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GENERALIZED NUMERICAL RANGES OF PERMANENTAL COMPOUNDS ARISING FROM QUANTUM SYSTEMS OF BOSONS*

NATÁLIA BEBIANO†, CHI-KWONG LI‡, AND JOÃO DA PROVIDÊNCIA§

Abstract. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an m-tuple of integers such that $1 \le \alpha_1 \le \cdots \le \alpha_m \le n$. Denote by per (X) the permanent of a square matrix. The permanental range $W_m^\alpha(A)$ of an $n \times n$ complex matrix A associated with α is the set of complex numbers of the form $\mathrm{per}(X^*AX)/\mathrm{per}(X^*X)$, where X is an $n \times m$ matrix with columns x_{α_j} for $j = 1, \ldots, m$ so that $(x_k, x_l) = \delta_{kl}$, the Kronecker delta. The set $W_m^\alpha(A)$ is related to quantum system of m bosons lying in single-particle states specified by α. Geometrical properties of $W_m^\alpha(A)$ are studied and their physical interpretations are given. Linear operators L on $n \times n$ matrices satisfying $W_m^\alpha(L(A)) = W_m^\alpha(A)$ are characterized. The permanental ranges associated with derivations are also discussed.

Key words. Generalized numerical range, permanent, quantum systems of Bosons

AMS subject classifications. 15A60, 15A90

1. Introduction. Let m and n be positive integers with $n \ge 2$. Denote by per (X) the permanent of a square matrix. The mth *permanental range* of $A \in \mathbb{C}^{n\times n}$ is defined by

$$W_m^*(A) = \{\mathrm{per}\,(X^*AX) : X \in \mathbb{C}^{n\times m},\ \mathrm{per}\,(X^*X) = 1\}.$$

In quantum physics, the state of a system of m Bosons, i.e., particles with integer spin, may be described by a matrix $X \in \mathbb{C}^{n\times m}$ satisfying per $(X^*X) = 1$ whose columns describe the individual states of the Bosons. If we associate the matrix $A \in \mathbb{C}^{n\times n}$ with the dynamics of the system, then $W_m^*(A)$ is the collection of all possible average energy values (of the system). In some models, one has to impose orthogonality conditions on the states of the particles; one then needs to consider the following variation

$$W_m^\perp(A) = \left\{ \frac{\mathrm{per}\,(X^*AX)}{\mathrm{per}\,(X^*X)} : X = [x_{\alpha_1}|\cdots|x_{\alpha_m}] \in \mathbb{C}^{n\times m},\ \alpha \in \Gamma_{m,n}, (x_k, x_l) = \delta_{kl} \right\},$$

where δ_{kl} denotes the Kronecker delta, and $\Gamma_{m,n}$ denotes the collection of all nondecreasing sequences $\alpha = (\alpha_1, \ldots, \alpha_m)$ of integers such that $1 \le \alpha_i \le n$ for all i. A further refinement of the model can be obtained by specifying some fixed orthogonal

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†Departamento de Matemática, Universidade de Coimbra, 3000 Coimbra, Portugal (bebiano@ci.uc.pt)

‡Department of Mathematics, College of William and Mary, P.O. Box 8795, Williamsburg, VA 23187-8795, USA (ckli@math.wm.edu)

§Departamento de Física, Universidade de Coimbra, 3000 Coimbra, Portugal (providencia@fteor5.fis.uc.pt)

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relations on the particles determined by a given $\alpha \in \Gamma_{m,n}$; we have

$$W_m^\alpha(A) = \left\{ \frac{\mathrm{per}\,(X^*AX)}{\mathrm{per}\,(X^*X)} : X = [x_{\alpha_1}|\cdots|x_{\alpha_m}] \in \mathbb{C}^{n\times m},\ (x_k, x_l) = \delta_{kl} \right\}.$$

Denote by k_j the number of occurrence of the number j in the sequence α. In physics terminology, k_j is the *occupation number* of level j, that is, the number of Bosons in state j. If $1 \leq m \leq n$, we always set $\iota = (1, \ldots, m) \in \Gamma_{m,n}$ and

$$W_m^\iota(A) = \{\mathrm{per}\,(X^*AX) : X \in \mathbb{C}^{n\times m},\ X^*X = I_m\}.$$

Evidently, $W_m^\iota(A)$ represents a system with all m particles lying in mutually distinct (orthogonal) single-particle states.

When $m = 1$, all the sets $W_m^*(A), W_m^\perp(A), W_m^\alpha(A), W_m^\iota(A)$ reduce to the *classical numerical range* of A defined by

$$W(A) = \{(Ax, x) : x \in \mathbb{C}^n,\ (x, x) = 1\},$$

which has been studied extensively in the last few decades because of its connections and applications to many pure and applied subjects; see [1, 9, 10, 11, 16].

While the sets $W_m^*(A), W_m^\perp(A)$ and $W_m^\iota(A)$ are quite well studied in the mathematical literature [14, 20, 23, 24], there are no results on the natural (from the physics point of view) object $W_m^\alpha(A)$. In this paper, we will fill this gap and develop some techniques to prove results on $W_m^\alpha(A)$ analogous to those on $W_m^\iota(A)$. Physical interpretations of the results will be discussed.

Our paper is organized as follows. Section 2 contains some preliminary results. Section 3 deals with permanental ranges of 2×2 matrices. In Section 4 we study special classes of matrices whose permanental ranges are more tractable. Section 5 concerns special boundary points of permanental ranges. Section 6 concerns matrices with degenerate permanental ranges. In Section 7, we study linear operators L on $\mathbb{C}^{n\times n}$ satisfying $W_m^\alpha(L(A)) = W_m^\alpha(A)$ for all $A \in \mathbb{C}^{n\times n}$. Section 8 is about derivations of permanental compounds, a concept more relevant to quantum physics.

2. Preliminaries. The mth permanental range can be regarded as the mth decomposable numerical range of A associated with the symmetric group of degree m and the constant character $\chi = 1$, i.e.,

$$W_m^*(A) = \{(P_m(A)x^*, x^*) : x^* \in \mathbb{C}^n_{(m)} \text{ is decomposable},\ (x^*, x^*) = 1\}.$$

Here $P_m(A)$ is the induced operator of A acting on decomposable vectors $x^* = x_1 * \cdots * x_m$ in the mth symmetric space $\mathbb{C}^n_{(m)}$ over $\mathbb{C}^n$, according to the formula

$$P_m(A)(x_1 * \cdots * x_m) = (Ax_1) * \cdots * (Ax_m).$$

One may see [17, 20] for the general mathematical background. In fact, if we identify $x^* = x_1 * \cdots * x_m$ with the $n \times m$ matrix X with columns $x_1, \ldots, x_m$, then

$$\|x^*\|^2 = (x^*, x^*) = \frac{1}{m!}\mathrm{per}\,(X^*X)$$

and

$$(P_m(A)x^*, x^*) = \frac{1}{m!}\mathrm{per}\,(X^*AX).$$

Note that $\mathbb{C}^n_{(m)}$ has dimension $\binom{m+n-1}{m}$. Suppose $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{C}^n$. Denote by $e_\alpha = e_{\alpha_1} * \cdots * e_{\alpha_m}$ for any $\alpha \in \Gamma_{m,n}$. Then

$$\left\{ \sqrt{\frac{m!}{\nu(\alpha)}}\, e_\alpha : \alpha \in \Gamma_{m,n} \right\},$$

is the standard orthonormal basis for $\mathbb{C}^n_{(m)}$, where

$$\nu(\alpha) = k_1! \cdots k_n!$$

such that k_j is the number of occurrence of j in the sequence α, i.e., the *occupation number* of level j. With this standard basis $P_m(A)$ has a matrix representation whose entry at the positions indexed by the sequences $\alpha, \beta \in \Gamma_{m,n}$ is given by

$$P_m(A)_{\alpha,\beta} = \frac{\mathrm{per}\,(E_\alpha^* A E_\beta)}{\sqrt{\mathrm{per}\,(E_\alpha^* E_\alpha)\mathrm{per}\,(E_\beta^* E_\beta)}} = \frac{\mathrm{per}\,(E_\alpha^* A E_\beta)}{\sqrt{\nu(\alpha)\nu(\beta)}},$$

where $E_\alpha, E_\beta \in \mathbb{C}^{n\times m}$ have columns $e_{\alpha_1}, \ldots, e_{\alpha_m}$, and $e_{\beta_1}, \ldots, e_{\beta_m}$, respectively. One can also describe $W_m^\perp(A), W_m^\alpha(A), W_m^\iota(A)$ in terms of the induced matrix $P_m(A)$. Using this formulation, one easily sees that

$$\bigcup_{\alpha\in\Gamma_{m,n}} W_m^\alpha(A) = W_m^\perp(A) \subseteq W_m^*(A) \subseteq W(P_m(A)).$$

We conclude this section with some simple observations about the permanental ranges that will be used in our subsequent discussion.

