The Electronic Journal of Linear Algebra.
A publication of the International Linear Algebra Society.
Volume 7, pp. 92-99, September 2000.
ISSN 1081-3810. http://math.technion.ac.il/iic/ela

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ON MATRICES WITH ALL MINORS NEGATIVE*

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Abstract. A matrix is called sign regular of order k if every minor of order i has the same sign for each $i = 1, 2, \ldots, k$. At one extreme of the sign regular matrices lies the well studied and important class of totally positive matrices. The purpose here is to initiate a study of the other extreme of sign regular matrices, namely the totally negative matrices (i.e., all minors negative). Many aspects of this class are considered including: existence, spectral properties, inverses, Schur complements, and factorizations.

Key words. Minors, Sign regular matrices, Totally positive, Totally negative.

AMS subject classifications. 15A48

1. Introduction. Let A be an $m \times n$ real matrix, $\alpha \subseteq \{1, \ldots, m\}$, $\beta \subseteq \{1, \ldots, n\}$ be index sets, and α^c denote the complement of α. Let $A[\alpha|\beta]$ denote the submatrix of A lying in rows indexed by α and columns indexed by β, with the principal submatrix $A[\alpha|\alpha]$ abbreviated to $A[\alpha]$. For $1 \leq k \leq \min\{m, n\}$, the k^{th} compound matrix A_k of A is the $\binom{m}{k} \times \binom{n}{k}$ matrix with α, β entry equal to $\det A[\alpha|\beta]$ where $|\alpha| = |\beta| = k$ and the index sets are ordered lexicographically. Let $1 \leq k \leq \min\{m, n\}$ and fix a k-vector of signs $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$ with $\epsilon_j \in \{\pm 1\}$; such a vector is called a *signature.* As in [1, (2.3)], [12, p. 12], an $m \times n$ matrix A is *(strictly) sign regular of order k with signature ϵ* if for each $p = 1, \ldots, k$, $\epsilon_p A_p$ is entrywise (positive) non-negative; see also [8], [9], [15]. When $k = \min\{m, n\}$, a (strictly) sign regular matrix of order k is referred to as simply a (strictly) sign regular matrix. Note that (strictly) sign regular matrices are called (strictly) fixed-sign matrices in [7, Chapter V].

In the case that $k = \min\{m, n\}$ and $\epsilon_j = 1$, for $j = 1, 2, \ldots, k$, then A is called *(strictly) totally positive* (usually abbreviated (STP) TP). By contrast, we focus on the case that $k = \min\{m, n\}$ and $\epsilon_j = -1$, for $j = 1, 2, \ldots, k$; and we call such a (strictly) sign regular matrix *(totally negative) totally non-positive*, which we abbreviate to (t.n.) t.n.p. Equivalently A is a (t.n.) t.n.p. matrix if the determinant of all its minors of all orders, including 1, are negative (non-positive). In [9, p. 138] square t.n. matrices are defined (they are referred to as strictly totally negative) and a characterization is given in terms of the parameters obtained from a Neville elimination.

For $m = n = k$, if the requirement of (negativity) non-positivity is placed only on the principal minors, then A is called (partially negative, p.n.) partially non-positive, p.n.p., by Johnson [11]. A p.n. matrix is called an N-matrix in economic models; see Bapat and Raghavan [2, p. 298], [14], and [16, Section 6], where p.n. matrices

*Received by the editors on 16 June 2000. Accepted for publication on 24 August 2000. Handling editor: Daniel Hershkowitz

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arise in conjunction with Lemke's algorithm for solving linear and convex quadratic programming problems. Evidently, for fixed n, {t.n. matrices} $\subseteq$ {t.n.p. matrices} $\subseteq$ {p.n.p. matrices}, and {t.n. matrices} $\subseteq$ {p.n. matrices}.

Whereas TP and STP matrices have been studied extensively (see, e.g., [1], [7] and references therein) and are used in such applications as economics, we know of no study of t.n. or t.n.p. matrices other than that in [9]. We begin such a study here, and focus mainly on t.n. matrices. In Section 2, we give some basic results, including spectral properties, in Section 3 we provide an algorithmic way of generating t.n. matrices and in Section 4 we prove results on triangular factorizations of t.n. matrices. Finally, in Section 5, we state some open problems and avenues for future research related to t.n. matrices.

