < k$*, then:*

(a) *If* $(\overline{\Omega}, \mu) \in \mathcal{C}_0$ *there exists a positive constant* $c_1 = c_1(K_0, \ldots, K_{k-1})$ *such that,*

$$c_1 \sum_{j=0}^{k-1} \|g^{(j+1)}\|_{L^1(K_j)} \le \|g\|_{W^{k,p}(\overline{\Omega},\mu)}, \qquad \forall g \in V^{k,p}(\overline{\Omega}, \mu).$$

(b) *If* $\left(\overline{\Omega}, \mu\right) \in \mathcal{C}$ *there exists a positive constant* $c_2 = c_2(K_0, \ldots, K_{k-1})$ *such that for every* $g \in V^{k,p}(\overline{\Omega}, \mu)$*, there exists* $g_0 \in V^{k,p}(\overline{\Omega}, \mu)$ (*the same function as in Theorem* 3.2), *with*

$$\|g_0 - g\|_{W^{k,p}(\overline{\Omega},\mu)} = 0\,, \qquad c_2 \sum_{j=0}^{k-1} \|g_0^{(j+1)}\|_{L^1(K_j)} \le \|g_0\|_{W^{k,p}(\overline{\Omega},\mu)} = \|g\|_{W^{k,p}(\overline{\Omega},\mu)}.$$

Furthermore, if g_0, f_0 *are the representatives of* g, f *respectively, we have for the same constant* c_2

$$c_2 \sum_{j=0}^{k-1} \|g_0^{(j+1)} - f_0^{(j+1)}\|_{L^1(K_j)} \le \|g - f\|_{W^{k,p}(\overline{\Omega},\mu)}.$$

As a consequence of theorems 3.2 and 3.1, we can prove the density of the space of polynomials in these Sobolev spaces (see [15], [16], [18], [1] and [19]) and the boundedness of the multiplication operator (see [15], [17] and [1]).

Proof of Theorem 3.1: Let $\{f_n\}$ be a Cauchy sequence in $W^{k,p}(\overline{\Omega}, \mu)$. Then, for each $0 \le j \le k$, $\{f_n^{(j)}\}$ is a Cauchy sequence in $L^p(\overline{\Omega}, \mu_j)$ and it converges to a function $g_j \in L^p(\overline{\Omega}, \mu_j)$.

First of all, let us show that g_j can be extended to a function in $C(\Omega^{(j)})$ (if $0 \le j < k$) and in $L^1_{loc}(\Omega^{(j-1)})$ (if $0 < j \le k$).

If $0 \le j < k$, let us consider any compact interval $K \subseteq \Omega^{(j)}$. By part (b) of Theorem 3.2 we know there exists a representative (independent of K) of the class of $f_n \in W^{k,p}(\overline{\Omega}, \mu)$ (which we also denote by f_n) and a positive constant c such that for every $n, m \in \mathbb{N}$

$$c \, \| f_n^{(j)} - f_m^{(j)} \|_{L^\infty(K)} \le \sum_{i=0}^{k} \| f_n^{(i)} - f_m^{(i)} \|_{L^p(\overline{\Omega}, \mu_i)} \, .$$

As $\{f_n^{(j)}\} \subseteq C(K)$, there exists a function $h_j \in C(K)$ such that

$$c \, \| f_n^{(j)} - h_j \|_{L^\infty(K)} \le \sum_{i=0}^{k} \| f_n^{(i)} - g_i \|_{L^p(\overline{\Omega}, \mu_i)} \, .$$

Since we can take as K any compact interval contained in $\Omega^{(j)}$, we obtain that the function h_j can be extended to $\Omega^{(j)}$ and we have in fact $h_j \in C(\Omega^{(j)})$. It is obvious that $g_j = h_j$ in $\Omega^{(j)}$ (except for at most a set of zero μ_j-measure), since $f_n^{(j)}$ converges to g_j in the norm of $L^p(\overline{\Omega}, \mu_j)$ and to h_j uniformly on each compact interval $K \subseteq \Omega^{(j)}$. Therefore we can assume that $g_j \in C(\Omega^{(j)})$.

If $0 < j \le k$, let us consider any compact interval $J \subseteq \Omega^{(j-1)}$. Now, part (b) of Corollary 3.3 gives

$$c \, \| f_n^{(j)} - f_m^{(j)} \|_{L^1(J)} \le \sum_{i=0}^{k} \| f_n^{(i)} - f_m^{(i)} \|_{L^p(\overline{\Omega}, \mu_i)} \, .$$

As $\{f_n^{(j)}\} \subseteq L^1(J)$, there exists a function $u_j \in L^1(J)$ such that

$$c \, \| f_n^{(j)} - u_j \|_{L^1(J)} \le \sum_{i=0}^{k} \| f_n^{(i)} - g_i \|_{L^p(\overline{\Omega}, \mu_i)} \, .$$

Since we can take as J any compact interval contained in $\Omega^{(j-1)}$, we obtain that the function u_j can be extended to $\Omega^{(j-1)}$ and we have in fact $u_j \in L^1_{loc}(\Omega^{(j-1)})$. It is obvious that $g_j = u_j$ in $\Omega^{(j)}$ (except for at most a set of zero Lebesgue measure), since $f_n^{(j)}$ converges to u_j in $L^1_{loc}(\Omega^{(j)})$ and to g_j locally uniformly in $\Omega^{(j)}$. Let us consider a set A which concentrates the mass of $(\mu_j)_s$, with $|A| = 0$; we can take $u_j = g_j$ in A. We only need to show $u_j = g_j$ in $\Omega_j \setminus (\Omega^{(j)} \cup A)$ (recall that by hypothesis $w_j = 0$ in $\mathbb{R} \setminus \Omega_j$), but this is immediate since $w_j \in B_p(\Omega_j)$ and the convergence in $L^p(\Omega_j, w_j)$ implies the convergence in $L^1_{loc}(\Omega_j)$. Therefore we can assume that $g_j \in L^1_{loc}(\Omega^{(j-1)})$.

In fact, we have seen that $\{f_n^{(j)}\}$ converges to g_j in $L^\infty_{loc}(\Omega^{(j)})$ (if $0 \le j < k$) and in $L^1_{loc}(\Omega^{(j-1)})$ (if $0 < j \le k$).

Let us see now that $g_j' = g_{j+1}$ in the interior of $\Omega^{(j)}$ for $0 \le j < k$. Let us consider a connected component I of $\operatorname{int}(\Omega^{(j)})$. Given $\varphi \in C_c^\infty(I)$, let us consider the convex hull K of supp φ. We have that K is a compact interval contained in $I \subseteq \Omega^{(j)}$. The uniform convergence of $\{f_n^{(j)}\}$ in K and the L^1 convergence of $\{f_n^{(j+1)}\}$ in K gives that

$$\int_K \varphi' \, g_j = \lim_{n\to\infty} \int_K \varphi' \, f_n^{(j)} = - \lim_{n\to\infty} \int_K \varphi \, f_n^{(j+1)} = - \int_K \varphi \, g_{j+1} \, .$$

Then $g_{j+1} = g_0^{(j+1)}$ in $\operatorname{int}(\Omega^{(j)})$ and $g_0^{(j)} \in AC_{loc}(\operatorname{int}(\Omega^{(j)}))$ for $0 \le j < k$. In order to see that $g_0^{(j)} \in AC_{loc}(\Omega^{(j)})$, it is enough to recall that $(g_0^{(j)})' = g_{j+1} \in L^1_{loc}(\Omega^{(j)})$. $\square$

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