PROPOSITION 2.1. *Let $A \in \mathbb{C}^{n\times n}$. If $\alpha, \beta \in \Gamma_{m,n}$ have the same sequence of occupation numbers when the entries are arranged in descending order, then $W_m^\alpha(A) = W_m^\beta(A)$.* By the above observation, we need only consider those $\alpha \in \Gamma_{m,n}$ with occupation numbers $k_1 \geq \cdots \geq k_n$.

We shall always let $\varepsilon = (1, \ldots, 1) \in \Gamma_{m,n}$ in the future discussion.

PROPOSITION 2.2. *Let $A \in \mathbb{C}^{n\times n}$. Then $W_m^\varepsilon(A) = \{z^m : z \in W(A)\}$.*

PROPOSITION 2.3. *The sets $W_m^\alpha(A)$, $W_m^\perp(A)$, $W_m^*(A)$, etc. are invariant if $A \in \mathbb{C}^{n\times n}$ is replaced by U^*AU for any unitary matrix U.*

This proposition makes perfect physical sense because a change of orthonormal bases should not affect the measurements on the system. In fact, the physical properties of any system are intrinsic to the system, which means that they are independent from the orthonormal bases chosen for their description.

PROPOSITION 2.4. *Let $A \in \mathbb{C}^{n\times n}$. If A is replaced by μA for some nonzero μ, then the sets $W_m^\alpha(A)$, $W_m^\perp(A)$, $W_m^*(A)$ are changed by a factor of μ^m.*

PROPOSITION 2.5. *Suppose $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Then*

$$W_m^\alpha(A) \subseteq W_m^\varepsilon(A) = W(P_m(A)) = [\lambda_n^m, \lambda_1^m].$$

3. Permanental ranges of 2×2 matrices. In this section, we study the permanental range of 2×2 matrices. A common technique in the study of numerical ranges is to reduce the dimension of the given operators to 2×2 matrices via compressions; see [18] and [11, Ch. 1]. Thus, the results in this section do not only provide a description of the permanental ranges for 2×2 matrices, but also have potential for further development.

It is well known [11, Theorem 1.3.6] that the classical numerical range of $A \in \mathbb{C}^{2\times 2}$ is an elliptical disk with its eigenvalues: λ_1 and λ_2 as foci, and $\{\mathrm{tr}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2\}^{1/2}$ as the length of minor axis. In the following, we show that $W_m^\alpha(A)$ is the image of $W(A)$ under a certain polynomial function.

THEOREM 3.1. *Let $\alpha \in \Gamma_{m,2}$ have occupation numbers $k_1 \geq k_2$. Then*

$$W_m^\alpha(A) = \{f(a) : a \in W(A)\},$$

where

$$f(a) = \sum_{t=k_1-k_2}^{k_1} \binom{k_1}{t}\binom{k_2}{k_1-t} a^t(\lambda_1 - a)^{k_1-t}(a-\lambda_2)^{k_1-t}(\lambda_1+\lambda_2-a)^{k_2-k_1+t}$$

is a polynomial of degree k_1+k_2 with leading coefficient $(-1)^{k_2}\binom{k_1+k_2}{k_1}$.

Proof. Let x_1 and x_2 be a pair of orthonormal vectors in $\mathbb{C}^2$, and let $X \in \mathbb{C}^{2\times m}$ so that the first k_1 columns equal to x_1 and the last k_2 columns equal to x_2. Then $\mathrm{per}\,(X^*AX)/(k_1!k_2!) \in W_m^\alpha(A)$, and every element in $W_m^\alpha(A)$ can be constructed in this way. Now X^*AX is of the form

$$\begin{pmatrix} aJ_{k_1,k_1} & bJ_{k_1,k_2} \\ cJ_{k_2,k_1} & dJ_{k_2,k_2} \end{pmatrix},$$

where $J_{p,q}$ is the $p \times q$ matrix with all entries equal to 1, and $a = x_1^*Ax_1$, $b = x_1^*Ax_2$, $c = x_2^*Ax_1$ and $d = x_2^*Ax_2$. A routine computation shows that

$$\mathrm{per}\,(X^*AX) = k_1!k_2! \sum_{t=k_1-k_2}^{k_1} \binom{k_1}{t}\binom{k_2}{k_1-t} a^t(bc)^{k_1-t}d^{k_2-k_1+t}.$$

Observe that $d = \mathrm{trA} - \mathrm{a} = \lambda_1 + \lambda_2 - \mathrm{a}$ and

$$bc = ad - \det(A) = a(\mathrm{trA}) - \mathrm{a}^2 - \det(\mathrm{A}) = (\lambda_1 - \mathrm{a})(\mathrm{a} - \lambda_2).$$

Thus, the set equality follows.

Note that the highest power of a in

$$a^t(\lambda_1 - a)^{k_1-t}(a-\lambda_2)^{k_1-t}(\lambda_1+\lambda_2-a)^{k_2-k_1+t}$$

is $k_1 + k_2$ and the coefficient is

$$(-1)^{k_2} \sum_{t=k_1-k_2}^{k_1} \binom{k_1}{t}\binom{k_2}{k_1-t} = (-1)^{k_2}\binom{k_1+k_2}{k_1}.$$

The assertion on f follows. □

By Proposition 2.1, there are basically only two types of $\alpha \in \Gamma_{2,2}$, namely, $\iota = (1,2)$ and $\varepsilon = (1,1)$. When $\alpha = \iota$, the polynomial $f(a)$ reduces to $f(a) = 2a(\mathrm{trA} - a) - \det \mathrm{A}$. Using this observation, one can prove the following.

COROLLARY 3.2. [15] *Let $m = 2$ and $A \in \mathbb{C}^{2\times 2}$ have eigenvalues λ_1 and λ_2. Set $b = \{\mathrm{tr}(\mathrm{A}^*\mathrm{A}) - |\lambda_1|^2 - |\lambda_2|^2\}^{1/2}$. Then $W_m^\iota(A)$ is the elliptical disk with foci at $\lambda_1\lambda_2$ and $(\lambda_1^2+\lambda_2^2)/2$, and major axis of length $|\lambda_1-\lambda_2|^2/2 + b^2$. In particular, $W_m^\iota(A)$ is a line segment if and only if $b = 0$; in such case, $W_m^\iota(A)$ is the line segment with end points $\lambda_1\lambda_2$ and $(\lambda_1^2+\lambda_2^2)/2$.*

The situation for $W_m^\varepsilon(A)$ is not so simple as shown in the following result.

PROPOSITION 3.3. *Let $\varepsilon = (1,1)$ and $A \in \mathbb{C}^{2\times 2}$.*

(a) *If* trA $= 0$, *then $W_m^\varepsilon(A)$ is an elliptical disk; in particular, $W(A)$ is a circular disk if and only if $W_m^\varepsilon(A)$ is.*

(b) *The set $W_m^\varepsilon(A)$ has no interior points if and only if one of the following holds.*

(b.i) *The matrix A is normal, but not a multiple of a Hermitian matrix, and $W_m^\varepsilon(A)$ is part of a parabola.*

(b.ii) *The matrix A is a multiple of a Hermitian matrix, and $W_m^\varepsilon(A)$ lies on a half line with the origin as the end point.*

(c) *The point $(\lambda_1+\lambda_2)^2/4 \in W_m^\iota(A) \cap W_m^\varepsilon(A) \subseteq W_m^\perp(A)$, and thus $W_m^\perp(A)$ is connected.*

Proof. (a) If trA $= 0$, then $W(A)$ is an elliptical disk centered at the origin, and there exists $t \in [0, 2\pi)$ such that

$$W(A) = e^{it}\{r(\alpha\cos s + i\beta\sin s) : 0 \le r \le 1,\ s \in [0,2\pi)\}.$$

Then

$$\begin{aligned} W_m^\varepsilon(A) &= e^{2it}\{r^2(\alpha^2\cos^2 s - \beta^2\sin^2 s + 2i\alpha\beta\sin s\cos s) : 0 \le r \le 1,\ s \in [0,2\pi)\} \\ &= e^{2it}\{(\alpha^2-\beta^2)/2 + r[(\alpha^2+\beta^2)/2\cos 2s + i\alpha\beta\sin 2s] : 0 \le r \le 1,\ s \in [0,\pi]\}, \end{aligned}$$

is an elliptical disk. Clearly, $W_m^\varepsilon(A)$ is a circular disk if and only if $W(A)$ is.