2. Basic Results. If A is a 2×2 t.n. matrix, then $-PA$ is a STP matrix, where P is the permutation matrix that reverses the rows. But for $n \geq 3$ there is no such easy relation. The Cauchy-Binet formula can be used to prove the following product results for matrices; see [1, Thm. 3.1].

PROPOSITION 2.1. *Let A be an $m \times n$ (t.n.) t.n.p. matrix and let B and C be $n \times p$ t.n. and STP matrices, respectively. Then AB is (strictly) sign regular of order $\min\{m,n,p\}$ with signature $\epsilon = (1,1,\ldots,1)$, and AC is (strictly) sign regular of order $\min\{m,n,p\}$ with signature $\epsilon = (-1,-1,\ldots,-1)$. In particular, if $m = n = p$, then AB is a (STP) TP matrix and AC is a (t.n.) t.n.p. matrix.*

It is well known that Fischer's inequality holds for TP matrices (see [1, (3.6)], [13]), and because of the signs it follows trivially with strict inequality for t.n. matrices.

PROPOSITION 2.2. *(Fischer's Inequality) Let $\alpha, \beta \subseteq \{1,\ldots,\min\{m,n\}\}$ with $\alpha \cap \beta = \phi$, If A is an $m \times n$ t.n.p. matrix, then*

$$\det A[\alpha \cup \beta] \leq \det A[\alpha] \ \det A[\beta],$$

with strict inequality holding if A is a t.n. matrix.

In the case $m = n$, the following result identifies some spectral properties of t.n. matrices.

THEOREM 2.3. *If A is an $n \times n$ t.n. matrix, then all the eigenvalues of A are real and distinct. Exactly one eigenvalue is negative, and this eigenvalue is simple, has the largest modulus and has an eigenvector with all entries positive.*

Proof. By considering the characteristic equation (which uses sums of only the principal minors), [11, Thm. 2] shows that if A is a p.n. matrix, then A has exactly one negative eigenvalue, with every other eigenvalue having no larger modulus. This result thus holds for t.n. matrices, and furthermore, by Perron Frobenius (since $-A$ is entrywise positive), this eigenvalue has modulus greater than that of every other eigenvalue, and its eigenvector can be taken entrywise positive. The fact that all eigenvalues of A are real and distinct follows from [1, Thm. 6.2], where this is proved for any square strictly sign regular matrix. □

By continuity it follows from Theorem 2.3, that if A is an $n\times n$ t.n.p. matrix, then the eigenvalues of A are real and at most one is negative. See [1, Section 6] for other observations including results on the eigenvectors of square sign regular matrices, and [4] where the distribution of eigenvalues of sign regular matrices is considered (although there appears to be an error in [4, Thm. 4] regarding the sign of the p^{th} eigenvalue of a sign regular matrix).

Let S be the $n \times n$ diagonal matrix with (i,i) entry equal to $(-1)^{i-1}$ for $i = 1, 2, \ldots, n$, thus $S = \text{diag}(1, -1, \ldots, \pm 1)$. It is well-known (see [1, Thm. 3.3(c)]) that if A is STP (TP, invertible), then $SA^{-1}S$ is STP (TP, invertible). If A is an $n \times n$ t.n. matrix, then A is nonsingular, and the sign of the (i,j) entry of A^{-1} is $(-1)^{i+j}$, i.e., A^{-1} has a "checkerboard pattern". However, a signature similarity of A^{-1} is sign regular, as the following result shows.

THEOREM 2.4. *Let A be an $n \times n$ t.n. matrix. Then $SA^{-1}S$ is strictly sign regular with signature $\epsilon = (1, \ldots, 1, -1)$.*

Proof. Since $\det A < 0$, it follows that $\det SA^{-1}S < 0$. Using Jacobi's identity as in [1, (1.32)],

$$\det SA^{-1}S[\alpha|\beta] = \frac{\det A[\beta^c|\alpha^c]}{\det A} > 0,$$

since A is t.n. and α and β are proper subsets of $\{1, 2, \ldots, n\}$. $\square$

By using an argument similar to that in the proof of Theorem 2.4, we have the following result, which is used in our first procedure for generating t.n. matrices.

REMARK 2.5. *If A is strictly sign regular with signature $\epsilon = (1, \ldots, 1, -1)$, then $SA^{-1}S$ is t.n.*

We also note that a result analogous to Theorem 2.4 holds for t.n.p. matrices; namely, if A is t.n.p. and invertible, then $SA^{-1}S$ is sign regular with signature $\epsilon = (1, 1 \ldots, 1, -1)$.

