The Electronic Journal of Linear Algebra.
A publication of the International Linear Algebra Society.
Volume 7, pp. 112-151, September 2000.
ISSN 1081-3810. http://math.technion.ac.il/iic/ela

ELA

CANONICAL FORMS FOR DOUBLY STRUCTURED MATRICES AND PENCILS*

CHRISTIAN MEHL†, VOLKER MEHRMANN‡, AND HONGGUO XU§

Abstract. In this paper, canonical forms under structure-preserving equivalence transformations are presented for matrices and matrix pencils that have a multiple structure, which is either an H-self-adjoint or H-skew-adjoint structure, where the matrix H is a complex nonsingular Hermitian or skew-Hermitian matrix. Matrices and pencils of such multiple structures arise, for example, in quantum chemistry in Hartree–Fock models or random phase approximation.

Key words. Indefinite inner product, Self-adjoint matrix, Skew-adjoint matrix, Matrix pencil, Canonical form.

AMS subject classifications. 15A21, 15A22, 15A57

1. Introduction. Canonical forms for matrices and matrix pencils have been studied for more than a hundred years since the work of Jordan, Kronecker, and Weierstrass; see [5]. In recent years, motivated by applications in control theory as well as quantum physics and quantum chemistry, there has been a revived interest in such canonical forms for matrices and pencils that have algebraic structures, such as Lie groups and Lie algebras. While the possible invariants were characterized some time ago [2], the emphasis in the new results lies on structure-preserving equivalence transformations; see, e.g., [1, 13, 14, 15, 16].

In this paper, we derive canonical forms under structure-preserving equivalence transformations for matrices and matrix pencils with multiple structure.

DEFINITION 1.1. Let $H \in \mathbb{C}^{n\times n}$ be a nonsingular Hermitian or skew-Hermitian matrix, and let $X \in \mathbb{C}^{n\times n}$.

1. X is called H-self-adjoint if $X^*H = HX$.
2. X is called H-skew-adjoint if $X^*H = -HX$.

Canonical forms for pairs (A, H), where H is Hermitian or skew-Hermitian nonsingular and A is H-self-adjoint or H-skew-adjoint, are well known in the literature; see, e.g., [7, 11]. These forms are obtained under transformations of the form

$$(A, H) \mapsto (P^{-1}AP,\ P^*HP),$$

*Received by the editors on 16 May 2000. Accepted for publication on 27 August 2000. Handling Editor: Daniel Hershkowitz.

†Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany (mehl@mathematik.tu-chemnitz.de). A large part of this author's work was performed while he was visiting the College of William and Mary, Department of Mathematics, P.O. Box 8795, Williamsburg, VA 23187-8795, USA, and while he was supported by Deutsche Forschungsgemeinschaft, Me 1797/1-1, Berechnung von Normalformen für strukturierte Matrizenbüschel.

‡Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany (mehrmann@mathematik.tu-chemnitz.de). Supported by Deutsche Forschungsgemeinschaft within SFB393, Project: Algebraische Zerlegungsmethoden.

§Department of Mathematics, University of Kansas, 405 Snow Hall, Lawrence, KS 66045, USA (xu@math.ukans.edu).

where P is nonsingular. Here, we are interested in canonical forms for matrix triples (A, H, G), where G and H are Hermitian or skew-Hermitian nonsingular and A is doubly structured with respect to G and H, i.e., A is H-self-adjoint or H-skew-adjoint and at the same time G-self-adjoint or G-skew-adjoint. We are also interested in the pencil case, i.e., we will also consider pencils $\varrho A - B$, where both A and B are doubly structured with respect to H and G.

The main motivation for our interest in these types of matrices and pencils arises from quantum chemistry. Response function models lead to the problem of solving the generalized eigenvalue problem with a matrix pencil of the form

$$\lambda \mathcal{E}_0 - \mathcal{A}_0 := \lambda \begin{bmatrix} C & Z \\ -Z & -C \end{bmatrix} - \begin{bmatrix} E & F \\ F & E \end{bmatrix}, \tag{1}$$

where $E, F, C, Z \in \mathbb{C}^{n\times n}, E = E^*, F = F^*, C = C^*, Z = -Z^*$; see [8, 17]. Furthermore, there are important special cases in which the pencil has even further structure. For example, the simplest response function model is the time-dependent Hartree–Fock model, also called the random phase approximation (RPA). In this case, C is the identity and Z is the zero matrix; see [8, 17]. Thus, the generalized eigenvalue problem (1) reduces to the problem of finding the eigenvalues of the matrix

$$\mathcal{L}_0 = \begin{bmatrix} E & F \\ -F & -E \end{bmatrix}, \tag{2}$$

where E, F are as in (1). For stable Hartree–Fock ground-state wave-functions, it is furthermore known that $E - F$ and $E + F$ are positive definite; see [8].

In other applications, however, such as in multiconfigurational RPA [8], it is not even guaranteed that the matrix $\mathcal{E}_0$ in (1) is nonsingular.

It is easy to see that the matrices $\mathcal{E}_0$, $\mathcal{A}_0$ in (1) and $\mathcal{L}_0$ in (2) are doubly structured. With

$$G = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}, \quad H = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

we have that $\mathcal{E}_0$ is I-self-adjoint and H-skew-adjoint, $\mathcal{A}_0$ is I-self-adjoint and H-self-adjoint, while $\mathcal{L}_0$ is G-self-adjoint and J-skew-adjoint.

When designing structure-preserving numerical methods for large-scale structure eigenvalue problems, difficulties in the convergence of the methods were sometimes observed in [3, 4] that have to do with the invariants of these pencils under structure-preserving equivalence transformations; see also [1]. It is another motivation for our work to derive canonical forms that allow a better understanding of those properties of the pencils that lead to these difficulties.

We will derive the canonical form for matrix triples (A, H, G) under structure-preserving transformations of the form

$$(A, H, G) \mapsto (P^{-1}AP,\ P^*HP,\ P^*GP),$$

where P is nonsingular. This preserves the (skew-) Hermitian structure of H and G and also the structure of A with respect to H and G. Based on the classical results (see

Section 2), we clearly have canonical forms for (A,H), (A,G), or the pencil $\varrho H-G$, and hence the invariants of the pairs (A,H) and (A,G) as well as the invariants of the pencil $\varrho H-G$ under congruence are invariants of the triple (A,H,G).

It is our goal to obtain a canonical form that displays simultaneously the Jordan structure of A and the invariants of the canonical forms of (A,H) and (A,G). In general this is a very difficult problem; such a form may not even exist. Consider the following example.

EXAMPLE 1.2. Consider matrices

$$A=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix},\quad G=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},\quad\text{and}\quad H=\begin{bmatrix}0&0&1\\0&1&1\\1&1&0\end{bmatrix}.$$

Then A is G-self-adjoint and H-self-adjoint. But it is impossible to simultaneously decompose A, H, and G further into smaller block-diagonal forms. This follows from the obvious fact that the pencil $\varrho G-H$ cannot be decomposed further. On the other hand, A has the Jordan canonical form

$$\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Hence, both (A,G) and (A,H) are decomposable into smaller blocks (see Theorems 3.1 and 3.3 below).