(b) If (b.i) or (b.ii) holds, one easily checks that $W_m^\varepsilon(A)$ has no interior. Now, suppose $W_m^\varepsilon(A)$ has no interior. Then $W(A)$ has no interior, i.e., $\beta = 0$. In such case (see, e.g., [11, 19]), $W(A)$ is a line segment and hence is a normal matrix. If A is not a multiple of a Hermitian matrix, then there exists $t \in [0, 2\pi)$ and $s_1 < s_2$ such that

$$W(A) = e^{2it}\{\alpha + si : s \in [s_1, s_2]\}.$$

Thus

$$W_m^\varepsilon(A) = e^{2it}\{\alpha^2 - s^2 + 2\alpha si : s \in [s_1, s_2]\},$$

which is part of a parabola as asserted. If A is a multiple of a Hermitian matrix, then one readily verifies that $W_m^\varepsilon(A)$ is a line segment lying on a half line with the origin as the end point.

(c) Note that $\mu = (\lambda_1+\lambda_2)/2 \in W(A)$. Thus $\mu^2 \in W_m^\varepsilon(A)$. Also, μ^2 is the center of the elliptical disk $W_m^\iota(A)$. The assertion follows. □

By Corollary 3.2 and Proposition 3.3, we see that a variety of shapes is possible for $W_2^{\perp}(A) = W_2^{\iota}(A) \cup W_2^{\varepsilon}(A)$. For $W_2^*(A)$, we have more information. Note that all vectors in the second symmetric space $\mathbb{C}^2_{(2)}$ over $\mathbb{C}^2$ are decomposable. In fact, an easy computation shows that any vector in $\mathbb{C}^3$ can be written in the form

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} * \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} \sqrt{2}x_1x_2 \\ x_1x_4 + x_2x_3 \\ \sqrt{2}x_3x_4 \end{pmatrix} \in \mathbb{C}^2_{(2)}.$$

Consequently, we have the following result.

PROPOSITION 3.4. *Suppose $A \in \mathbb{C}^{2\times 2}$. Then $W_2^*(A) = W(P_2(A))$. In particular, if $A = \operatorname{diag}(a, b)$, then $W(P_2(A)) = W_2^*(A)$ is the triangular region with vertices a^2, ab, b^2.*

4. Special classes of matrices. In general, it is difficult to determine $W_m^{\alpha}(A)$ for general A. Here we consider some special classes of matrices. First, we show that the permanental ranges of rank one matrices admit a relatively simple description.

PROPOSITION 4.1. *Suppose $A \in \mathbb{C}^{n\times n}$ has rank one and $\alpha \in \Gamma_{m,n}$ has occupation numbers $k_1 \geq \cdots \geq k_n$. Then*

$$W_m^{\alpha}(A) = \left\{ \frac{m!}{\nu(\alpha)} \prod_{j=1}^{n} \mu_j^{k_j} : \mu_1, \ldots, \mu_n \in \mathbb{C},\ \sum_{j=1}^{n} \mu_j = \mathrm{trA},\ \sum_{\mathrm{j=1}}^{\mathrm{n}} |\mu_{\mathrm{j}}| \leq \sqrt{\mathrm{trA^*A}} \right\}.$$

Proof. Suppose $A = (a_1, \ldots, a_n)^t(b_1, \ldots, b_n)$. First, observe that there is a unitary matrix U such that $UAU^* = (u_1, \ldots, u_n)^t(v_1, \ldots, v_n)$ if and only if $\sum_{j=1}^n u_jv_j = \mathrm{trA}$ and $(\sum_{j=1}^n |u_j|^2)(\sum_{j=1}^n |v_j|^2) = \mathrm{trA^*A}$. Furthermore, for a unitary matrix U such that $UAU^* = (u_1, \ldots, u_n)^t(v_1, \ldots, v_n)$, if $X \in \mathbb{C}^{n\times m}$ is constructed from U by using its columns with multiplicities according to α, and if $D_u = u_1 \otimes I_{k_1} \oplus \cdots \oplus u_n \otimes I_{k_n}$ and $D_v = v_1 \otimes I_{k_1} \oplus \cdots \oplus v_n \otimes I_{k_n}$, then

$$\mathrm{per}\,(X^*AX) = \mathrm{per}\,(D_uJ_{m,m}D_v) = m! \prod_{j=1}^{n} (u_jv_j)^{k_j}.$$

Hence, $W_m^{\alpha}(A)$ consists of complex numbers of the form:

$$\mu = \frac{m!}{\nu(\alpha)} \prod_{j=1}^{n} (u_jv_j)^{k_j}.$$

Now, if we set $\mu_j = u_jv_j$ for $j = 1, \ldots, n$, then $\mu = \frac{m!}{\nu(\alpha)} \prod_{j=1}^n \mu_j^{k_j}$, where

(i) $\sum_{j=1}^n \mu_j = \mathrm{trA}$, and (ii) $\{\sum_{j=1}^n |\mu_j|\}^2 \leq \left(\sum_{j=1}^n |u_j|^2\right)\left(\sum_{j=1}^n |v_j|^2\right) = \mathrm{trA^*A}$.

Conversely, suppose $\mu = \frac{m!}{\nu(\alpha)} \prod_{j=1}^n \mu_j^{k_j}$, where $\mu_1, \ldots, \mu_n \in \mathbb{C}$ satisfy $\sum_{j=1}^n \mu_j = \mathrm{trA}$ and $\{\sum_{j=1}^n |\mu_j|\}^2 \leq \mathrm{trA^*A}$. If $\mu_j = 0$ for all j, then set $u_1 = \sqrt{\mathrm{trA^*A}}$, $v_n = 1$, and

$0 = u_2 = \cdots = u_n = v_1 = \cdots = v_{n-1}$. If $k > 0$, $k < n$ is the smallest integer such that $\mu_k \neq 0$, set $u_j = v_j = \mu_j^{1/2}$ for $j \neq k$, $u_k = t\mu_k^{1/2}$ and $v_k = \mu_k^{1/2}/t$ for a suitable $t \geq 1$ so that the equality in (ii) holds. If $k = n$ is the smallest integer such that $\mu_k \neq 0$, set $u_n = v_n = \mu_n^{1/2}$, $v_{n-1} = \cdots = v_1 = u_{n-2} = \cdots = u_1 = 0$, $u_{n-1} = \sqrt{(\mathrm{tr A^* A} - |\mu_\mathrm{n}|^2)/|\mu_\mathrm{n}|}$ so that the equality in (ii) holds. Then $\mu = \frac{m!}{\nu(\alpha)} \prod_{j=1}^n (u_j v_j)^{k_j}$. □

By Proposition 4.1, if A is a rank one nilpotent matrix, then $W_m^\alpha(A)$ is a circular disk centered at the origin. In fact, we have the following more general result which follows from [2, 3.2].

PROPOSITION 4.2. *If $A \in \mathbb{C}^{n\times n}$ is unitarily similar to a matrix in shift block form, i.e., a matrix in block form $(B_{ij})_{1\leq i,j\leq k}$ such that all the diagonal blocks are square matrices and $B_{ij} = 0$ if $j \neq i+1$, then all the permanental ranges are circular disks centered at the origin.*

It is easy to construct examples so that $W(A)$ is a circular disk, but A is not in shift block form. Thus, the converse of the Proposition 4.2 is not true if $m = 1$. For $m > 1$, we have the following result showing that the converse of Proposition 4.2 is not valid in general.