For an $n \times n$ matrix A with $A[\alpha]$ invertible, the *Schur complement of $A[\alpha]$ in A*, denoted by $A/A[\alpha]$, is defined as

$$A/A[\alpha] = A[\alpha^c] - A[\alpha^c|\alpha](A[\alpha])^{-1}A[\alpha|\alpha^c].$$

It is well-known (see [1, (1.29)]) that if A is invertible, then the inverse of $A/A[\alpha]$ is equal to $A^{-1}[\alpha^c]$. We now present a result about the Schur complement of a t.n. matrix.

THEOREM 2.6. *Let A be an $n \times n$ t.n. matrix and $\alpha \subset \{1, 2, \ldots, n\}$. Then the Schur complement $A/A[\alpha]$ is similar to a STP matrix via a diagonal matrix with diagonal entries ± 1.*

Proof. From the previous remarks $A/A[\alpha] = (A^{-1}[\alpha^c])^{-1}$. Since A is t.n., the result of Theorem 2.4 implies that $A^{-1} = SBS$, where $S = \text{diag}(1, -1, \cdots, \pm 1)$ and B is strictly sign regular with signature $\epsilon = (1, 1, \ldots, 1, -1)$. Then $A/A[\alpha] = (SBS[\alpha^c])^{-1} = (S[\alpha^c]B[\alpha^c]S[\alpha^c])^{-1} = S[\alpha^c](B[\alpha^c])^{-1}S[\alpha^c]$. The signature of B implies that $B[\alpha^c]$ is STP (as α is nonempty), and hence by [1, Thm. 3.3], $(B[\alpha^c])^{-1} = S'CS'$, where $S' = \text{diag}(1, -1, \ldots, \pm 1)$ and C is a STP matrix. Therefore

$$A/A[\alpha] = S[\alpha^c]S'CS'S[\alpha^c] = S''CS'',$$

where $S'' = S[\alpha^c]S'$ is a diagonal matrix with diagonal entries ± 1. $\square$

The next observation follows immediately from the proof of the previous result.

COROLLARY 2.7. *Let A be an $n \times n$ t.n. matrix and let $\alpha \subset \{1, 2, \ldots, n\}$. If α^c is an index set based on consecutive indices, then $A/A[\alpha]$ is a totally positive matrix.*

Proof. Observe from the proof of Theorem 2.6 that

$$A/A[\alpha] = (S[\alpha^c]S')C(S'S[\alpha^c]),$$

where C is STP and $S = \mathrm{diag}(1, -1, \ldots, \pm 1)$ $(n \times n)$ and $S' = \mathrm{diag}(1, -1, \ldots, \pm 1)$ $(|\alpha^c| \times |\alpha^c|)$. Thus, if α^c is based on consecutive indices, then $S[\alpha^c]S' = \pm I$. In either case, $A/A[\alpha]$ is STP. □

3. Generating t.n. Matrices. At first glance, it is not immediately clear that t.n. matrices exist for all values of m and n. However, this is indeed true, and t.n. matrices are relatively easy to generate. We consider here two generating schemes both involving STP matrices. Thus we assume that STP matrices are easy to generate; indeed (see [6, Thm. 5]) $n \times n$ STP matrices can be parameterized via bidiagonal factorizations and the rectangular case is an easy extension [5, Lemma 2.3]. We note here that we do not claim nor expect that these procedures generate all (or even a dense subset) of the t.n. matrices for given m and n. We view these procedures as a means of verifying the existence of this particular class of matrices. To begin we state a result for STP matrices; see [5, Thm. 4.2]. Matrix E_{ij} denotes the $n \times n$ standard basis matrix whose only nonzero entry is a 1 in the $(i, j)^{th}$ position.

LEMMA 3.1. *Let A be an $n \times n$ STP matrix. Then $\tilde{A} = A - xE_{11}$, where $x = \frac{\det A}{\det A[\{2,\ldots,n\}]}$ is singular and all of its proper minors are positive.*

We now provide our first procedure for generating (square) t.n. matrices. Recall that $S = \mathrm{diag}(1, -1, \ldots, \pm 1)$.

