Electronic Transactions on Numerical Analysis.
Volume 24, pp. 66-73, 2006.

ISSN 1068-9613.

ETNA
Kent State University
etna@mcs.kent.edu

ANOTHER APPROACH TO VIBRATION ANALYSIS OF STEPPED STRUCTURES*

IGOR FEDOTOV†, STEVE JOUBERT†, JULIAN MARAIS,† AND MICHAEL SHATALOV‡

Abstract. In this paper a model of an N–stepped bar with variable Cross-sections coupled with foundation by means of lumped masses and springs is studied. It is assumed that the process of vibrations in each section of the bar is described by a wave equation. The analytical tools of vibration analysis are based on finding eigenfunctions with piecewise continuous derivatives, which are orthogonal with respect to a generalized weight function. These eigenfunctions automatically satisfy the boundary conditions at the end points as well as the non-classical boundary conditions at the junctions. The solution of the problems is formulated in terms of Green function. By means of the proposed algorithm a problem of arbitrary complexity could be considered in the same terms as a single homogeneous bar. This algorithm is efficient in design of low frequency transducers. An example is given to show the practical application of the algorithm to a two-stepped transducer.

Key words. PDE with discontinuous coefficients, numerical approximation of eigenvalues, stepped structure, transducers, waveguide, variable cross-section, non-classical boundary conditions, Green function, resonance

AMS subject classifications. 35B34, 35R05, 34B27, 34L16

1. Introduction.

Classical models of stepped bar vibrations are based on solution of the wave equation. Typical models include low-frequency underwater transducers, acoustic horns and waveguides, stepped shafts and electromagnetic waveguides with variable cross-sections. Conventional approaches mainly consist of formulating an equivalent electric circuit (Mason, Redwood, KLM equivalent circuits [1]) or on state flow models [2], for individual segments with their subsequent integration into the system [3].

Despite of the advantages of these approaches it is necessary to stress the lack of physical clarity in interpretation of non-classical boundary conditions. These models were substantially based on the fact that each section is described by the wave equation. However, these models cannot be easily generalized to more sophisticated, which make use of more complicated equations models such as Rayleigh and Bishop models [4], whereas our model can be generalized easily.

The purpose of the paper is to formulate an algorithm based on the analysis of an N–stepped bar in terms of a single homogeneous bar. To this end the system of boundary conditions was derived, which has an obvious physical interpretation and could be automatically generalized for a stepped bar with variable cross sections of an arbitrary complexity.

The eigenvalues of this system are determined by solution of a transcendential equation. The next step consists of derivation of a system of eigenfunctions corresponding to the eigenvalues, which automatically satisfy the boundary conditions and are orthogonal with a generalized weight function.

The closed form solution of this system is derived using Green function and an example of application of the above-mentioned algorithms, is considered for the case of a two-stepped bar with cylindrical and conical cross-sections.

2. Model of stepped bar with variable cross-sections governing equations and boundary conditions.

Let us consider an N–stepped bar with variable cross-sections (Fig. 2.1) coupled with foundation by lumped masses and springs. Suppose that the length of each

*Received January 7, 2005. Accepted for publication November 17, 2005. Recommended by J. Arvesú.
†Department of Mathematical Technology, P.B.X680, Pretoria 0001 FIN-40014 Tshwane University of Technology, South Africa (igor@techpta.ac.za).
‡CSIR Manufacturing and Materials P.O. Box 395, Pretoria 0001, CSIR, South Africa and Department of Mathematical Technology P.B.X680, Pretoria 0001 FIN-40014 Tshwane University of Technology, South Africa (mshatlov@csir.co.za).