Due to this difficulty, we restrict ourselves to an important special case. In most applications, the matrices H and G that induce the structure are contained in the set

$$\mathcal{S}=\left\{I_n,\quad\begin{bmatrix}0&I_m\\-I_m&0\end{bmatrix},\quad\begin{bmatrix}I_m&0\\0&-I_{n-m}\end{bmatrix},\begin{bmatrix}0&I_m\\I_m&0\end{bmatrix},m,n\in\mathbb{N}\right\}.$$

If this is the case, then the pencil $\varrho H-G$ is nondefective.

DEFINITION 1.3. Let $\varrho A-B\in\mathbb{C}^{n\times n}$ be a matrix pencil. We say that $\varrho A-B$ is nondefective if there exist nonsingular matrices $P,Q\in\mathbb{C}^{n\times n}$ such that both PAQ and PBQ are diagonal.

We will show that if G,H are Hermitian nonsingular such that the pencil $\varrho H-G$ is nondefective, then a canonical form for the triple (A,H,G) exists, which is also unique except for the permutation of blocks. In particular, this canonical form includes the Jordan structure of A, and also the canonical forms of the pairs (A,H) and (A,G) and the pencil $\varrho H-G$ can be easily read off.

The paper is organized as follows. After providing some preliminary results in Section 2, we review canonical forms for matrices that are structured with respect to only one Hermitian matrix in Section 3. In Section 4, we discuss doubly structured matrices, and in Section 5, we discuss canonical forms for structured pencils of the form $\lambda\mathcal{A}-\mathcal{B}$, where both $\mathcal{A}$ and $\mathcal{B}$ are singly or doubly structured matrices.

2. Preliminaries. Throughout the paper, we use the following notation.

By $\sigma(A)$ we denote the spectrum of the matrix A. $\mathcal{J}_p(\lambda)$ denotes the $p\times p$ upper-triangular Jordan block with eigenvalue λ. By $\operatorname{sign}(t)$ we mean the sign of a real number $t \in \mathbb{R}\backslash\{0\}$. $A = A_1 \oplus \cdots \oplus A_m$ stands for the block-diagonal matrix A with diagonal blocks $A_1, \ldots, A_m$. Moreover, we use the abbreviation A^{-*} for $(A^*)^{-1}$.

Furthermore, we introduce the following $p \times p$ matrices:

$$Z_p := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}, \quad D_p := \begin{bmatrix} (-1)^2 & & 0 \\ & \ddots & \\ 0 & & (-1)^{p+1} \end{bmatrix},$$

$$\text{and} \quad F_p := \begin{bmatrix} 0 & & (-1)^2 \\ & \iddots & \\ (-1)^{p+1} & & 0 \end{bmatrix}.$$

Note that $F_p \in \mathbb{R}^{p \times p}$ is symmetric if p is odd and skew-symmetric if p is even, whereas Z_p and D_p are symmetric for all p. We list some properties of these matrices and the matrix $\mathcal{J}_p(0)$, which can be easily verified, and will be used in what follows.

LEMMA 2.1. *Let $p \in \mathbb{N}$.*

1. $Z_p^2 = I_p, \quad D_p^2 = I_p, \quad F_p^2 = (-1)^{p+1} I_p$.
2. $F_p Z_p = D_p = (-1)^{p+1} Z_p F_p, \quad D_p F_p = Z_p = (-1)^{p+1} F_p D_p$.
3. $D_p Z_p = F_p = (-1)^{p+1} Z_p D_p, \quad F_p Z_p F_p = Z_p$.
4. $Z_p^{-1} \mathcal{J}_p(0) Z_p = \mathcal{J}_p(0)^*$.
5. $D_p^{-1} \mathcal{J}_p(0) D_p = -\mathcal{J}_p(0)$.
6. $F_p^{-1} \mathcal{J}_p(0) F_p = -\mathcal{J}_p(0)^*$.

Another important and well-known result that will frequently be used throughout the paper is the following [5].

LEMMA 2.2. *Let A, B, X be square matrices such that the spectra of A and B are disjoint. If $AX = XB$, then $X = 0$.*

Finally, we review the canonical forms for regular Hermitian pencils, i.e., regular pencils $\varrho H - G$, where both H and G are Hermitian. An arbitrary matrix pencil $\lambda A - B$ is called regular if $\det(\varrho A - B) \not\equiv 0$. In this case, it makes sense to speak of eigenvalues of a pencil. Introducing homogeneous parameters $\alpha A - \beta B$ [6], the eigenvalues of $\alpha A - \beta B$ can be defined as pairs $(\alpha, \beta) \in \mathbb{C}^2\backslash\{0\}$ such that

$$\alpha A x - \beta B x = 0 \quad \text{for an } x \in \mathbb{C}^n\backslash\{0\}.$$

Obviously, $(t\alpha, t\beta)$ represents the same eigenvalue for all $t \in \mathbb{C}\backslash\{0\}$; thus, we denote them by $\lambda = \frac{\alpha}{\beta}$ if $\beta \neq 0$. Pairs $(\alpha, 0), \alpha \neq 0$, represent the eigenvalue infinity of $\alpha A - \beta B$ that we will denote by ∞.

The following result goes back to results from Weierstrass [19] and Kronecker [10].

THEOREM 2.3. *Let $\varrho H - G$ be a regular Hermitian pencil. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that*

$$P^*(\varrho H - G)P = (\varrho H_1 - G_1) \oplus \cdots \oplus (\varrho H_l - G_l), \tag{3}$$

where the blocks $\varrho H_j - G_j$ have one and only one of the following forms:

1. blocks associated with paired nonreal eigenvalues λ, $\overline{\lambda}$, *where* $\mathrm{Im}(\lambda) > 0$*:*

$$\varrho H_j - G_j = \varrho \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mathcal{J}_r(\lambda) \\ \mathcal{J}_r(\lambda)^* & 0 \end{bmatrix}; \tag{4}$$

2. blocks associated with real eigenvalues λ *and sign* $\varepsilon \in \{1, -1\}$*:*

$$\varrho H_j - G_j = \varrho\varepsilon Z_r - \varepsilon Z_r \mathcal{J}_r(\lambda) = \varrho\varepsilon \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} 0 & & & \lambda \\ & & \lambda & 1 \\ & \iddots & \iddots & \\ \lambda & 1 & & 0 \end{bmatrix}; \tag{5}$$

3. blocks associated with the eigenvalue ∞ *and sign* $\varepsilon \in \{1, -1\}$*:*

$$\varrho H_j - G_j = \varrho\varepsilon Z_r \mathcal{J}_r(0) - \varepsilon Z_r = \varrho\varepsilon \begin{bmatrix} 0 & & & 0 \\ & & 0 & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}.$$

Moreover, the decomposition (3) is unique up to a block permutation that exchanges blocks $\rho H_i - G_i$*.*

Proof. For a full proof, see [18], Lemmas 1–4. There, the result is shown without the additional condition $\mathrm{Im}(\lambda) > 0$ for the blocks associated with nonreal eigenvalues, but applying a permutation, we may always place the block that is associated with the eigenvalue λ in the $(1, 2)$-block position of the form in (4). □

If $\varrho H - G$ is nondefective, then we immediately have the following corollary.