PROPOSITION 3.2. *Let A be an $n \times n$ STP matrix, and define $\tilde{A} = A - xE_{11}$, where $x = \frac{\det A}{\det A[\{2,\ldots,n\}]}$. Then there exists $t > 0$ (small) such that $B = \tilde{A} - tE_{11}$ is nonsingular and $SB^{-1}S$ is t.n.*

Proof. By Lemma 3.1, $\tilde{A}$ is singular and all of its proper minors are positive. Thus, by continuity, there exists $t > 0$ (small) such that $B = \tilde{A} - tE_{11}$ has negative determinant and all proper minors positive. Hence B is strictly sign regular with signature $\epsilon = (1, 1, \ldots, 1, -1)$. Therefore, by Remark 2.5, $SB^{-1}S$ is t.n. □

The above procedure is illustrated in the following example.

EXAMPLE 3.3. Let $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. Then A is STP with $\det A = 8$ and $\det A[\{2,3\}] = 5$ (so $x = 8/5$). Then $\tilde{A} = A - xE_{11} = \begin{bmatrix} \frac{7}{5} & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. If we choose $t = .05$, then

$$S(\tilde{A} - tE_{11})^{-1}S = \begin{bmatrix} -20 & -16 & -4 \\ -16 & -12.2 & -2.8 \\ -4 & -2.8 & -.2 \end{bmatrix},$$

which is t.n.

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Our second procedure uses Proposition 2.1 to generate t.n. matrices. First we need the following lemma.

LEMMA 3.4. *Let A be an $n \times n$ matrix with* $\det A \leq 0$ *and all proper minors negative. If $C = [c_{ij}] = A + tE_{11}$ $(t > 0)$ has $c_{11} < 0$, then C is t.n.*

Proof. Suppose $C = [c_{ij}] = A + tE_{11}$, where $t > 0$ is chosen so that $c_{11} < 0$. Then for any $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$ with $|\alpha| = |\beta|$

$$\det C[\alpha|\beta] = \begin{cases} \det A[\alpha|\beta] + t \det A[\alpha \setminus \{1\}|\beta \setminus \{1\}], & \text{if } 1 \in \alpha \cap \beta \\ \det A[\alpha|\beta], & \text{if } 1 \notin \alpha \cap \beta. \end{cases}$$

Thus $\det C[\alpha|\beta] \leq \det A[\alpha|\beta]$, for all α, β, except when $\alpha = \beta = \{1\}$. But in this case $c_{11} < 0$ by hypothesis. Also $\det C < 0$ since $t > 0$. Thus every minor of C is negative. □

Note that there is a result analogous to Lemma 3.4 for increasing the (n, n) entry.

The following procedure is inductive in nature. Consequently, we assume (using the previous procedure, for example) that t.n. matrices have already been generated for small orders.

THEOREM 3.5. *Let $n \geq 2$. Assume that A is an $(n-1) \times (n-1)$ t.n. matrix, and that B, C are two $n \times (n-1)$ STP matrices. Then $BAC^T + t_1E_{11}$ and $BAC^T + t_nE_{nn}$ are $n \times n$ t.n. matrices for sufficiently small positive t_j with $j = 1$ or n.*

Proof. Matrix $F = BAC^T$ is an $n \times n$ matrix with all proper minors negative (by Cauchy-Binet) and $\det F = 0$. By Lemma 3.4 increasing the (1,1) (or (n, n)) entry of F, while keeping it negative, yields an $n \times n$ t.n. matrix, as desired. □

This procedure is illustrated in the following example.

EXAMPLE 3.6. Let $A = \begin{bmatrix} -2 & -3 \\ -2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$. Then A is t.n. and

$$BAB^T = \begin{bmatrix} -9 & -14 & -19 \\ -13 & -20 & -27 \\ -17 & -26 & -35 \end{bmatrix}$$

has zero determinant. Increasing the (1,1) (or (3,3)) entry (keeping it negative) of BAB^T yields a t.n. matrix.

We close this section by noting that if $n \times n$ t.n. matrices exist, then $m \times n$ t.n. matrices also exist. This can be seen by multiplying an $m \times n$ STP matrix and an $n \times n$ t.n. matrix and using Proposition 2.1.

4. Triangular Factorizations. There have been several investigations into LU (or LDU) factorizations of STP and TP matrices into TP matrices; see, e.g., [1], [3]. In this section we explore the properties of triangular factorizations of t.n. matrices; see also [9]. An $n \times n$ triangular matrix is said to be ΔSTP if all minors that are not zero by virtue of the zero pattern are necessarily positive. If A is STP, then A can be written as $A = LDU$ where L (U) are lower (upper) ΔSTP (unit diagonal) and D is a positive diagonal matrix; see [1, Thm. 3.5], [3, Thm. 1.1]. We use this result to aid in a factorization result for t.n. matrices.