ETNA
Kent State University
etna@mcs.kent.edu

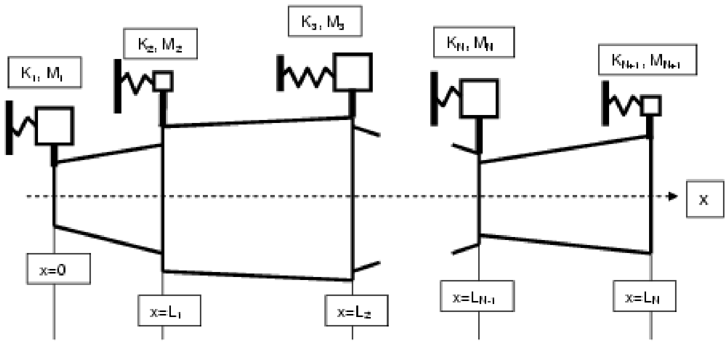

FIG. 2.1.

section of the bar is much greater than the linear dimension of its cross-section. In this case the elementary theory of vibration of bars, based on wave equations, describes a longitudinal motion of the stepped bar.

Suppose that ρ_j – mass density, E_j – modulus of elasticity, $A_j(x)$ – area of cross-section of jth section. By means of lumped masses (M_{j+1}) and springs with stiffness K_{j+1} that are located at junctions of jth and $(j+1)$th sections the bar is attached to an immovable foundation.

Equations of motion could be obtained as:

$$\rho_j A_j(x)\frac{\partial^2 u_j}{\partial t^2} - E_j\frac{\partial}{\partial x}\left(A_j(x)\frac{\partial u_j}{\partial x}\right) = f_j(t,x),\ x \in [L_{j-1}, L_j],\ (j = 1, 2, \ldots, N) \tag{2.1}$$

with the following system of boundary conditions:

$$x = 0: \quad E_1 A_1(0)u_1'(0,t) - \left[M_1\ddot{u}_1(0,t) + K_1 u_1(0,t)\right] = 0;$$

$$\begin{aligned} x = L_j: \quad & u_j(L_j,t) - u_{j+1}(L_j,t) = 0; \qquad (j = 1, 2, \ldots, N-1) \\ & E_j A_j(L_j)u_j'(L_j,t) - E_{j+1}A_{j+1}(L_j)u_{j+1}'(L_j,t) \\ & \quad + \left[M_{j+1}\ddot{u}_{j+1}(L_j,t) + K_{j+1}u_{j+1}(L_j,t)\right] = 0; \end{aligned} \tag{2.2}$$

$$\begin{aligned} x = L_N: \quad & E_N A_N(L_N)u_N'(L_N,t) \\ & \quad + \left[M_{N+1}\ddot{u}_N(L_N,t) + K_{N+1}u_N(L_N,t)\right] = 0; \end{aligned}$$

and initial conditions:

$$t = 0: \ u_j(x,0) = g_j(x); \quad \dot{u}_j(x,0) = h_j(x) \quad (j = 1, 2, \ldots, N). \tag{2.3}$$

Let us introduce the following notation for system (2.1):

$$\rho(x)A(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left[E(x)A(x)\frac{\partial u}{\partial x}\right] = F(x,t), \quad x \in [L_0, L_N]. \tag{2.4}$$

In this equation we suppose that

$$u = u(x,t) = \sum_{j=1}^{N} u_j(x,t)\cdot\Delta H_j(x) = \sum_{j=1}^{N} X_j(x)\cdot\Delta H_j(x)\cdot\exp(i\omega t)\ (i^2 = -1);$$

ETNA
Kent State University
etna@mcs.kent.edu

$$\Delta H_j(x) = H(x - L_{j-1}) - H(x - L_j);$$

$$H(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{(Heaviside function);}$$

$$\rho(x) = \sum_{j=1}^{N} \rho_j(x) \cdot \Delta H_j(x); \qquad E(x) = \sum_{j=1}^{N} E_j(x) \cdot \Delta H_j(x);$$

$$A(x) = \sum_{j=1}^{N} A_j(x) \cdot \Delta H_j(x); \quad f(t,x) = \sum_{j=1}^{N} f_j(t,x) \cdot \Delta H_j(x).$$