PROPOSITION 4.3. *Let $A \in \mathbb{C}^{n\times n}$. Then $W_m^\varepsilon(A) = \{z^m : z \in W(A)\}$ is a circular disk centered at the origin if and only if $W(A)$ is.*

Proof. The ($\Leftarrow$) part is clear. To prove the converse, assume that $W(A)$ is not a circular disk. Then there cannot be more than n points in $W(A)$ attaining the numerical radius; see [5, Theorem 2.2]. Thus, there cannot be more than n points in $W_m^\varepsilon(A)$ attaining the maximum modulus. Hence $W_m^\varepsilon(A)$ cannot be a circular disk centered at the origin. □

5. Special boundary points. In this section , we study special boundary points of permanental ranges. Some of these have special physical interpretations. We begin with the following result.

PROPOSITION 5.1. *Suppose $A \in \mathbb{C}^{n\times n}$ and $\alpha \in \Gamma_{m,n}$ has occupation numbers $k_1 \geq \cdots \geq k_n$. If A has eigenvalues $\lambda_1, \ldots, \lambda_n$, then*

$$\lambda_1^{k_1} \cdots \lambda_n^{k_n} \in W_m^\alpha(A).$$

Proof. By a unitary similarity transform, we can put A in triangular form with $\lambda_1, \ldots, \lambda_n$ on the diagonal. Then use the standard vectors $e_1, \ldots, e_n$ to construct X to get $\mathrm{per}\,(X^*AX)/\mathrm{per}\,(X^*X) = \lambda_1^{k_1} \cdots \lambda_n^{k_n} \in W_m^\alpha(A)$ as asserted. □

Note that we can put the eigenvalues in any order we like. The elements in $W_m^\alpha(A)$ of the form $\lambda_1^{k_1} \cdots \lambda_n^{k_n}$ will be called *λ-points.*

THEOREM 5.2. *Suppose A is in upper triangular form with diagonal entries $\lambda_1, \ldots, \lambda_n$. Let $\alpha \in \Gamma_{m,n}$ have occupation numbers $k_1 \geq \cdots \geq k_t > 0 = k_{t+1} = \cdots = k_n$. If $\prod_{j=1}^t \lambda_j^{k_j}$ is nonzero and lies on the boundary of $W_m^\alpha(A)$, then $A = A_1 \oplus A_2$ with $A_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_t)$. Furthermore, if A_2 is non-trivial, i.e., $t < n$, then λ_j is not in the interior of $W(A_2)$ for any $j = 1, \ldots, t$.*

Proof. Suppose $A = (a_{jk})$ is in upper triangular form as asserted. We show that $a_{ij} = 0$ if $\{i,j\} \cap \{1, \ldots, t\} \neq \emptyset$; otherwise, $\prod_{j=1}^t \lambda_j^{k_j}$ will fail to be a boundary point. Note that the proof has to be done in a certain order of i and j.

Suppose $1 \le i < j \le n$. Denote by $B[i,j]$ be the submatrix of $B \in \mathbb{C}^{n\times n}$ lying in the rows and columns indexed by i and j.

We first show that $A = A_1 \oplus A_2$ with $A_1 \in \mathbb{C}^{t\times t}$ if $t < n$. We start with row t of A. Suppose $t < j \le n$ and $a_{tj} \ne 0$, then λ_t is an interior point of the elliptical disk $W(A[t,j])$. For any $\mu \in W(A[t,j])$, there exists a 2×2 unitary matrix U so that the $(1,1)$ entry of $U^*A[t,j]U$ equals μ. Let $\tilde{U}$ be obtained from I_n by replacing $I_n[t,j]$ with U. Then the leading $t \times t$ principal submatrix of $\tilde{U}^*A\tilde{U}$ is in upper triangular form with diagonal entries $\lambda_1, \dots, \lambda_{t-1}$ and μ. Construct X using the first k columns of $\tilde{U}$ with multiplicities according to α. Then

$$\mathrm{per}\,(X^*AX)/\mathrm{per}\,(X^*X) = \prod_{j=1}^{t-1} \lambda_j^{k_j} \mu^{k_t} \in W_m^\alpha(A).$$

Hence, $\prod_{j=1}^{t} \lambda_j^{k_j}$ is an interior point of

$$\left\{ \prod_{j=1}^{t-1} \lambda_j^{k_j} \mu^{k_t} : \mu \in W(A[t,j]) \right\} \subseteq W_m^\alpha(A),$$

which is a contradiction. Thus, we see that $a_{t,j} = 0$ for $j > t$.

Next, we consider row $t-1$ of A. Suppose $t < j \le n$ and $a_{t-1,j} \ne 0$. Then λ_{t-1} is an interior point of the elliptical disk $W(A[t-1,j])$. For any $\mu \in W(A[t-1,j])$, there exists a 2×2 unitary matrix U so that the $(1,1)$ entry of $U^*A[t-1,j]U$ equals μ. Let $\tilde{U}$ be obtained from I_n by replacing $I_n[t-1,j]$ with U. By the fact that $a_{tj} = 0$ for $j > t$, we see that the leading $t \times t$ principal submatrix of $\tilde{U}^*A\tilde{U}$ is in upper triangular form with diagonal entries $\lambda_1, \dots, \lambda_{t-2}$, μ and λ_t. Using arguments similar to those in the preceding paragraph, we conclude that $\prod_{j=1}^{t} \lambda_j^{k_j}$ is an interior point of $W_m^\alpha(A)$, which is a contradiction. Thus, we see that $a_{t-1,j} = 0$ for $j > t$.

One can repeat the above arguments to show that for each of the rows $i = t-2, \dots, 1$, we have $a_{ij} = 0$ if $j > t$. Hence $A = A_1 \oplus A_2$.

Next, we show that A_1 is in diagonal form. Again, we have to argue $a_{ij} = 0$ for $1 \le i < j \le t$ in a special order of i and j.

First, suppose there exists i with $1 \le i < t$ such that $a_{i,i+1} \ne 0$. Then λ_i is an interior point of $W(A[i,i+1])$. For any $\mu \in W(A[i,i+1])$, there exists a 2×2 unitary matrix U so that the $(1,1)$ entry of $U^*A[i,i+1]U$ equals μ. Let $\tilde{U}$ be obtained from I_n by replacing $I_n[i,i+1]$ with U. Then the leading $t \times t$ principal submatrix of $\tilde{U}^*A\tilde{U}$ is almost in upper triangular form except that the $(i+1,i)$ position may be nonzero. Construct X using the first t columns of $\tilde{U}$ with multiplicities according to α. Then

$$\mathrm{per}\,(X^*AX)/\mathrm{per}\,(X^*X) = \prod_{j\ne i,i+1} \lambda_j^{k_j} f(\mu) \in W_m^\alpha(A),$$

where $f(\mu)$ is defined as in Theorem 3.1 with respect to the 2×2 matrix $A[i,i+1]$ and occupation numbers k_i and k_{i+1}. By the Open Mapping Theorem [6, p. 99], we conclude that $f(\lambda_i) = \lambda_i^{k_i}\lambda_{i+1}^{k_{i+1}}$ is an interior point of $f(W(A[i,i+1]))$. Hence $\prod_{j=1}^t \lambda_j^{k_j}$ is an interior point of

$$\prod_{j\neq i,i+1} \lambda_j^{k_j} f(W(A[i,i+1])) \subseteq W_m^\alpha(A),$$

which is a contradiction.

Now, using the fact that $a_{i,i+1} = 0$ for all $1 \leq i < t$, one can show that $a_{i,i+2} = 0$ for all $1 \leq i < t-1$ by arguments similar to the previous case. Repeating this process, we conclude that A_1 is in diagonal form.