Corollary 2.4. *Let* $\varrho H - G$ *be a nondefective Hermitian pencil, where both* H *and* G *are nonsingular. Then there exists a nonsingular matrix* $P \in \mathbb{C}^{n\times n}$ *such that*

$$P^*(\varrho H - G)P = (\varrho H_1 - G_1) \oplus \cdots \oplus (\varrho H_l - G_l),$$

where the spectra of $\varrho H_j - G_j$ *and* $\varrho H_l - G_l$ *are disjoint for* $j \neq l$*, and where each block* $\varrho H_j - G_j$ *has either only one pair of complex conjugate eigenvalues or only one single real eigenvalue. Moreover, the block* $\varrho H_j - G_j$ *has one and only one of the following forms:*

1. blocks with nonreal eigenvalues λ, $\overline{\lambda}$, *where* $\mathrm{Im}\lambda > 0$ *and* $q \in \mathbb{N}$*:*

$$\varrho H_j - G_j = \varrho \begin{bmatrix} 0 & I_q \\ I_q & 0 \end{bmatrix} - \begin{bmatrix} 0 & \lambda I_q \\ \overline{\lambda} I_q & 0 \end{bmatrix}; \tag{6}$$

2. blocks with real eigenvalue λ, *where* $q, p \in \mathbb{N}, p \leq q$*:*

$$\varrho H_j - G_j = \begin{bmatrix} I_p & 0 \\ 0 & -I_{q-p} \end{bmatrix} - \lambda \begin{bmatrix} I_p & 0 \\ 0 & -I_{q-p} \end{bmatrix}. \tag{7}$$

Moreover, the decomposition is unique up to a block permutation.

Proof. By Theorem 2.3, there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that the pencil $P^*(\varrho H - G)P$ is in canonical form (3). Since the pencil is nondefective and

H is nonsingular, only blocks of the forms (4) and (5) can appear and the parameters r in these blocks all have to be equal to one. A number of q blocks of the form (4) with the same eigenvalue $\lambda \in \mathbb{C}\backslash\mathbb{R}$ produce one block of the form (6) after an appropriate permutation that combines the q blocks in a single block is applied. On the other hand, q blocks of the form (5) with the same eigenvalue $\lambda \in \mathbb{R}$ (p blocks with sign $\varepsilon = +1$ and $q-p$ blocks with sign $\varepsilon = -1$) produce a block of the form (7), again after an appropriate permutation is applied. ☐

REMARK 2.5. Following Theorem 2.3 and Corollary 2.4, it is obvious that if $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is an eigenvalue of $\varrho H - G$, then so is $\overline{\lambda}$, and both eigenvalues have the same Jordan structure.

3. Singly structured matrices. In this section, we review the well-known canonical forms for H-self-adjoint matrices and H-skew-adjoint matrices, where H always denotes a complex nonsingular Hermitian $n \times n$ matrix.

THEOREM 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be H-self-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$P^{-1}AP = A_1 \oplus \cdots \oplus A_k \quad \textit{and} \quad P^*HP = H_1 \oplus \cdots \oplus H_k, \tag{8}$$

where A_j and H_j are of the same size and the pair (A_j, H_j) has one and only one of the following forms:

1. blocks associated with real eigenvalues:

$$A_j = \mathcal{J}_p(\lambda) \quad \textit{and} \quad H_j = \varepsilon Z_p,$$

where $\lambda \in \mathbb{R}$, $p \in \mathbb{N}$, and $\varepsilon \in \{1, -1\}$;

2. blocks associated with a pair of nonreal eigenvalues:

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\overline{\lambda}) \end{bmatrix} \quad \textit{and} \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \tag{9}$$

where $\lambda \in \mathbb{C}\backslash\mathbb{R}$ with $\mathrm{Im}(\lambda) > 0$ and $p \in \mathbb{N}$.

Moreover, the form $(P^{-1}AP, P^*HP)$ of (A, H) is uniquely determined up to the permutation of blocks.

Proof. See, e.g., [7]. ☐

Even though (8) is unique only up to a permutation of blocks, we call it a *canonical form* of the pair (A, H).

REMARK 3.2. In some instances, it will turn out be useful to use a slightly different form for the blocks of type (9) in (8). Multiplying the matrices from both sides by $I_p \oplus Z_p$, one finds that (9) takes the form

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\lambda)^* \end{bmatrix} \quad \text{and} \quad H_j = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}. \tag{10}$$

Using the same transformation, we can also get back from the form (10) to the form (9). This transformation will frequently be used in the following and its application will be called the *Z-trick*.

Apart from the eigenvalues of an H-self-adjoint matrix A, the parameters ε that are associated with blocks to real eigenvalues are invariants of the pair (A,H). The collection of these parameters is sometimes referred to as the *sign characteristic*; see, e.g., [7] and [11]. To highlight that these parameters are related to the matrix H (we will soon have to deal with two structures), we will use the term *H-structure indices* in what follows.

THEOREM 3.3. *Let $S\in\mathbb{C}^{n\times n}$ be H-skew-adjoint. Then there exists a nonsingular matrix $P\in\mathbb{C}^{n\times n}$ such that*

$$P^{-1}SP = S_1\oplus\cdots\oplus S_k \quad \textit{and} \quad P^*HP = H_1\oplus\cdots\oplus H_k, \tag{11}$$

where S_j and H_j are of the same size and each pair (S_j,H_j) has one and only one of the following forms:

1. blocks associated with purely imaginary eigenvalues:

$$S_j = i\mathcal{J}_p(\lambda) \quad \textit{and} \quad H_j=\varepsilon Z_p, \tag{12}$$

where $\lambda\in\mathbb{R}$, $p\in\mathbb{N}$, and $\varepsilon\in\{1,-1\}$;

2. blocks associated with a pair of non-purely imaginary eigenvalues:

$$S_j = \begin{bmatrix} i\mathcal{J}_p(\lambda) & 0 \\ 0 & i\mathcal{J}_p(\overline{\lambda}) \end{bmatrix} \quad \textit{and} \quad H=\begin{bmatrix} 0 & Z_p \\ Z_p & 0\end{bmatrix},$$

where $\lambda\in\mathbb{C}\backslash\mathbb{R}$ with $\mathrm{Im}(\lambda)>0$ and $p\in\mathbb{N}$.

Moreover, the form $(P^{-1}SP, P^*HP)$ of (S,H) is uniquely determined up to a permutation of blocks.

Proof. This follows directly from Theorem 3.1, considering the H-self-adjoint matrix iS. ∎

Again, we will call the parameter ε in (12) the H-structure index of the block S_j in (12). Moreover, the form (11) will be called the canonical form of the pair (S,H).

REMARK 3.4. From Theorems 3.1 and 3.3, it is easy to find the following symmetries in the spectra of H-self-adjoint and H-skew-adjoint matrices. If $\lambda\notin\mathbb{R}$ is an eigenvalue of the H-self-adjoint matrix A, then so is $\overline{\lambda}$, and both eigenvalues have the same Jordan structure. If $\lambda\notin i\mathbb{R}$ is an eigenvalue of the H-skew-adjoint matrix A, then so is $-\overline{\lambda}$, and both eigenvalues have the same Jordan structure.

4. Doubly structured matrices. In this section, we give canonical forms for matrices that are doubly structured with respect to Hermitian or skew-Hermitian nonsingular matrices G and H. First, we note that by Theorem 3.1, Jordan blocks associated with real eigenvalues in the self-adjoint case (or purely imaginary eigenvalues in the skew-adjoint case) have structure indices with respect to G and/or H. We will call these indices the G- and H-structure indices of A, respectively.

Moreover, we may always assume that G and H are Hermitian. Otherwise, we may consider iG or iH, respectively, keeping in mind the following remark.