We have to solve (2.4), satisfying boundary conditions (2.2) and initial conditions (2.3), written as:

$$u(x,0) = g(x) = \sum_{j=1}^{N} g_j(x) \cdot \Delta H_j; \qquad \dot{u}(x,0) = h(x) = \sum_{j=1}^{N} h_j(x) \cdot \Delta H_j.$$

To solve problem (2.4) – (2.2) – (2.3) the Fourier method is used. For each section we represent solution (2.4) as the sum of $X_j(x) \cdot \exp(i\omega t)$, where ω – eigenvalue and $X_j(x)$ – eigenfunctions of the problem, satisfying the equation:

$$X_j'' + \frac{A_j'(x)}{A_j(x)} X_j' + \frac{\omega^2}{c_j^2} X_j = 0, \quad \left(c_j = \sqrt{\frac{E_j}{\rho_j}} \right) \tag{2.5}$$

and satisfies boundary conditions for jth section. Let us suppose that $A_j(x) = d_j \cdot (x - l_j)^{n_j}$, where n_j – real number. In this case the eigenfunction of jth section is

$$X_j(x) = a_j \frac{J_{\frac{n_j-1}{2}} \left[\frac{\omega}{c_j} \cdot (x - l_j) \right]}{(x - l_j)^{\frac{n_j-1}{2}}} + b_j \frac{Y_{\frac{n_j-1}{2}} \left[\frac{\omega}{c_j} \cdot (x - l_j) \right]}{(x - l_j)^{\frac{n_j-1}{2}}}, \tag{2.6}$$

where $J_{\frac{n_j-1}{2}} \left[\frac{\omega}{c_j} \cdot (x - l_j) \right]$, $Y_{\frac{n_j-1}{2}} \left[\frac{\omega}{c_j} \cdot (x - l_j) \right]$ – Bessel functions of the first and second kinds of order $\frac{n_j-1}{2}$.

In the particular case of cylindrical cross-section ($n_j = 0 \Rightarrow A_j(x) = A_j = const$) the solution (2.6) could be represented as:

$$X_j(x) = a_j \cos \left[\frac{\omega}{c_j} \cdot (x - L_{j-1}) \right] + b_j \sin \left[\frac{\omega}{c_j} \cdot (x - L_{j-1}) \right],$$

where L_{j-1} – coordinate of junction of "$j-1$" and "j" – sections.

For a conical cross-section ($n_j = 2 \Rightarrow A_j(x) = d_j^2 (x - l_j)^2$, where l_j – vertex of jth – cone) solution (2.6) could be transformed into

$$X_j(x) = a_j \frac{\cos \left[\frac{\omega}{c_j} \cdot (x - l_j) \right]}{(x - l_j)} + b_j \frac{\sin \left[\frac{\omega}{c_j} \cdot (x - l_j) \right]}{(x - l_j)}. \tag{2.7}$$

Note that the cross-section of arbitrary complexity could be approximated by conical sections with eigensolutions (2.7) and hence, the methods developed in present report are applicable for approximate analysis of steeped bars with arbitrary cross-sections.

ETNA
Kent State University
etna@mcs.kent.edu

The eigenvalues can be obtained by substitution of $X_j(x)\exp(i\omega t)$ in (2.2) and the characteristic system could be obtained. It is necessary to keep in mind that:

$$X_j = a_j X_{1j}(x) + b_j X_{2j}(x),$$

$$X_{1j}(x) = \begin{cases} \cos\left[\frac{\omega}{c_j}\cdot(x - L_{j-1})\right], & \text{if } n_j = 0 \\ \frac{\cos\left[\frac{\omega}{c_j}\cdot(x-l_j)\right]}{(x-l_j)}, & \text{if } n_j = 2 \\ \frac{J_{\frac{n_j-1}{2}}\left[\frac{\omega}{c_j}\cdot(x-l_j)\right]}{(x-l_j)^{\frac{n_j-1}{2}}}, & \text{if } n_j \text{ – arbitrary real number} \end{cases}$$