Finally, suppose $t < n$ and λ_r is an interior point of $W(A_2)$ for some $1 \leq r \leq t$. Then one easily checks that $\prod_{j=1}^t \lambda_j^{k_j}$ is an interior point of

$$\left\{\prod_{j\neq r} \lambda_j^{k_j} \mu^{k_r} : \mu \in W(A_2)\right\} \subseteq W_m^\alpha(A),$$

which is a contradiction. Thus the last assertion of the theorem follows. □

COROLLARY 5.3. *Let $\alpha \in \Gamma_{m,n}$ have occupation numbers $k_1 \geq \cdots \geq k_t > 0 = k_{t+1} = \cdots = k_n$. Suppose*

$$g(n,\alpha) = \begin{cases} 0 & \text{if } t \geq n-1, \\ (n-2)!/(n-2-t)! & \text{otherwise.} \end{cases}$$

If $A \in \mathbb{C}^{n\times n}$ and there are more than $g(n,\alpha)$ nonzero λ-points on the boundary of $W_m^\alpha(A)$, then A is normal.

Proof. First, let A be in triangular form with diagonal entries $\lambda_1, \ldots, \lambda_n$ so that $\prod_{j=1}^t \lambda_j^{k_j}$ is nonzero and lying on the boundary. Then $A = \mathrm{diag}(\lambda_1,\ldots,\lambda_t) \oplus A_2$. If $t \geq n-1$, then A is normal. If not, any $(n-2)\times(n-2)$ leading principal submatrix of A can accommodate at most $(n-2)!/(n-2-t)!$ so many nonzero λ-points. Thus, the result follows. □

Recall that a boundary point z of a subset K of $\mathbb{C}$ is called a *corner* if there exists a sufficiently small $\epsilon > 0$ such that the intersection of K and the circular disk

$$D = \{\nu \in \mathbb{C} : |\nu - z| < \epsilon\}$$

is contained in a sector of D of degree less than π.

In physics, corners correspond to particularly important states. For example, the *vacuum* state produces a corner in the numerical range (classical or decomposable) of the energy operator. For the classical numerical range, corners of $W(A)$ must be reducing eigenvalues of A; see [11]. This result has been extended to other types of generalized numerical ranges; see [2, 3]. Based on these results, one may guess that corners of $W_m^\alpha(A)$ are λ-points. However, it is not true for $W_m^\alpha(A)$ except when $\alpha = \varepsilon$ as shown in the following result.

PROPOSITION 5.4. *Let $\nu \in W(A)$. If ν^m is a corner of $W_m^\varepsilon(A)$, then ν is a corner of $W(A)$ and hence a reducing eigenvalue of A. If $\alpha \neq \varepsilon$, then for $A =$ $\mathrm{diag}(1,0,\ldots,0)$, the set $W_m^\alpha(A)$ has a corner which is not a λ-point.*

Proof. If z is not a corner of $W(A)$, then there is a smooth curve C in $W(A)$ passing through z. Thus $\tilde{C} = \{\mu^m : \mu \in C\}$ is a smooth curve in $W_m^\varepsilon(A)$ passing through z^m. Thus z^m is not a corner of $W_m^\varepsilon(A)$. Consequently, if z^m is a corner of $W_m^\varepsilon(A)$, then z is a corner of $W(A)$, and hence $Az = \lambda z$ for some reducing eigenvalue of A and the first assertion follows.

For the second assertion, note that $W_m^\alpha(A)$ is a nondegenerate line segment by Theorems 6.1 and 6.2 in Section 6, but 0 is the only λ-point of $W_m^\alpha(A)$. Thus one of the end points of the line segment is a corner, which is not a λ-point. □

Note that even if ν is a corner point of $W(A)$, it does not follow automatically that ν^m is a corner point of $W_m^\varepsilon(A)$ except for some degenerate cases such as $m = n = 2$. For example, if $n \geq 3$, $A = \mathrm{diag}(e^{i\pi/3}, e^{-i\pi/3}) \oplus 0_{n-2}$. Then 0 is a corner of $W(A)$, but 0 is not a corner of $W_m^\varepsilon(A)$. In fact, 0 is not even a boundary point if $m > 2$.

6. Matrices with degenerate permanental ranges. Hu and Tam [14] have studied those matrices with $W_m^\iota(A)$ included in a straight line. In the following, we show that similar results hold for $W_m^\alpha(A)$. We will first state the results with a corollary and some remarks, and then present the intricate proofs.

THEOREM 6.1. *Let $A \in \mathbb{C}^{n\times n}$ and $\alpha \in \Gamma_{m,n}$. Then $W_m^\alpha(A) = \{\mu\}$ if and only if $A = \lambda I$ such that $\lambda^m = \mu$.*

THEOREM 6.2. *Let $A \in \mathbb{C}^{n\times n}$, and $\alpha \in \Gamma_{m,n}$ be such that $\alpha \neq \iota$. The following conditions are equivalent.*

(a) *The set $W_m^\alpha(A)$ is a line segment.*

(b) *The set $W_m^\alpha(A)$ is a line segment lying on a straight line passing through the origin.*

(c) *$e^{it}A$ is a Hermitian matrix for some $t \in [0, 2\pi)$.*

Several remarks are in order in connection with Theorems 6.1 and 6.2. First, Theorem 6.2 is also true for $\alpha = \iota$ except when $n = m = 2$ as shown in [15]. Second, one can deduce the corresponding theorems for $W_m^*(A)$ and $W_m^\perp(A)$. Third, we can deduce the following corollary from Theorem 6.2.

COROLLARY 6.3. *Let $A \in \mathbb{C}^{n\times n}$. Suppose $\alpha \in \Gamma_{m,n}$, where $\iota \neq \alpha$, has occupation numbers $k_1 \geq \cdots \geq k_n$ so that at least one of them is odd. Then $W_m^\alpha(A) \subseteq (0, \infty)$ (respectively, $[0, \infty)$) if and only if μA is positive (semi-)definite for some complex number μ such that $\mu^m = 1$.*

Proof. We prove the positive definite case. The positive semi-definite case follows from a continuity argument.

The ($\Leftarrow$) part is clear. For the converse, we prove the result by induction on $n \geq 2$. First of all, by Theorem 6.2 there exists some $\mu \in \mathbb{C}$ with $|\mu| = 1$ such that μA is Hermitian with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Since all the λ-points are positive, we see that $\lambda_j \neq 0$ for all j.

Suppose $\lambda_1 > 0$. Otherwise, replace μ by $-\mu$. If $n = 2$, the result follows readily from Theorem 3.1. Now consider $n > 2$, and assume that the result is true for matrices of sizes smaller than n. Consider the occupation numbers $k_1 \geq \cdots \geq k_n$ of α. If $k_2 = 0$, then $k_1 = m$ is odd. We have $\mu^{-m}\lambda_j^m$ is positive for all j, and the result follows.

Suppose $k_2 > 0$ and k_t is odd. Since $\alpha \neq \iota$, we see that $k_1 > 1$. As $n \geq 3$, we can construct $\tilde{\alpha}$ from α be removing k_r with $r \neq t$ and $r \neq 1$. Then $\tilde{\alpha} \neq \iota$ and k_t is one of the occupation numbers of $\tilde{\alpha}$. Let $\tilde{m} = m - k_r$ and let $\tilde{A}_j$ be the diagonal matrix obtained from μA by deleting the jth row and column. Then

$$\{\mu^{-m}\lambda_j^{k_r} z : z \in W_{\tilde{m}}^{\tilde{\alpha}}(\tilde{A}_j)\} \subseteq W_m^{\alpha}(A) \subseteq (0, \infty).$$

By the induction assumption and the fact that $\lambda_1 > 0$, we see that $\tilde{A}_2$ and $\tilde{A}_3$ are positive definite matrices. It is then easy to check that $\mu^m = 1$. The result follows. □

We remark that our proof of Corollary 6.3 can handle the case $\alpha = \iota$ if $n \geq 3$. In such case, the induction has to start from $n = 3$.

The rest of this section is devoted to proving Theorems 6.1 and 6.2. We divide the proofs into several lemmas.

LEMMA 6.4. *Suppose $f(z)$ is a non-constant function analytical on the entire plane. Then for any $b > 0$, the set $S = \{f(be^{it}) : t \in [0, 2\pi)\}$ cannot be a subset of a line segment in $\mathbb{C}$.*

Proof. By contradiction. The derivative of $f(e^{it})$ is equal to $if'(z)z$, where $z = e^{it}$. Derivation is in the sense of complex derivation. If the image of the circle belongs to some line, hence $f'(z)z = Zr(z)$, where Z is some constant nonzero complex number and $r(z)$ is real. Thus, by the Cauchy-Riemann equations, $f'(z)z$ is constant on the entire plane. Since f and f' are analytical on the entire plane, we have only the possibility $f'(z) = 0$ everywhere, which contradicts the hypothesis. Thus, the set S cannot have points with constant slope, and hence cannot lie on a straight line. □

The next lemma generalizes the result in [14, Lemma 4].