REMARK 4.1. Let $H\in\mathbb{C}^{n\times n}$ be nonsingular and Hermitian or skew-Hermitian and let $A\in\mathbb{C}^{n\times n}$. Then the following conditions hold.

1. A is H-self-adjoint if and only if A is iH-self-adjoint.

2. A is H-skew-adjoint if and only if A is iH-skew-adjoint.

3. A is H-self-adjoint if and only if iA is H-skew-adjoint.

Remark 4.1 implies in particular that we may assume that the structure on A induced by one of the matrices G and H, say H, is the structure of a self-adjoint matrix. In other words, we may assume that A is H-self-adjoint. Otherwise, we may consider iA. Hence, it remains to discuss the following cases:

(i) matrices that are H-self-adjoint and G-self-adjoint (Section 4.1), and

(ii) matrices that are H-self-adjoint and G-skew-adjoint (Section 4.2).

Finally, we always assume that the pencil $\varrho H - G$ is nondefective.

REMARK 4.2. Instead of requiring $\varrho H - G$ to be nondefective, we may as well consider the generalization of this case, that for the matrices A, G, H, at least one of the three pencils $\varrho H - G$, $\varrho H - HA$, and $\varrho G - GA$ is nondefective. For example, if A is nonsingular and both H- and G-self-adjoint, then we can consider the matrix triple $(H^{-1}G, H, HA)$ for which H, HA are Hermitian and $H^{-1}G$ is H- and HA-self-adjoint, since $(H^{-1}G)^* = GH^{-1}$. Thus, if $\varrho H - HA$ is nondefective, then we can get the canonical form of this new triple. But once we have this, we can easily get the canonical form of the original triple (A, H, G). So our results will cover more general cases.

4.1. Matrices that are H-self-adjoint and G-self-adjoint. In this section, we will derive a canonical form for matrices that are self-adjoint with respect to nonsingular Hermitian matrices H and G such that the pencil $\varrho H - G$ is nondefective. For the proof of our main result, the following lemma will be needed. Note that the lemma is also true for the case that the pencil $\varrho H - G$ is not nondefective.

LEMMA 4.3. *Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian and nonsingular. Let $A \in \mathbb{C}^{n\times n}$ be H-self-adjoint and G-self-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \cdots \oplus A_k, \\ P^*HP &= H_1 \oplus \cdots \oplus H_k, \\ P^*GP &= G_1 \oplus \cdots \oplus G_k, \end{aligned}$$

where A_j, H_j, and G_j have corresponding sizes. Moreover, each pencil $\varrho H_j - G_j$ has as spectrum either $\{\gamma_j, \overline{\gamma}_j\}$ for some $\gamma_j \in \mathbb{C}\backslash\mathbb{R}$ or $\{\gamma_j\}$ for some $\gamma_j \in \mathbb{R}$, and the spectra of two subpencils $\varrho H_j - G_j$ and $\varrho H_l - G_l$, $j \neq l$, are disjoint.

Proof. By Theorem 2.3 and by applying an appropriate permutation that combines blocks that display the same eigenvalues $\{\gamma_1, \overline{\gamma}_1\}$ in one large block $\varrho H_1 - G_1$, there exists a nonsingular matrix $Q \in \mathbb{C}^{n\times n}$ such that

$$Q^*GQ = \begin{bmatrix} G_1 & 0 \\ 0 & \tilde{G}_2 \end{bmatrix}, \quad Q^*HQ = \begin{bmatrix} H_1 & 0 \\ 0 & \tilde{H}_2 \end{bmatrix}, \quad \text{and} \quad Q^{-1}AQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where the pencil $\varrho H_1 - G_1$ has as spectrum either $\{\gamma_1, \overline{\gamma}_1\}$ for some $\gamma_1 \in \mathbb{C}\backslash\mathbb{R}$ or $\{\gamma_1\}$ for some $\gamma_1 \in \mathbb{R}$ and such that the spectra of the pencils $\varrho H_1 - G_1$ and $\varrho \tilde{H}_2 - \tilde{G}_2$ are disjoint. Since A is H-self-adjoint and G-self-adjoint, we obtain that

$$\begin{bmatrix} A_{11}^*H_1 & A_{21}^*\tilde{H}_2 \\ A_{12}^*H_1 & A_{22}^*\tilde{H}_2 \end{bmatrix} = \begin{bmatrix} H_1A_{11} & H_1A_{12} \\ \tilde{H}_2A_{21} & \tilde{H}_2A_{22} \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} A_{11}^*G_1 & A_{21}^*\tilde{G}_2 \\ A_{12}^*G_1 & A_{22}^*\tilde{G}_2 \end{bmatrix} = \begin{bmatrix} G_1A_{11} & G_1A_{12} \\ \tilde{G}_2A_{21} & \tilde{G}_2A_{22} \end{bmatrix}.$$

Since with G also $\tilde{G}_2$ is nonsingular, this implies that

$$A_{21}^*\tilde{H}_2\tilde{G}_2^{-1} = H_1A_{12}\tilde{G}_2^{-1} = H_1G_1^{-1}G_1A_{12}\tilde{G}_2^{-1} = H_1G_1^{-1}A_{21}^*.$$

Since the pencils $\varrho H_1 - G_1$ and $\varrho\tilde{H}_2 - \tilde{G}_2$ have disjoint spectra, we obtain that $A_{21}^* = 0$ and therefore $A_{12} = H_1^{-1}A_{21}^*\tilde{H}_2 = 0$. The remainder of the proof follows by induction. □

THEOREM 4.4. *Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian and nonsingular such that the pencil $\varrho H - G$ is nondefective. Let $A \in \mathbb{C}^{n\times n}$ be H-self-adjoint and G-self-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \cdots \oplus A_k, \\ P^*GP &= G_1 \oplus \cdots \oplus G_k, \\ P^*HP &= H_1 \oplus \cdots \oplus H_k, \end{aligned} \tag{13}$$

where the blocks A_j, G_j, H_j have corresponding sizes and are of one and only one of the following forms.

Type (1):

$$A_j = \mathcal{J}_p(\lambda), \quad H_j = \varepsilon Z_p, \quad \text{and} \quad G_j = \varepsilon\gamma Z_p,$$

where $\lambda \in \mathbb{R}$, $p \in \mathbb{N}$, $\varepsilon \in \{1, -1\}$, and $\gamma \in \mathbb{R}\backslash\{0\}$. The H-structure index of A_j is ε and the G-structure index of A_j is $\operatorname{sign}(\varepsilon\gamma)$.

Type (2):

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\lambda) \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \quad \text{and} \quad G_j = \begin{bmatrix} 0 & \gamma Z_p \\ \overline{\gamma} Z_p & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{R}$, $p \in \mathbb{N}$, and $\gamma \in \mathbb{C}$, $Im(\gamma) > 0$. The H-structure indices of A_j are $1, -1$ and the G-structure indices of A_j are $1, -1$.

Type (3):

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\overline{\lambda}) \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \quad \text{and} \quad G_j = \begin{bmatrix} 0 & \gamma Z_p \\ \overline{\gamma} Z_p & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{C}\backslash\mathbb{R}$, $p \in \mathbb{N}$, and $\gamma \in \mathbb{C}\backslash\{0\}$, where $Im(\gamma) \geq 0$.

Moreover, the canonical form (13) is unique up to permutation of blocks.