$$X_{2j}(x) = \begin{cases} \sin\left[\frac{\omega}{c_j}\cdot(x - L_{j-1})\right], & \text{if } n_j = 0 \\ \frac{\sin\left[\frac{\omega}{c_j}\cdot(x-l_j)\right]}{(x-l_j)}, & \text{if } n_j = 2 \\ \frac{Y_{\frac{n_j-1}{2}}\left[\frac{\omega}{c_j}\cdot(x-l_j)\right]}{(x-l_j)^{\frac{n_j-1}{2}}}, & \text{if } n_j \text{ – arbitrary real number} \end{cases}$$

and

$$X'_{1j}(x) = \begin{cases} -\left(\frac{\omega}{c_j}\right)\sin\left[\frac{\omega}{c_j}\cdot(x - L_{j-1})\right], & \text{if } n_j = 0 \\ -\left(\frac{\omega}{c_j}\right)\frac{\sin\left[\frac{\omega}{c_j}\cdot(x-l_j)\right]}{(x-l_j)} - \frac{\cos\left[\frac{\omega}{c_j}\cdot(x-l_j)\right]}{(x-l_j)^2} & \text{if } n_j = 2 \\ -\left(\frac{\omega}{c_j}\right)\frac{J_{\frac{n_j+1}{2}}\left[\frac{\omega}{c_j}\cdot(x-l_j)\right]}{(x-l_j)^{\frac{n_j-1}{2}}}, & \text{if } n_j \text{ – arbitrary real number} \end{cases}$$

$$X'_{2j}(x) = \begin{cases} \left(\frac{\omega}{c_j}\right)\cos\left[\frac{\omega}{c_j}\cdot(x - L_{j-1})\right], & \text{if } n_j = 0 \\ \left(\frac{\omega}{c_i}\right)\frac{\cos\left[\frac{\omega}{c_j}\cdot(x-l_j)\right]}{(x-l_j)} - \frac{\sin\left[\frac{\omega}{c_j}\cdot(x-l_j)\right]}{(x-l_j)^2} & \text{if } n_j = 2 \quad (j = 1, 2, \ldots, N-1) \\ -\left(\frac{\omega}{c_j}\right)\frac{Y_{\frac{n_i+1}{2}}\left[\frac{\omega}{c_j}\cdot(x-l_j)\right]}{(x-l_j)^{\frac{n_j-1}{2}}}, & \text{if } n_j \text{ – arbitrary real number.} \end{cases}$$

The resulting system has non-trivial solution if and only if the main determinant of the system is equal to zero:

$$D(\omega) = 0. \tag{2.10}$$

ETNA
Kent State University
etna@mcs.kent.edu

This is the system of transcendential equations, which have enumerable solutions – eigenvalues ω_k $(k = 1, 2, \ldots)$. Let us find eigenfunctions $\varphi_k(x)$, corresponding to each eigenvalue ω_k :

$$\varphi_k(x) = \sum_{j=1}^{N} \left\{ a_j^{(k)} \frac{J_{\frac{n_j-1}{2}}\left[\frac{\omega_k}{c_j}\cdot(x-l_j)\right]}{(x-l_j)^{\frac{n_j-1}{2}}} + b_j^{(k)} \frac{Y_{\frac{n_j-1}{2}}\left[\frac{\omega_k}{c_j}\cdot(x-l_j)\right]}{(x-l_j)^{\frac{n_j-1}{2}}} \right\} \cdot \Delta H_j(x),$$

where $a_1^{(k)} = 1$ and $b_1^{(k)}, a_j^{(k)}, b_j^{(k)},\ (j = 2, \ldots, N)$ are derived from the first $(2N-1)$ equations of system (2.10) for every particular eigenvalue ω_k.