LEMMA 6.5. *Suppose $A \in \mathbb{C}^{n\times n}$ and $\alpha \in \Gamma_{m,n}$. If $W_m^{\alpha}(A)$ is a subset of a line segment, then A is normal.*

Proof. We prove the lemma by induction on $n \geq 2$. If $n = 2$, by Theorem 3.1, $W_m^{\alpha}(A) = f(W(A))$ for a degree m polynomial. Now, $W_m^{\alpha}(A) \subseteq \mathbb{C}$ has empty interior, and so must $W(A)$. It follows that $W(A)$ is a subset of a line segment, and hence A is normal.

Now, suppose $n > 2$, and the lemma is true for matrices of sizes less than n. Let $A \in \mathbb{C}^{n\times n}$ with $W_m^{\alpha}(A)$ contained in a line segment.

By a suitable unitary similarity transform, we may assume that A is in upper triangular form with diagonal entries $\lambda_1, \ldots, \lambda_n$ so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$.

If the occupation number k_n of α is zero, then one may assume that $\alpha \in \Gamma_{m,n-1}$ and consider $W_m^{\alpha}(A_j)$, where A_j is obtained from A by deleting the jth row and jth column. Clearly, we have

$$W_m^{\alpha}(A_j) \subseteq W_m^{\alpha}(A),$$

and hence $W_m^{\alpha}(A_j)$ is contained in a line segment for $j = 1, \ldots, n$. By the induction assumption, A_j is normal, and hence is a diagonal matrix for all j. Thus A is a diagonal matrix.

Now suppose $k_n > 0$. We continue to use the convention that A_j is obtained from A by deleting the jth row and column. Let $\tilde{m} = m - k_1$ and let $\tilde{\alpha} \in \Gamma_{\tilde{m},n-1}$ have occupation numbers $k_2 \geq \cdots \geq k_n$. Then

$$W_j = \{\lambda_j^{k_1} z : z \in W_{\tilde{m}}^{\tilde{\alpha}}(A_j)\} \subseteq W_m^{\alpha}(A)$$

for all j. If $\lambda_j \neq 0$, one can conclude that $W_{\tilde{m}}^{\tilde{\alpha}}(A_j)$ is contained in a line segment, and hence A_j is normal by the induction assumption. We consider several situations.

First, if $\lambda_3 \neq 0$, then A_1, A_2 and A_3 are all normal. It follows that A is normal.

Second, if $|\lambda_2| > 0 = \lambda_3$, then A_1 and A_2 are normal matrices and hence $A = \begin{pmatrix} \lambda_1 & b \\ 0 & \lambda_2 \end{pmatrix} \oplus 0_{n-2}$. We may replace A by A/λ_1 and assume that $\lambda_1 = 1$. Then $W_m^\alpha(A)$ is the collection of points of the form

$$\mu = \nu(\alpha)^{-1}\mathrm{per}\,(X^*(e_1e_1^* + \lambda_2 e_2 e_2^* + b e_1 e_2^*)X)$$

for some $X \in \mathbb{C}^{n\times m}$ generated from a unitary matrix U using its columns with multiplicities according to α. Let U be a unitary matrix such that the first row of U is $(1,\ldots,1)/\sqrt{n}$ and the second row of U is $(v_1,\ldots,v_n)$ so that $v_j \neq 0$ for all j. Then μ can be written in the form

$$\mu = \nu(\alpha)^{-1}\sum_{j=0}^{m} b^j f_j(v)$$

for some polynomial $f_j(v)$ on the real parts and imaginary parts of the entries of v for $j = 0,\ldots,m$. In particular,

$$f_m(v) = \mathrm{per}\,(J_{m,m}D_v/\sqrt{n}) = (\sqrt{n})^{-m}(m!)\prod_{j=1}^{n} \bar{v}_j^{k_j} \neq 0,$$

where $D_v = \bar{v}_1 I_{k_1} \oplus \cdots \oplus \bar{v}_n I_{k_n}$. For each $t \in [0, 2\pi)$ if we replace v by $e^{-it}v$, the value μ will change to

$$\mu(t) = \nu(\alpha)^{-1}\sum_{j=0}^{m}(e^{it}b)^j f_j(v).$$

By Lemma 6.4, these points in $W_m^\alpha(A)$ cannot all lie in a line segment, unless $b = 0$, i.e., A is normal.

Third, if $|\lambda_1| > 0 = \lambda_2$, then A_2 is normal. Thus A is rank one and $A = \lambda e_1 e_1^* + b e_1 e_2^*$ by a suitable unitary similarity transform. It is normal if and only if $b = 0$. Suppose $b \neq 0$. Let $u = (1,\ldots,1)^t/\sqrt{n}$ and $v = (v_1,\ldots,v_n)^t$ be a unit vector in $\{u\}^\perp$ such that $v_j \neq 0$ for all j. Then there exists a unitary U such that $UAU^* = \lambda_1 uu^* + buv^*$. Construct $X \in \mathbb{C}^{n\times m}$ from U using its columns with multiplicities according to α. Then $\mu = \mathrm{per}\,(X^*AX)/\mathrm{per}\,(X^*X) = \nu(\alpha)^{-1}\sum_{j=0}^m b^j f_j(v)$, for some polynomial $f_j(v)$ on the real parts and imaginary parts of the entries of v for $j = 0,\ldots,m$. In particular,

$$f_m(v) = \mathrm{per}\,(J_{m,m}D_v/\sqrt{n}) = (\sqrt{n})^{-m}(m!)\prod_{j=1}^{n} \bar{v}_j^{k_j} \neq 0,$$

where $D_v = \bar{v}_1 I_{k_1} \oplus \cdots \oplus \bar{v}_n I_{k_n}$. For each $t \in [0, 2\pi)$ if we replace v by $e^{-it}v$, the value per $(X^*AX)/\text{per}\,(X^*X)$ will change to

$$\mu(t) = \nu(\alpha)^{-1} \sum_{j=0}^{m} (e^{it}b)^j f_j(v).$$

By Lemma 6.4, these points cannot lie on a line in $\mathbb{C}$. Thus $W_m^\alpha(A)$ is not a subset of a line segment, which is a contradiction. So, A must be normal.

Finally, suppose $\lambda_j = 0$ for all j. Assume that A is not normal, then we may assume (see [19]) that A is in upper triangular form with the $(1,2)$ entry equal to $b \neq 0$. Let $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$ so that B is 2×2. If D is not normal, then $D \neq 0_{n-2}$. We can decompose α into $\beta_1 \in \Gamma_{m_1,2}$ and $\beta_2 \in \Gamma_{m_2,n-2}$ so that β_1 has occupation numbers $k_1 \geq k_2$ and β_2 has occupation numbers $k_3 \geq \cdots \geq k_n$. Then

$$W = \{z_1 z_2 : z_1 \in W_{m_1}^{\beta_1}(B), z_2 \in W_{m_2}^{\beta_2}(D)\}$$

is a subset of $W_m^\alpha(A)$. By the induction assumption, none of $W_{m_1}^{\beta_1}(B)$ or $W_{m_2}^{\beta_2}(D)$ is contained in a line segment, and neither is W, which is a contradiction. Thus, we see that D must be zero. If $n \geq 4$, one can find a unitary matrix $V \in \mathbb{C}^{(n-2)\times(n-2)}$ such that CV only has nonzero entries in the $(1,1)$, $(2,1)$ and $(2,2)$ entries. Replacing A by $(I_2 \oplus V^*)A(I_2 \oplus V)$, we may assume that

$$A = be_1e_2^* + ce_1e_3^* + d_1e_2e_3^* + d_2e_2e_4^*.$$

If $n = 3$, we can use the same representation with $d_2 = 0$. Let $\{u_1, u_2, u_3\}$ be an orthonormal set in $\mathbb{C}^n$ so that $u_1 = (1, \ldots, 1)^t/\sqrt{n}$ and $(bu_2^* + cu_3^*)/\sqrt{|b|^2 + |c|^2} = (v_1, \ldots, v_n)$ is a unit vector with nonzero entries. Then there exists a unitary matrix U so that