Proof. By Lemma 4.3, we may assume that the pencil $\varrho H - G$ has as eigenvalues either $\gamma, \overline{\gamma}$ for some $\gamma \in \mathbb{C}\backslash\mathbb{R}$ or γ for some $\gamma \in \mathbb{R}$.

Case 1: $\gamma \in \mathbb{R}$.

Since the pencil $\varrho H - G$ is nondefective, by Corollary 2.4, there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$P^*(\varrho H - G)P = \varrho \begin{bmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{bmatrix} - \gamma \begin{bmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{bmatrix},$$

i.e., in particular that G is a scalar multiple of H. Applying Theorem 3.1, we find that there exists a nonsingular matrix $Q \in \mathbb{C}^{n\times n}$ such that $(Q^{-1}AQ, Q^*HQ)$ is in canonical form (8). Since $G = \gamma H$, we obtain that A, H, and G can be reduced simultaneously to block-diagonal form with diagonal blocks of Types 1 and 3.

Case 2: $\gamma, \overline{\gamma} \in \mathbb{C}\backslash\mathbb{R}$.

In this case, we obtain from Corollary 2.4 that there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$P^*(\varrho H - G)P = \varrho \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} - \begin{bmatrix} 0 & \gamma I_m \\ \overline{\gamma} I_m & 0 \end{bmatrix},$$

where $2m = n$ and $\mathrm{Im}(\gamma) > 0$. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be partitioned conformably. Then we obtain from $A^*H = HA$ and $A^*G = GA$ that

$$A_{12}^* = A_{12} \quad \text{and} \quad \gamma A_{12}^* = \overline{\gamma} A_{12}.$$

Since $\gamma \neq \overline{\gamma}$, this implies that $A_{12} = 0$. In an analogous way we show that $A_{21} = 0$, and moreover, we have $A_{22} = A_{11}^*$ by symmetry. Let Q_1 be such that $Q_1^{-1}A_{11}Q_1$ is in Jordan canonical form and set

$$Q = P \begin{bmatrix} Q_1 & 0 \\ 0 & Q_1^{-*} \end{bmatrix}.$$

Then we obtain

$$Q^{-1}AQ = \begin{bmatrix} Q_1^{-1}A_{11}Q_1 & 0 \\ 0 & Q_1^* A_{11}^* Q_1^{-*} \end{bmatrix},$$

$$Q^*HQ = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}, \quad \text{and} \quad Q^*GQ = \begin{bmatrix} 0 & \gamma I_m \\ \overline{\gamma} I_m & 0 \end{bmatrix}.$$

After an appropriate block permutation, we obtain that A, H, and G can be reduced simultaneously to block-diagonal form with diagonal blocks of the forms

$$\tilde{A} = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\lambda)^* \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}, \quad \text{and} \quad \tilde{G} = \begin{bmatrix} 0 & \gamma I_p \\ \overline{\gamma} I_p & 0 \end{bmatrix},$$

respectively, where $p \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. The result then follows by applying the Z-trick; see Remark 3.2.

Uniqueness: Suppose that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}, \quad \text{and}$$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} \tilde{H}_1 & 0 \\ 0 & \tilde{H}_2 \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} \tilde{G}_1 & 0 \\ 0 & \tilde{G}_2 \end{bmatrix},$$

are in canonical form, where H, G, $\tilde{H}$, $\tilde{G}$ are Hermitian nonsingular, A is H-self-adjoint and G-self-adjoint, $\tilde{A}$ is $\tilde{H}$-self-adjoint and $\tilde{G}$-self-adjoint, and all matrices have corresponding block structures. If $P^{-1}AP = \tilde{A}$, $\sigma(A_1) = \sigma(\tilde{A}_1)$, and $\sigma(A_2) = \sigma(\tilde{A}_2)$ such that the spectra of A_1 and A_2 are disjoint, then it follows immediately that P has a corresponding block-diagonal structure. Analogously, assuming that the spectra of $\varrho H_1 - G_1$ and $\varrho \tilde{H}_2 - \tilde{G}_2$ (and of $\varrho H_2 - G_2$ and $\varrho \tilde{H}_1 - \tilde{G}_1$, respectively) are disjoint and that $P^*HP = \tilde{H}$ and $P^*GP = \tilde{G}$, where P is nonsingular, we obtain again that P has a corresponding block-diagonal structure. Indeed, partitioning

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad \text{and} \quad P^{-*} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

conformably with H, we obtain that

$$G_{11}P_{12} = Q_{12}\tilde{G}_{22} \quad \text{and} \quad H_{11}P_{12} = Q_{12}\tilde{H}_{22}.$$

This implies that

$$H_{11}^{-1}G_{11}P_{12} = P_{12}\tilde{H}_{22}^{-1}\tilde{G}_{22},$$

and from that, we obtain $P_{12} = 0$, since the spectra of $H_{11}^{-1}G_{11}$ and $\tilde{H}_{22}^{-1}\tilde{G}_{22}$ are disjoint. Analogously, we show $P_{21} = 0$.

Hence, it is sufficient to prove the uniqueness for the case that A has only one pair of eigenvalues $\lambda, \overline{\lambda}$ and that $\varrho G - H$ has only a pair of eigenvalues $\gamma, \overline{\gamma}$. But then the uniqueness is clear, since we obtain from Theorem 3.1 the uniqueness of the canonical form for the pair (A, H). Note that the structure of G is then uniquely defined by the invariant γ with $Im(\gamma) \geq 0$.

In both cases, it is easy to verify that the H- and G-structure indices of each block are as claimed in the theorem. □

An important special case is the case $H = I$, i.e., A is Hermitian and G-self-adjoint. This leads to the well-known fact that two commuting Hermitian matrices are simultaneously diagonalizable.

Corollary 4.5. *Let $G \in \mathbb{C}^{n\times n}$ be Hermitian nonsingular and let $A \in \mathbb{C}^{n\times n}$ be Hermitian and G-self-adjoint. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that both U^*GU and U^*AU are diagonal.*

Proof. Every Hermitian, G-self-adjoint matrix A satisfies $AG = A^*G = GA$. Then, it is well known that A and G are simultaneously unitarily diagonalizable. But the result is also a special case of Theorem 4.4. To see this, we first note that it follows from Sylvester's law of inertia (see, e.g., [5]) applied to $H = I$ that only blocks of Theorem 4.4 may appear in which the matrix H_j has only positive eigenvalues. Thus, the only possible blocks are those of Type (1), where the parameters p, ε satisfy $\varepsilon = +1$ and $p = 1$. Then Theorem 4.4 implies that there exists a nonsingular matrix $U \in \mathbb{C}^{n\times n}$ such that $U^{-1}AU$ and U^*GU are diagonal and that $U^*HU = H = I$. The latter condition says that U is unitary. □

REMARK 4.6. The result can be generalized to the case that H is positive definite. In this case, there exists a nonsingular matrix P such that $P^*HP = I$. Then we can apply Corollary 4.5 to $P^{-1}AP$ and P^*GP.