THEOREM 4.1. *Let A be an $n \times n$ t.n. matrix. Then A can be written as $A = UDL$, where L (U) are lower (upper) ΔSTP matrices (unit diagonal) and D is a diagonal matrix with all diagonal entries positive except for a negative (n, n) entry.*

Proof. Let $B = SA^{-1}S$, for $S = \text{diag}(1, -1, \dots, \pm 1)$. Then, by Theorem 2.4 B is strictly sign regular with signature $\epsilon = (1, 1, \dots, 1, -1)$. Choose $x > 0$ large enough so that $C = B + xE_{nn}$ is STP (it is clear that such an x exists). Then $C = L'D'U'$ where L', U' are ΔSTP (unit diagonal) and D is a positive diagonal matrix. Consider

$$\begin{aligned} B &= L'D'U' - xE_{nn} \\ &= \begin{bmatrix} L_1 & 0 \\ l_1 & 1 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & d_{nn} \end{bmatrix} \begin{bmatrix} U_1 & u_1 \\ 0 & 1 \end{bmatrix} - xE_{nn} \\ &= \begin{bmatrix} L_1 & 0 \\ l_1 & 1 \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & d_{nn} - x \end{bmatrix} \begin{bmatrix} U_1 & u_1 \\ 0 & 1 \end{bmatrix} \\ &= L'D''U'. \end{aligned}$$

Since D' is a positive diagonal matrix, D_1 is a positive diagonal matrix, and since $\det B < 0$, it follows that $d_{nn} - x < 0$. Also $A^{-1} = SBS = SL'D''U'S$, and hence

$$\begin{aligned} A &= S(U')^{-1}(D'')^{-1}(L')^{-1}S \\ &= (S(U')^{-1}S)(D'')^{-1}(S(L')^{-1}S) \\ &= UDL, \end{aligned}$$

where $U = S(U')^{-1}S$ and $L = S(L')^{-1}S$ and $D = (D'')^{-1}$. By the remarks preceding Theorem 2.4 it follows that U, L are ΔSTP and D has all diagonal entries positive except for the (n, n) entry, which is negative. ☐

It is well-known (and not difficult to verify) that if ρ is the reverse permutation matrix induced by the map $i \to n-i+1$, then $\rho A \rho$ is STP (TP) if and only if A is STP (TP). Not surprisingly, a similar result holds for t.n. and t.n.p. matrices. Namely, $\rho A \rho$ is t.n. (t.n.p.) if and only if A is t.n. (t.n.p.). A consequence of this simple fact and the result of Theorem 4.1 is that any t.n. matrix A can be written as $A = LDU$ where L (U) are lower (upper) ΔSTP matrices (unit diagonal) and D is a diagonal matrix with diagonal entries positive except for a negative $(1, 1)$ entry.

The converse to Theorem 4.1 does not hold in general; for example,

$$UDL = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} 3/4 & -1/2 \\ -1/2 & -1 \end{bmatrix},$$

which is not t.n. It is true that any matrix UDL satisfying the conditions in Theorem 4.1 will have, for example, $\det UDL[\{k, \dots, n\}] < 0$, for $k = 1, 2, \dots, n$. However, the product UDL need not even be entry-wise negative. In the above example, only the (1,1) entry is not negative and all remaining minors are negative. This observation leads to our next result; see [9] for a similar result, in which properties of the Neville elimination are used in the proof.

THEOREM 4.2. *Suppose A is an $n \times n$ matrix with $a_{11} < 0$ written as $A = UDL$, where L (U) are lower (upper) ΔSTP matrices (unit diagonal) and D is a diagonal* *matrix with all diagonal entries positive except for a negative (n,n) entry. Then A is totally negative.*