$$U^*AU = bu_1u_2^* + cu_1u_3^* + d_1u_2u_3^* + d_2u_2u_4^*,$$

where u_4^* is the fourth row of U if $n \geq 4$ and the term $d_2u_2u_4^*$ does not exist if $n = 3$. Let $X \in \mathbb{C}^{n\times m}$ be generated from U according to α. Then $W_m^\alpha(A)$ contains the point

$$\begin{aligned}
\mu &= \text{per}\,(X^*AX)/\text{per}\,(X^*X) \\
&= \nu(\alpha)^{-1}\text{per}\,(X^*(e_1(be_2^* + ce_3^*) + d_1e_2e_3^* + d_2e_2e_4^*)X) \\
&= \nu(\alpha)^{-1} \sum_{j=0}^{m} (|b|^2 + |c|^2)^{j/2} f_j(U),
\end{aligned}$$

for some polynomial expression $f_j(U)$ involving the real parts and imaginary parts of the entries of U. In particular, we have

$$f_m(U) = \text{per}\,(J_{m,m}D_v/\sqrt{n}) = (\sqrt{n})^{-m}(m!)\prod_{j=1}^{n} v_j^{k_j} \neq 0.$$

If we replace u_1 by $e^{it}u_1$, then μ will change to

$$\mu(t) = \sum_{j=0}^{m} (e^{it}(|b|^2 + |c|^2)^{1/2})^j f_j(U).$$

By Lemma 6.4, these points in $W_m^\alpha(A)$ cannot all lie in a line segment, which is a contradiction. Hence, A must be normal. □

REMARK 6.6. Note that if $\lambda_j \neq 0$ for all j, then Lemma 6.5 follows easily from Corollary 5.3. If A is singular, then the situation is complicated. One cannot just prove the result for invertible matrices and deduce the result by continuity because of the following reason: if A is singular such that $W_m^\alpha(A)$ is a subset of a line segment, it is unclear how to construct a sequence of invertible matrices $\{A_j\}$ to approach A so that each $W_m^\alpha(A_j)$ is a subset of a line segment.

We finish the proof of Theorem 6.1 and 6.2 by the following lemma.

LEMMA 6.7. *Suppose $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $\alpha \in \Gamma_{m,n}$. If $W_m^\alpha(A) = \{\mu\}$, then $A = \lambda I$ with $\lambda^m = \mu$. If $\alpha \neq \iota$ and if $W_m^\alpha(A)$ is a subset of a line segment, then all λ_j lie in a line passing through the origin.*

Proof. The result can be proved by induction on $n \geq 2$. When $n = 2$, the result follows from Theorem 3.1. If $k_n = 0$, the induction is easy. If $k_n \neq 0$, one can show that any two eigenvalues of A lie on a line passing through the origin. For example, to prove this for λ_1 and λ_2, we consider U so that the jth column of U is e_j for $j \geq 3$. We get a subset of $W_m^\alpha(A)$ lying on a line and from Theorem 3.1 we can conclude that λ_1 and λ_2 lie on a line passing through the origin. □

7. Linear preservers. In [23] and [24], it was shown that a linear operator L on $\mathbb{C}^{n\times n}$ satisfies $W_m^\perp(L(A)) = W_m^\perp(A)$ for all $A \in \mathbb{C}^{n\times n}$ if and only if there exist an mth root of unity μ and a unitary matrix U such that L is of the form $A \mapsto \mu UAU^*$ or $A \mapsto \mu UA^tU^*$. The same result is true for $W_m^\iota(A)$ except when $m = n = 2$ (see [15]); in the special case, there are additional linear preservers for $W_m^\iota(A)$ (see [8], and note that the statement in [15] is not accurate). The purpose of this section is to show that the same result holds for $W_m^\alpha(A)$ for all other α.

THEOREM 7.1. *Let $\alpha \in \Gamma_{m,n}$ be such that $\alpha \neq \iota$. A linear operator L on $\mathbb{C}^{n\times n}$ satisfies $W_m^\alpha(L(A)) = W_m^\alpha(A)$ for all $A \in \mathbb{C}^{n\times n}$ if and only if there exist an mth root of unity μ and a unitary matrix U such that L is of the form $A \mapsto \mu UAU^*$ or $A \mapsto \mu UA^tU^*$.*

Proof. The ($\Leftarrow$) part is clear. We prove the converse in the following. First of all, if $L(A) = 0$, then $\{0\} = W_m^\alpha(L(A)) = W_m^\alpha(A)$. By Theorem 6.1, $A = 0$. Thus L is invertible.

Next, note that $\{1\} = W_m^\alpha(I) = W_m^\alpha(L(I))$. By Theorem 6.1, we see that $L(I) = \mu I$ for some μ with $\mu^m = 1$.

Replace L by $\mu^{-1}L$. Then the modified operator L preserves I and $W_m^\alpha(A)$. We shall show that this modified operator L will map the set of positive definite matrices onto itself. Thus, the modified operator L will map the set of positive definite matrices onto itself. To this end, let A be positive definite. Then $W_m^\alpha(L(A)) = W_m^\alpha(A) \subset (0, \infty)$ by Proposition 2.5. By Theorem 6.2, $e^{it}L(A)$ is Hermitian for some $t \in [0, 2\pi)$.

If $e^{it} \neq \pm 1$, then $I + L(A)$ is not Hermitian. However, $W_m^\alpha(I+A) = W_m^\alpha(L(I+A)) = W_m^\alpha(I+L(A))$ is a real line segment, which is impossible. So, $L(A)$ must be Hermitian. Suppose $L(A)$ is not positive definite. Then there exists $r \geq 0$ so that $rI + L(A)$ is singular. By Proposition 5.1, we have $0 \in W_m^\alpha(rI + L(A)) = W_m^\alpha(L(rI + A)) = W_m^\alpha(rI + A)$. However, $rI + A$ is positive definite, and hence $W_m^\alpha(rI + A) \subseteq (0, \infty)$ by Proposition 2.5. Thus, $L(A)$ must be positive definite.

Notice that L^{-1} also preserves I and $W_m^\alpha(A)$. Thus L^{-1} also maps the set of positive definite matrices into itself. Consequently, L maps the set of positive definite matrices onto itself.

Now, by a result of Schneider [22], there exists an invertible matrix U such that the modified operator L is of the form $A \mapsto UAU^*$ or $A \mapsto UA^tU^*$. Since $L(I) = I$, we see that $UU^* = I$.

Consequently, the original operator L is of the asserted form. □

As mentioned in Section 2, unitary similarity transforms that correspond to a change of the reference frames of the state spaces will not change the permanental ranges $W_m^\alpha(A)$, $W_m^\perp(A)$, $W_m^*(A)$. By Theorem 6.1, we see that these are basically the only transforms that will preserve the permanental ranges - the ranges of possible average values.

The following physical interpretation for A^t occurs. Physically relevant operators are Hermitian and so $A^t = \bar{A}$. The transformation $A \to \bar{A}$ is associated with time reflexion. Invariance under time reflexion implies $A^t = \bar{A} = A$.

8. Derivations of Permanental Compounds. In physics, the concept of derivation is more relevant than the concept of induced operator, although the study of the trace of a particular (positive definite) induced operator may be recognized in the statistical mechanics of independent Bose (or Fermi) systems [4, 21].