It is possible to show that these eigenfunctions are orthogonal on $[L_0, L_N]$ with weight

$$w(x) = \sum_{j=1}^{N} \Big[\rho_j A_j(x) \Delta H_j(x) + M_j \cdot \delta(x - L_{j-1})\Big] + M_{N+1} \cdot \delta(x - L_N),$$

where $\delta(x)$ – Dirac delta function.

3. Solution of the problem. Transforming equation (2.4) as follows:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{1}{A(x)}\frac{\partial A(x)}{\partial x}\frac{\partial u}{\partial x}\right] = \frac{F(x,t)}{\rho A(x)} \equiv f(x,t), \ \left(c^2 = \frac{E}{\rho}\right) \tag{3.1}$$

one can obtain solution of the problem as follows:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \cdot \varphi_k(x), \tag{3.2}$$

where $u_k(t)$ – unknown functions. The force of excitation in the right hand side of equation (3.1) is

$$f(x,t) = \sum_{k=1}^{\infty} f_k(t) \cdot \varphi_k(x), \tag{3.3}$$

where

$$f_k(t) = \frac{1}{\|\varphi_k\|^2} \cdot \int_0^{L_N} x(y) f(y,t) \varphi_k(y) dy.$$

After substitution of (3.2) – (3.3) into (3.1) we obtain:

$$\ddot{u}_k + \omega_k^2 u_k = f_k(t) \tag{3.4}$$

with initial conditions (at $t = 0$):

$$u_k(0) = \frac{1}{\|\varphi_k\|^2} \cdot \int_0^{L_N} w(y) g(y) \varphi_k(y) dy; \quad \dot{u}_k(0) = \frac{1}{\|\varphi_k\|^2} \cdot \int_0^{L_N} w(y) h(y) \varphi_k(y) dy \tag{3.5}$$

where $u(x,0) = g(x) = \sum_{k=1}^{\infty} u_k(0) \cdot \varphi_k(x)$; $\dot{u}(x,0) = h(x) = \sum_{k=1}^{\infty} \dot{u}_k(0) \cdot \varphi_k(x)$.

The solution of the problem (3.4) – (3.5) is

$$u_k(t) = u_k(0)\cdot\cos(\omega_k t) + \frac{1}{\omega_k}\dot{u}_k(0)\cdot\sin(\omega_k t) + \frac{1}{\omega_k}\cdot\int\limits_0^t f_k(\tau)\cdot\sin\left[\omega_k(t-\tau)\right]d\tau$$

$$(3.6)\qquad = \frac{1}{||\varphi_k||^2}\int\limits_0^{L_N} w(y)g(y)\varphi_k(y)\cos(\omega_k t)dy + \frac{1}{||\varphi_k||^2}\int\limits_0^{L_N} w(y)h(y)\varphi_k(y)\frac{\sin(\omega_k t)}{\omega_k}dy$$
$$+\frac{1}{\omega_k||\varphi_k||^2}\int\limits_0^t\int\limits_0^{L_N} w(y)\varphi_k(y)f(y,\tau)\sin\left[\omega_k(t-\tau)\right]dy\,d\tau,$$

where the square of the norm $||\varphi_k||^2$ is

$$||\varphi_k||^2 = \int\limits_0^{L_N} w(x)\varphi_k^2(x)dx.$$

Substituting (3.6) into (3.2) yields

$$u(x,t) = \int\limits_0^{L_N}\frac{\partial G(x,y,t)}{\partial t}w(y)g(y)dy + \int\limits_0^{L_N} G(x,y,t)w(y)h(y)dy$$
$$+\int\limits_0^t\int\limits_0^{L_N} G(x,y,t-\tau)f(y,\tau)dy\,d\tau,$$

where the Green function $G(x,y,t)$ is:

$$G(x,y,t) = \sum_{k=1}^{\infty}\frac{1}{||\varphi_k||^2}\varphi_k(x)w(y)\varphi_k(y)\frac{\sin\omega_k t}{\omega_k}.$$

4. Example. Let us consider a two-stepped bar, consisting of cylinder and cone (Fig. 4.1) with lumped mass (M = 0.2 kg) between the sections, attached to an immovable foundation by a lumped spring (K = 10^{10} Nm^{-1}) that is attached to the lumped mass.