4.2. Matrices that are H-self-adjoint and G-skew-adjoint. In this section, we present a canonical form for a matrix A that is H-self-adjoint and G-skew-adjoint, where H and G are Hermitian nonsingular matrices such that the pencil $\varrho H - G$ is nondefective. By Remark 3.4, the eigenvalues of A satisfy more symmetry properties. If $\lambda \in \mathbb{C}$ is an eigenvalue of A, then, because A is G-skew-adjoint, so is $-\overline{\lambda}$, having the same Jordan structure as λ. On the other hand, A is H-self-adjoint and thus, with λ, $-\overline{\lambda}$ also $\overline{\lambda}$, $-\lambda$ are eigenvalues of A, having the same Jordan structures as λ. Thus, the eigenvalues of A occur in quadruples $\{\lambda, \overline{\lambda}, -\overline{\lambda}, -\lambda\}$, where all these eigenvalues have the same Jordan structure. If λ is real or purely imaginary, this set is equal to $\{\lambda, -\lambda\}$, and if $\lambda = 0$, this set is just $\{0\}$.

The following lemmas will be needed for constructing the canonical form.

LEMMA 4.7. *Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian and nonsingular. Furthermore, let $A \in \mathbb{C}^{n\times n}$ be H-self-adjoint and G-skew-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \cdots \oplus A_k, \\ P^*HP &= H_1 \oplus \cdots \oplus H_k, \\ P^*GP &= G_1 \oplus \cdots \oplus G_k, \end{aligned}$$

where A_j, H_j, and G_j have corresponding sizes. Moreover, each matrix A_j has the spectrum $\{\lambda_j, \overline{\lambda}_j, -\lambda_j, -\overline{\lambda}_j\}$ and the spectra of two matrices A_j and A_l, where $j \neq l$, are disjoint.

Proof. By using the eigenvalue properties of A mentioned above, one can find a matrix $Q \in \mathbb{C}^{n\times n}$ such that

$$Q^*GQ = \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^* & G_{22} \end{bmatrix}, \ Q^*HQ = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix}, \ \text{and} \ \ Q^{-1}AQ = \begin{bmatrix} A_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix},$$

where A_1 has the spectrum $\{\lambda_1, \overline{\lambda}_1, -\lambda_1, -\overline{\lambda}_1\}$ for some $\lambda_1 \in \mathbb{C}$ such that the spectra of A_1 and $\tilde{A}_2$ are disjoint. Then we obtain from $A^*H = HA$ and $-A^*G = GA$ that

$$A_1^*H_{12} = H_{12}\tilde{A}_2 \quad \text{and} \quad -A_1^*G_{12} = G_{12}\tilde{A}_2.$$

By construction, the spectra of $\pm A_1^*$ and $\tilde{A}_2$ are disjoint. This implies $H_{12} = 0$ and $G_{12} = 0$. The proof then follows by induction. □

LEMMA 4.8. *Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian and nonsingular. Furthermore, let $A \in \mathbb{C}^{n\times n}$ be H-self-adjoint and G-skew-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \cdots \oplus A_k, \\ P^*HP &= H_1 \oplus \cdots \oplus H_k, \\ P^*GP &= G_1 \oplus \cdots \oplus G_k, \end{aligned}$$

where A_j, H_j, and G_j have corresponding sizes. The spectrum of each pencil $\varrho H_j - G_j$ is contained in $\{\gamma_j, -\gamma_j, \overline{\gamma}_j, -\overline{\gamma}_j\}$ for some $\gamma_j \in \mathbb{C}$ and the spectrum of $\varrho H_l - G_l$ is disjoint from the set $\{\gamma_j, -\gamma_j, \overline{\gamma}_j, -\overline{\gamma}_j\}$ if $j \neq l$.

Proof. The proof proceeds analogously to the proof of Lemma 4.3 using the equations $A^*H = HA$ and $-A^*G = GA$. □

Note that, in contrast to the eigenvalues of A, the eigenvalues of the pencil $\varrho H - G$ need not occur in quadruples $\{\gamma_j, -\gamma_j, \overline{\gamma}_j, -\overline{\gamma}_j\}$. If γ_j is an eigenvalue of $\varrho H - G$, then from Theorem 2.3, we only know that $\overline{\gamma}_j$ is also an eigenvalue, but $-\gamma_j$ and $-\overline{\gamma}_j$ need not be. However, to get corresponding block-diagonal forms of A, G, H, we have to group γ_j and $\overline{\gamma}_j$ together with $-\gamma_j$ and $-\overline{\gamma}_j$ if they are also eigenvalues of $\varrho H - G$.

In view of Lemma 4.8, it is sufficient to consider pencils $\varrho H - G$ whose spectrum is contained in $\{\gamma, -\gamma, \overline{\gamma}, -\overline{\gamma}\}$. Therefore, a discussion of the properties of such pencils will be helpful.

LEMMA 4.9. *Let G, $H \in \mathbb{C}^{n\times n}$ be nonsingular and Hermitian such that the pencil $\varrho H - G$ is nondefective.*

(i) If the spectrum of $\varrho H - G$ is contained in $\{\gamma, -\gamma\}$, where $\gamma^2 \in \mathbb{R}\backslash\{0\}$, then

$$H^{-1}GH^{-1}G = \gamma^2 I_n.$$

(ii) If the spectrum of $\varrho H - G$ is contained in $\{\gamma, -\gamma, \overline{\gamma}, -\overline{\gamma}\}$, where $\gamma^2 \in \mathbb{C}\backslash\mathbb{R}$, then there exists a matrix P such that for $\tilde{H} = P^*HP$, $\tilde{G} = P^*GP$, and $\tilde{A} = P^{-1}AP$,

$$\tilde{H}^{-1}\tilde{G}\tilde{H}^{-1}\tilde{G} = \begin{bmatrix} \overline{\gamma}^2 I_m & 0 \\ 0 & \gamma^2 I_m \end{bmatrix}. \tag{14}$$

Moreover,

$$\tilde{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_1^* \end{bmatrix}.$$

Proof. (i) We consider the problem in two cases.

Case (1): $\mathrm{Im}(\gamma) = 0$.

Since the pencil $\varrho H - G$ is nondefective and its spectrum is contained in $\{\gamma, -\gamma\} \subseteq \mathbb{R}$, by Corollary 2.4 there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ and numbers $p, q, r, s \in \mathbb{N}$ such that

$$H = P^* \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & -I_q & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & -I_s \end{bmatrix} P \quad \text{and} \quad G = P^* \begin{bmatrix} \gamma I_p & 0 & 0 & 0 \\ 0 & -\gamma I_q & 0 & 0 \\ 0 & 0 & -\gamma I_r & 0 \\ 0 & 0 & 0 & \gamma I_s \end{bmatrix} P.$$

This implies $H^{-1}GH^{-1}G = P^{-1}(\gamma^2 I_n)P = \gamma^2 I_n$.

Case (2): $\mathrm{Re}(\gamma) = 0$.

Since the pencil $\varrho H - G$ is nondefective and has only the eigenvalues $\gamma, -\gamma \in i\mathbb{R}$, by Corollary 2.4, there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$H = P^* \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} P \quad \text{and} \quad G = P^* \begin{bmatrix} 0 & \gamma I_m \\ -\gamma I_m & 0 \end{bmatrix} P,$$

where $m = \frac{n}{2} \in \mathbb{N}$. This implies $H^{-1}GH^{-1}G = P^{-1}(\gamma^2 I_n)P = \gamma^2 I_n$.