Proof. First observe that A is invertible and $\det A < 0$. To prove A is t.n., we show that $B = SA^{-1}S$, for $S = \mathrm{diag}(1, -1, \ldots \pm 1)$ is strictly sign regular with signature $\epsilon = (1, 1, \ldots, 1, -1)$, and apply Remark 2.5. If $B = SA^{-1}S$, then B can be written as $B = L'D'U'$, where L' (U') are lower (upper) ΔSTP matrices (unit diagonal) and D' is a diagonal matrix with all diagonal entries positive except for a negative (n,n) entry. Consider

$$\begin{aligned}&B[\{1,\ldots,n-1\}|\{1,\ldots,n\}]\\&= L'[\{1,\ldots,n-1\}]D'[\{1,\ldots,n-1\}]U'[\{1,\ldots,n-1\}|\{1,\ldots,n\}].\end{aligned}$$

Extending these $(n-1)\times(n-1)$ submatrices of L' and D' to $n \times n$ while keeping them lower ΔSTP and positive diagonal, respectively, and applying the result of [3, Thm. 1.1], shows that the above matrix is STP. Similarly, $B[\{1,\ldots,n\}|\{1,\ldots,n-1\}]$ is STP. The only remaining minors to verify are of the form $\det B[\alpha|\beta]$ with $n \in \alpha \cap \beta$. By [1, Thm. 2.5] it is enough to check minors with row and column index sets based on consecutive indices. Hence among the remaining minors to be checked, it is enough to determine the sign of $\det B[\{k,\ldots,n\}]$, for $k = 1, 2, \ldots, n$. For $k = 1$, $\det B < 0$. For $k = 2$,

$$\det B[\{2,\ldots,n\}] = \det SA^{-1}S[\{2,\ldots,n\}] = \det A^{-1}[\{2,\ldots,n\}] = \frac{\det A[\{1\}]}{\det A} > 0,$$

since a_{11} is negative (the last equality following from Jacobi's identity). Consider the case $k = 3$. By Sylvester's identity (see [10, 0.8.6] and [5])

$$\begin{aligned}&0 < \det B[\{2,\ldots n\}]\det B[\{3,\ldots,n-1\}]\\&= \det B[\{3,\ldots,n\}]\det B[\{2,\ldots,n-1\}]\\&\quad -\det B[\{2,\ldots,n-1\}|\{3,\ldots,n\}]\det B[\{3,\ldots,n\}|\{2,\ldots,n-1\}].\end{aligned}$$

Hence it follows that $\det B[\{3,\ldots,n\}] > 0$. The remaining cases follow inductively using the above relation as k increases to n. Hence B is strictly sign regular with signature $\epsilon = (1,\ldots,1-1)$. Thus A is t.n., as desired. □

Combining the results of Theorems 4.1 and 4.2 yields the following result (see also [9, Thm. 3.4]).

COROLLARY 4.3. *An $n \times n$ matrix A with $a_{11} < 0$ is totally negative if and only if A can be written as $A = UDL$, where L (U) are lower (upper) ΔSTP matrices (unit diagonal) and D is a diagonal matrix with all diagonal entries positive except for a negative (n,n) entry.*

5. Open Issues. We conclude with a variety of open problems and unresolved issues related to totally negative and sign regular matrices. Recall from Proposition 2.1 that the product of any two square t.n. matrices is STP. We are interested in a converse to this result, and propose the following problem.

PROBLEM 5.1. *Can any $n \times n$ totally positive matrix be written as the product of two totally negative matrices?*

There is also an analogous problem for totally nonnegative matrices; namely, can any totally nonnegative matrix be written as a product of two totally nonpositive matrices?

An interesting special case of Problem 5.1 deals with square roots of STP matrices.

PROBLEM 5.2. *Does an $n \times n$ totally positive matrix have a totally negative square root (i.e., can a STP matrix A be written as $A = B^2$, where B is t.n.)?*

We now turn to a problem dealing with Fischer's inequality for sign-regular matrices. Recall that in Proposition 2.2 it is noted that Fischer's inequality holds for TP matrices and trivially for all t.n. matrices. However, Fischer's inequality does not hold in general for every strictly sign regular class of matrices, even when there is no obvious sign contradiction by virtue of the sign regularity. For example, consider

$$A = \begin{bmatrix} -0.03 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -0.01 \end{bmatrix}.$$

Then A is strictly sign regular with $\epsilon = (-1, -1, 1)$, i.e., A has all proper minors negative, but $\det A$ is positive. However (to 2 d.p.), $0.04 = \det A > a_{33}\ \det A[\{1,2\}] = 0.01$.

PROBLEM 5.3. *Describe all the signatures ϵ of order n for which all $n \times n$ sign regular matrices with signature ϵ satisfy Fischer's inequality.*

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