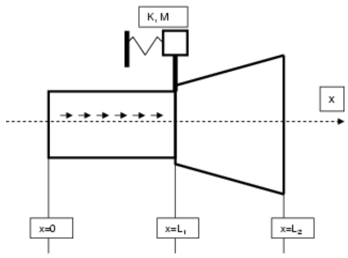

FIG. 4.1.

A time dependent force excites the cylindrical section of this bar

$$(4.1)\qquad f(t) = \left|\begin{array}{ll}\sin\left\{\left[(\omega_1-\Delta\omega)+\dfrac{2\cdot\Delta\omega}{T}t\right]\cdot t\right\} & \text{if } 0\le t<T\\ 0 & \text{otherwise.}\end{array}\right.$$

TABLE 4.1

Number of section	1	2
Mass density (kg/m^3)	7850	2700
Modulus of elasticity (GPa)	200	70
Length (m)	0.159	0.141
Radius (m)	0.04	0.055 - 0.075

where $\Delta\Omega = 0.1 \cdot \omega_1$; $T = 2.10^{-3}$ sec. that is a finite almost periodic wave packet in the vicinity of a first longitudinal resonance of the stepped bar (see below).

Suppose that mass densities, modules of elasticity, lengths and radii of the sections are:

In this case the first five eigenvalues of the bar, defined by (2.5) – (2.6) are as follows: ${}^{\omega_0}/_{2\pi} \approx 4102$ Hz, ${}^{\omega_1}/_{2\pi} \approx 7933$ Hz, ${}^{\omega_2}/_{2\pi} \approx 17760$ Hz, ${}^{\omega_3}/_{2\pi} \approx 25520$ Hz, ${}^{\omega_4}/_{2\pi} \approx 33470$ Hz.

First resonance is a pendulum mode at which the stepped bar vibrates as “rigid” pendulum. All other modes are “longitudinal”. Geometry of the bar was specially chosen so that vibration amplitude of the junction between the cylindrical and conical sections is small comparison to end amplitudes at the second resonance frequency. This is especially important for transducers, because it is possible to use this junction for suspension of the transducer in the housing. In this case the mechanical energy of the transducer does not leak to the housing. Corresponding eigenfunctions for the first five eigenvalues are shown in Fig. 4.2.

The wave packet (4.1), is shown in Fig. 4.3.

Initial conditions are supposed to be zero. Motion of the planes $x = L_1$ and $x = L_2 = L$ are shown in Fig. 4.4.

One can see that as the excitation frequency approaches the resonance frequency ω_1, the amplitude of oscillations increases (approximately) linearly. After it has passed through the resonance frequency at $t = 0.001$ s, the amplitude of oscillations decreases. The presence of additional harmonics in the solution then results in beats, which are evident in Fig. 4.4 for $0.001 < t < 0.002$ s. At $t = 0.002$ s, $f(t, x) = 0$ and no additional energy is added to the system. The beats are mainly result of the first and second harmonic superposition. The oscillations continue until infinity since damping has been neglected.

5. Conclusion. It was shown that vibrations of stepped bar, with various cross-sections coupled with foundation by means of lumped masses and springs, could be considered in terms of piecewise continuous eigenfunctions, which are orthogonal with respect to the generalized weight and automatically satisfy the boundary conditions at the ends as well as the non-classical boundary conditions at the junctions. The algorithm of calculation was formulated, which is applicable to stepped bars of arbitrary complexity. This algorithm could easily be generalized for more sophisticated models of bars, which are not described by the wave equation. An example of application of the algorithm was considered for the case of a near resonance excitation.