In the context of induced operators, the kth derivation $P_m^{(k)}(A)$ of $A \in \mathbb{C}^{n \times n}$ acting on $\mathbb{C}^n_{(m)}$ is determined by the formula

$$P_m(I + tA) = \sum_{k=0}^{m} t^k P_m^{(k)}(A),$$

and the decomposable numerical range of the kth derivation is defined by

$$W_{m,k}^*(A) = \{(P_m^{(k)}(A)x^*, x^*) : x^* \in \mathbb{C}^n_{(m)} \text{ is decomposable, } (x^*, x^*) = 1\}.$$

Similarly, one can define $W_{m,k}^\perp(A)$ and $W_{m,k}^\alpha(A)$. When $k = m$, these sets reduce to $W_m^*(A)$, $W_m^\perp(A)$ and $W_m^\alpha(A)$, respectively. These sets are related to an $m-$particle Boson system in which any k particles interact according to A, and we have

$$W_{m,k}^\perp(A) = \bigcup_{\alpha \in \Gamma_{m,k}} W_{m,k}^\alpha(A).$$

Some physical interpretations are presented in the following. Suppose there are m particles (obeying Bose-Einstein statistics) in the system with (normalized) state vectors $x_1, \ldots, x_n$ such that $\{x_1, \ldots, x_n\}$ is an orthonormal set. Assume there are m_j

particles in the state x_j, for $j = 1, \ldots, n$, where $m_1 + \cdots + m_n = m$. Then their joint state is represented by $x_\alpha = x_{\alpha(1)} * \cdots * x_{\alpha(m)}$, where $\alpha \in \Gamma_{m,n}$ has occupation numbers $m_1, \ldots, m_n$. In physics, first derivations are particularly relevant. For instance, the energy of a collection of photons is the sum of the energies of individual photons, not the product. Observable quantities of an additive nature, such as the kinetic energy, linear momentum or angular momentum, frequently occur. If $A \in \mathbb{C}^{n\times n}$ represents such a quantity for a one-particle system, then the corresponding quantity for an m-particle system is represented by $P_m^{(1)}(A)$. The average value of measurements of such a quantity in the state $x_\alpha = x_{\alpha_1} * \cdots * x_{\alpha_m}$, where $\alpha \in \Gamma_{m,n}$ has occupation numbers $m_1, \ldots, m_n$, is computed by

$$\begin{aligned} \frac{(P_m^{(1)}(A)x_\alpha, x_\alpha)}{(x_\alpha, x_\alpha)} &= \sum_{i=1}^{m} \frac{(x_{\alpha_1} * \cdots * Ax_{\alpha_i} * \cdots * x_{\alpha_m}, x_\alpha)}{(x_\alpha, x_\alpha)} \\ &= \sum_{i=1}^{m} (Ax_{\alpha_i}, x_{\alpha_i}) = \sum_{i=1}^{n} m_i (Ax_i, x_i). \end{aligned} \tag{1}$$

Recall that for $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$, the c-numerical range of A is defined by

$$W_c(A) = \left\{ \sum_{j=1}^{n} c_j (Ax_j, x_j) : \{x_1, \ldots, x_n\} \text{ is an orthonormal basis} \right\}.$$

Thus the collection of all possible average values of measurements of $P_m^{(1)}(A)$ for the system characterized by the occupation numbers $m_1, \cdots, m_n$ is the c-numerical range of A with $c = (m_1, \ldots, m_n)$.

The energy operator of a system of independent Bosons, being the sum of the energy operator of each Boson, is the first derivation of a single Boson energy operator. An important physical effect characteristic of Bose Systems is the well known Einstein condensation, see [4, 21] for some interesting treatments of the standard quantum formalisms. In terms of the matrix model, this fact is equivalent to the inclusion $W^*_{m,1}(A) \subseteq W^\varepsilon_{m,1}(A)$. This, and Proposition 2.5, suggests that the following inclusion occurs, at least for A positive semi-definite,

$$W^*_{m,k}(A) \subseteq W^\varepsilon_{m,k}(A). \tag{2}$$

Whether (2) holds for a general A is an open problem. Nonetheless, we have the following result confirming the set equality in (2) when $k = 1$.

PROPOSITION 8.1. *Let $A \in \mathbb{C}^{n\times n}$. We have*

$$W^*_{m,1}(A) = W^\varepsilon_{m,1}(A) = mW(A).$$

Proof. The equality $mW(A) = W^\varepsilon_{m,1}(A)$ follows readily from (1). Also, the inclusion $mW(A) \subseteq W^*_{m,1}(A)$ is clear. We consider the reverse inclusion. Let

$$z(x^*) = \frac{(P_m^{(1)}(A)x^*, x^*)}{(x^*, x^*)} \in W^*_{m,1}(A), \tag{3}$$

where the decomposable tensor $x^* = x_1 * \cdots * x_m$ need not have unit length. Consider the $n \times n$ complex matrix $C = [c_{\mu\nu}]$

$$c_{\mu\nu} = \frac{1}{\operatorname{per} Z} \sum_{ij} (E_{\nu\mu} x_j, x_i) \operatorname{per} Z(i,j),$$

where $Z = [(x_j, x_i)] \in \mathbb{C}^{m\times m}$ and $Z(i,j)$ denotes the submatrix obtained from Z deleting the ith row and the jth column. Since C depends on x^*, the notation $C(x^*)$ will occasionally be used for clarity. It is straightforward to prove that trC = m. We observe that the matrices $C(x^*)$ and $C(x_\varepsilon)$ are positive semidefinite. The argument is based on the following fact: If $\operatorname{tr}(\mathrm{AC}) \geq 0$ for any positive semidefinite A, then C is positive semidefinite.

Since

$$(P_m^{(1)}(A)x^*, x^*) = \sum_{ij} (Ax_j, x_i)(\operatorname{per}(Z(i,j))),$$

and

$$\sum_{\mu\nu} a_{\mu\nu} \sum_{ij} (E_{\mu\nu} x_j, x_i)(\operatorname{per}(Z(i,j))) = \operatorname{tr}(\mathrm{AC}),$$

from (3) we conclude that $z(x^*) = \operatorname{tr}(\mathrm{AC})$. Analogously, we have that $W^{\varepsilon}_{m,1}(A) = W_{C(x_\varepsilon)}(A)$.

Recall that $mW(A) = W^{\varepsilon}_{m,1}(A)$. The n-tuple of the eigenvalues of $C(x^*)$ is majorized by the n-tuple of the eigenvalues of $C(x_\varepsilon)$ which is equal to $(m, 0, \cdots, 0)$. Now, by a result in [7]

$$z(x^*) \in W_{C(x_\varepsilon)}(A) = mW(A).$$

The proof is complete. $\square$

We close with a description of the physical interpretation of the decomposable numerical range of $W^{\perp}_{m,k}(A)$. Quantities of an interactive nature, such as a 2-body, 3-body or many-body interactions (or many-body operators), also occur in physics. A k-particle interaction is only effective in an m-particle system if $m \geq k$ and, in some simple circumstances, may be described as a derivation $P_m^{(k)}(A)$. (Under more complex circumstances it may be necessary to consider different operators $A_1, \ldots, A_l$ acting simultaneously on different particles, or linear combinations of derivations.) Denote by $M(m'_1, \cdots, m'_n; m_1, \cdots, m_n)$ the matrix which is obtained from M by repeating row j, m'_j times for $j = 1, \ldots, n$. The average value of measurements of $P_m^{(k)}(A)$ in the state $x_\alpha = x_{\alpha_1} * \cdots * x_{\alpha_m}$, where $\alpha \in \Gamma_{m,n}$ has occupation numbers $m_1, \ldots, m_n$, is then computed by

$$\frac{(P_m^{(k)}(A)x_\alpha, x_\alpha)}{(x_\alpha, x_\alpha)} = \sum_{\omega \in Q_{k,m}} \frac{(y_1 * \cdots * Ay_{\omega(1)} * \cdots * Ay_{\omega(q)} * \cdots * y_p, x_\alpha)}{(x_\alpha, x_\alpha)}$$

$$= \sum_{(m'_1,\cdots,m'_n)\in \mathcal{Q}(m_1,\cdots,m_n)} \prod_{i=1}^{n} \binom{m_i}{m'_i} \frac{1}{m'_i!} \operatorname{per}(A(m'_1, \cdots, m'_n; m'_1, \cdots, m'_n)),$$

where $(y_1, \ldots, y_m) = (x_{\alpha_1}, \ldots, x_{\alpha_m})$, $Q_{k,m}$ denotes the set of increasing sequences of k integers between 1 and m and $\mathcal{Q}(m_1, \cdots, m_n)$ is the set of sequences $(m_1', \cdots, m_n')$ such that $m_j' \leq m_j$ for $j = 1, \ldots, n$, such that $m_1' + \cdots + m_n' = k$. Thus $W_{m,k}^{\perp}(A)$ can be regarded as the collection of all possible average values of measurements of $P_m^{(k)}(A)$ in the state x_α of such a system and is very useful in the study of it.

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