(ii) By Corollary 2.4, there exists a nonsingular matrix P such that

$$\varrho\tilde{H} - \tilde{G} = \varrho P^*HP - P^*GP = \varrho \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} - \begin{bmatrix} 0 & \gamma\Sigma \\ \overline{\gamma}\Sigma & 0 \end{bmatrix},$$

where $m = \frac{n}{2} \in \mathbb{N}$ and $\Sigma = I_p \oplus (-I_{m-p})$, $0 \le p \le m$. We then obtain that

$$\tilde{H}^{-1}\tilde{G} = \begin{bmatrix} \overline{\gamma}\Sigma & 0 \\ 0 & \gamma\Sigma \end{bmatrix}, \tag{15}$$

and hence we have (14). Note that $\tilde{A}$ is $\tilde{H}$-self-adjoint and $\tilde{G}$-skew-adjoint. This implies that

$$\tilde{A}(\tilde{H}^{-1}\tilde{G}) = \tilde{H}^{-1}\tilde{A}^*\tilde{G} = -(\tilde{H}^{-1}\tilde{G})\tilde{A}.$$

Since in this case $\gamma \pm \overline{\gamma} \neq 0$, from the block form (15) we get $\tilde{A} = A_1 \oplus A_2$. Since $\tilde{A}$ is $\tilde{H}$-self-adjoint, we obtain that $A_2 = A_1^*$. □

THEOREM 4.10. *Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian nonsingular such that the pencil $\varrho H - G$ is nondefective. Furthermore, let $A \in \mathbb{C}^{n\times n}$ be H-self-adjoint and G-skew-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \cdots \oplus A_k, \\ P^*GP &= G_1 \oplus \cdots \oplus G_k, \\ P^*HP &= H_1 \oplus \cdots \oplus H_k, \end{aligned} \tag{16}$$

where, for each j, the blocks A_j, G_j, H_j have corresponding sizes and are of one and only one of the following forms.

Type (1a):

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}_p(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}_p(\overline{\lambda}) & 0 \\ 0 & 0 & 0 & -\mathcal{J}_p(\overline{\lambda}) \end{bmatrix},$$

$$H_j = \begin{bmatrix} 0 & 0 & Z_p & 0 \\ 0 & 0 & 0 & Z_p \\ Z_p & 0 & 0 & 0 \\ 0 & Z_p & 0 & 0 \end{bmatrix}, \quad \textit{and} \quad G_j = \begin{bmatrix} 0 & 0 & 0 & \gamma Z_p \\ 0 & 0 & \gamma Z_p & 0 \\ 0 & \overline{\gamma} Z_p & 0 & 0 \\ \overline{\gamma} Z_p & 0 & 0 & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda)\mathrm{Im}(\lambda) > 0$, $p \in \mathbb{N}$, and $\gamma^2 \in \mathbb{R}\backslash\{0\}$, $\mathrm{Re}(\gamma), \mathrm{Im}(\gamma) \ge 0$.

Type (1b):

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & -\mathcal{J}_p(\lambda) \end{bmatrix}, \quad H_j = \varepsilon \begin{bmatrix} Z_p & 0 \\ 0 & \left(\frac{\gamma}{|\gamma|}\right)^2 Z_p \end{bmatrix}, \quad G_j = \begin{bmatrix} 0 & \gamma Z_p \\ \overline{\gamma} Z_p & 0 \end{bmatrix},$$

where $\lambda > 0$, $p \in \mathbb{N}$, and $\gamma^2 \in \mathbb{R}\backslash\{0\}$, $\mathrm{Re}(\gamma), \mathrm{Im}(\gamma) \geq 0$. The H_j-structure index of λ is ε and the H_j-structure index of $-\lambda$ is $(-1)^{p+1}\varepsilon(\gamma/|\gamma|)^2$.

Type (1c):

$$A_j = i\begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & -\mathcal{J}_p(\lambda) \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \quad G_j = \varepsilon|\gamma| \begin{bmatrix} Z_p & 0 \\ 0 & \left(\frac{|\gamma|}{\gamma}\right)^2 Z_p \end{bmatrix},$$

where $\lambda > 0$, $p \in \mathbb{N}$, and $\gamma^2 \in \mathbb{R}\backslash\{0\}$, $\mathrm{Re}(\gamma), \mathrm{Im}(\gamma) \geq 0$. The G_j-structure index of λ is ε and the G_j-structure index of $-\lambda$ is $(-1)^{p+1}\varepsilon(|\gamma|/\gamma)^2$.

Type (1d1):

$$A_j = \mathcal{J}_p(0), \quad H_j = \varepsilon Z_p, \quad \text{and} \quad G_j = \tilde{\varepsilon}\gamma F_p,$$

where $\gamma^2 \in \mathbb{R}\backslash\{0\}$, $\mathrm{Re}(\gamma), \mathrm{Im}(\gamma) \geq 0$, and $p \in \mathbb{N}$ is odd if $\gamma \in \mathbb{R}$ and even if $\gamma \in i\mathbb{R}$. Moreover, the eigenvalue $\lambda = 0$ has the H_j-structure index ε and the G_j-structure index $\tilde{\varepsilon}\frac{\gamma}{|\gamma|}i^{p-1}$.

Type (1d2):

$$A_j = \begin{bmatrix} \mathcal{J}_p(0) & 0 \\ 0 & \mathcal{J}_p(0) \end{bmatrix}, \; H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \quad \text{and} \quad G_j = \begin{bmatrix} 0 & \gamma F_p \\ -\gamma F_p & 0 \end{bmatrix},$$

where $\gamma^2 \in \mathbb{R}\backslash\{0\}$, $\mathrm{Re}(\gamma), \mathrm{Im}(\gamma) \geq 0$, and $p \in \mathbb{N}$ is even if $\gamma \in \mathbb{R}$ and odd if $\gamma \in i\mathbb{R}$. Moreover, the eigenvalue $\lambda = 0$ has the H_j-structure indices $+1, -1$ and the G_j-structure indices $+1, -1$.

Type (2a):

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}_p(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}_p(\overline{\lambda}) & 0 \\ 0 & 0 & 0 & -\mathcal{J}_p(\overline{\lambda}) \end{bmatrix},$$

$$H_j = \begin{bmatrix} 0 & 0 & Z_p & 0 \\ 0 & 0 & 0 & Z_p \\ Z_p & 0 & 0 & 0 \\ 0 & Z_p & 0 & 0 \end{bmatrix}, \quad \text{and} \quad G_j = \begin{bmatrix} 0 & 0 & 0 & \gamma Z_p \\ 0 & 0 & \gamma Z_p & 0 \\ 0 & \overline{\gamma} Z_p & 0 & 0 \\ \overline{\gamma} Z_p & 0 & 0 & 0 \end{bmatrix},$$

where $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda)\mathrm{Im}(\lambda) > 0$, $p \in \mathbb{N}$, and $\gamma^2 \in \mathbb{C}$ with $\mathrm{Re}(\gamma)\mathrm{Im}(\gamma) > 0$.

Type (2b):

$$A_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}_p(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}_p(\lambda) & 0 \\ 0 & 0 & 0 & -\mathcal{J}_p(\lambda) \end{bmatrix},$$

$$H_j = \begin{bmatrix} 0 & 0 & Z_p & 0 \\ 0 & 0 & 0 & Z_p \\ Z_p & 0 & 0 & 0 \\ 0 & Z_p & 0 & 0 \end{bmatrix}, \quad \text{and} \quad G_j = \begin{bmatrix} 0 & 0 & 0 & \gamma Z_p \\ 0 & 0 & \gamma Z_p & 0 \\ 0 & \overline{\gamma} Z_p & 0 & 0 \\ \overline{\gamma} Z_p & 0 & 0 & 0 \end{bmatrix},$$

where $\lambda > 0$, $p \in \mathbb{N}$, *and* $\gamma^2 \in \mathbb{C}$ *with* $\mathrm{Re}(\gamma)\mathrm{Im}(\gamma) > 0$. *The* H_j*-structure indices of* λ *are* $+1, -1$ *and the* H_j*-structure indices of* $-\lambda$ *are* $+1, -1$.