REFERENCES

[1] G. S. KINO, *Acoustic Waves: Devices, Imaging and Analog Signal Processing*, Prentice Hall, 1987.
[2] W.-J. HSUEH, *Free and forced vibrations of stepped rods and coupled systems*, J. Sound Vibration, 226 (1999), pp. 891–904.
[3] C. N. BAPAT AND N. BHUTANI, *General approach for free and forced vibrations of stepped systems governed by the one-dimensional wave equations with non-classical boundary conditions*, J. Sound Vibration, 172 (1994), pp. 1–22.
[4] J. S. RAO, *Advanced Theory of Vibration*, John Wiley & Sons, 1992.

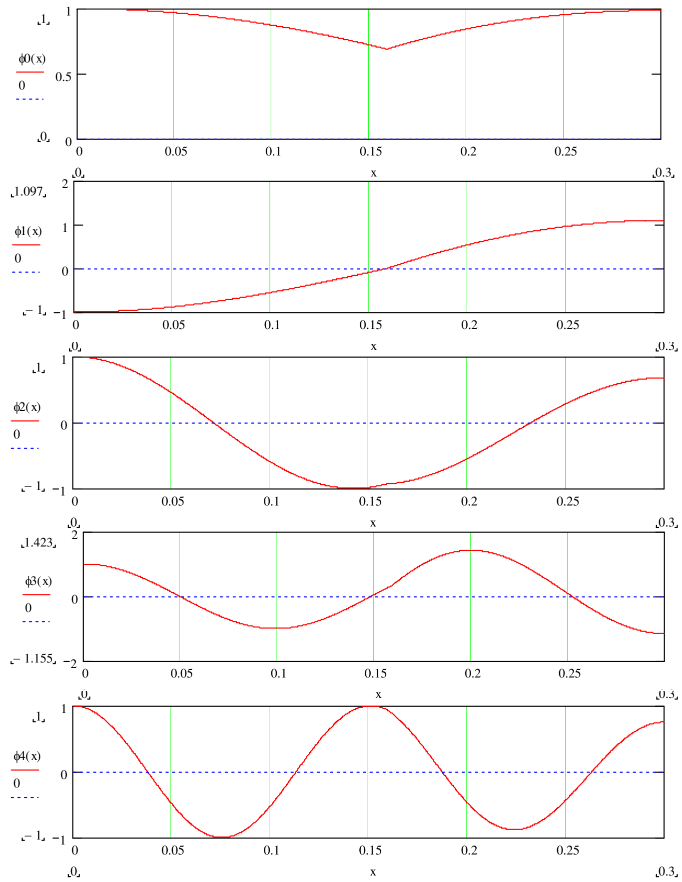

FIG. 4.2.

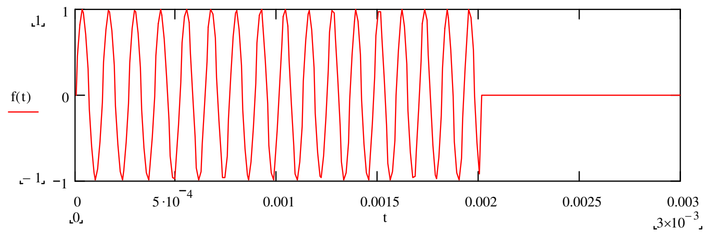

FIG. 4.3.

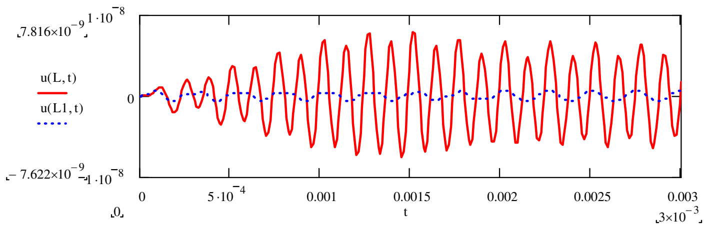

FIG. 4.4.