Type (2c):

$$A_j = i \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}_p(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}_p(\lambda) & 0 \\ 0 & 0 & 0 & -\mathcal{J}_p(\lambda) \end{bmatrix},$$

$$H_j = \begin{bmatrix} 0 & 0 & 0 & Z_p \\ 0 & 0 & Z_p & 0 \\ 0 & Z_p & 0 & 0 \\ Z_p & 0 & 0 & 0 \end{bmatrix}, \quad \textit{and} \quad G_j = \begin{bmatrix} 0 & 0 & \gamma Z_p & 0 \\ 0 & 0 & 0 & \gamma Z_p \\ \overline{\gamma} Z_p & 0 & 0 & 0 \\ 0 & \overline{\gamma} Z_p & 0 & 0 \end{bmatrix},$$

where $\lambda > 0$, $p \in \mathbb{N}$, *and* $\gamma^2 \in \mathbb{C}$ *with* $\mathrm{Re}(\gamma)\mathrm{Im}(\gamma) > 0$. *The* G_j*-structure indices of* λ *are* $+1, -1$ *and the* G_j*-structure indices of* $-\lambda$ *are* $+1, -1$.

Type (2d):

$$A_j = \begin{bmatrix} \mathcal{J}_p(0) & 0 \\ 0 & \mathcal{J}_p(0) \end{bmatrix}, H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, G_j = \varepsilon \begin{bmatrix} 0 & \gamma F_p \\ (-1)^{p+1}\overline{\gamma} F_p & 0 \end{bmatrix},$$

where $p \in \mathbb{N}$, $\varepsilon \in \{+1, -1\}$, *and* $\gamma^2 \in \mathbb{C}$ *with* $\mathrm{Re}(\gamma)\mathrm{Im}(\gamma) > 0$. *Moreover, the eigenvalue* $\lambda = 0$ *has the* H_j*-structure indices* $+1, -1$ *and the* G_j*-structure indices* $+1, -1$.

In the blocks of Types (1a)–(1d), the subpencil $\varrho H_j - G_j$ *has only real or purely imaginary eigenvalues. Those eigenvalues are* γ *and* $-\gamma$, *except for blocks of Type (1d1) when* $p = 1$. *Then the pencil* $\varrho H_j - G_j$ *has the eigenvalue* $\varepsilon\tilde{\varepsilon}\gamma$.

In the blocks of Types (2a)–(2d), the subpencil $\varrho H_j - G_j$ *has only eigenvalues that are neither real nor purely imaginary. Those eigenvalues are* γ, $-\gamma$, $\overline{\gamma}$, *and* $-\overline{\gamma}$, *except for blocks of Type (2d) when* $p = 1$. *Then the pencil* $\varrho H_j - G_j$ *has the eigenvalues* $\varepsilon\gamma$ *and* $\varepsilon\overline{\gamma}$.

Moreover, the canonical form (16) is unique up to permutation of blocks.

Proof. In view of Lemma 4.8, we may assume that the spectrum of the pencil $\varrho H - G$ is contained in $\{\gamma, -\gamma, \overline{\gamma}, -\overline{\gamma}\}$ for some $\gamma \in \mathbb{C}\backslash\{0\}$, $\mathrm{Re}(\gamma), \mathrm{Im}(\gamma) \geq 0$, and it is sufficient to distinguish the following two cases.

Case (1): $\mathrm{Re}(\gamma)\mathrm{Im}(\gamma) = 0$.

In view of Lemma 4.7, we may distinguish the following four subcases.

Subcase (1a): The spectrum of A is $\{\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}\}$, where $\mathrm{Re}(\lambda)\mathrm{Im}(\lambda) > 0$.

Since A is H-self-adjoint and G-skew-adjoint, it follows from Remark 3.4 that λ $\overline{\lambda}$, $-\overline{\lambda}$, and $-\lambda$ have the same Jordan structure. Applying Theorem 3.1, the Z-trick, and a block permutation, we may assume that A and H have the following forms:

$$(17) \qquad A = \begin{bmatrix} \mathcal{J}(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}(\lambda)^* & 0 \\ 0 & 0 & 0 & -\mathcal{J}(\lambda)^* \end{bmatrix}, \ H = \begin{bmatrix} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & I_m \\ I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{bmatrix},$$

where $m = \frac{n}{4} \in \mathbb{N}$ and $\mathcal{J}(\lambda)$ is an $(m \times m)$ matrix in Jordan canonical form only having the eigenvalue λ. Then the equation $-A^*G = GA$ and the fact that λ, $-\lambda$, $\overline{\lambda}$, and $-\overline{\lambda}$ are pairwise distinct imply that G necessarily has the form

$$G = \begin{bmatrix} 0 & 0 & 0 & G_2 \\ 0 & 0 & G_3 & 0 \\ 0 & G_3^* & 0 & 0 \\ G_2^* & 0 & 0 & 0 \end{bmatrix}, \tag{18}$$

where $G_2, G_3 \in \mathbb{C}^{m\times m}$. By Lemma 4.9, we obtain that $H^{-1}GH^{-1}G = \gamma^2 I_n$. This implies in particular that

$$G_3G_2 = \gamma^2 I_m. \tag{19}$$

Note that the equation $-A^*G = GA$ also implies that $\mathcal{J}(\lambda)^*G_2 = G_2\mathcal{J}(\lambda)^*$, i.e., G_2 commutes with $\mathcal{J}(\lambda)^*$. Hence, setting

$$Q := \begin{bmatrix} \overline{\gamma^{\frac{1}{2}}} G_2^{-*} & 0 & 0 & 0 \\ 0 & \overline{\gamma^{-\frac{1}{2}}} I_m & 0 & 0 \\ 0 & 0 & \gamma^{-\frac{1}{2}} G_2 & 0 \\ 0 & 0 & 0 & \gamma^{\frac{1}{2}} I_m \end{bmatrix},$$

we obtain that $Q^{-1}AQ = A$, $Q^*HQ = H$, and

$$Q^*GQ = \begin{bmatrix} 0 & 0 & 0 & \gamma I_m \\ 0 & 0 & \gamma^{-1}G_3G_2 & 0 \\ 0 & \overline{\gamma^{-1}} G_2^*G_3^* & 0 & 0 \\ \overline{\gamma} I_m & 0 & 0 & 0 \end{bmatrix}. \tag{20}$$

Then it follows from (19) and (20) that the triple (A, H, G) can be reduced to blocks of Type (1a) by applying an appropriate block permutation and the Z-trick.

Subcase (1b): The spectrum of A is $\{\lambda, -\lambda\}$, where $\lambda > 0$.

Theorem 3.3 implies that λ and $-\lambda$ have the same Jordan structure. Moreover, applying Theorem 3.1, we may assume that A, H, and G have the following forms:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & -A_1 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}, \quad \text{and} \quad G = \begin{bmatrix} G_1 & G_2 \\ G_2^* & G_3 \end{bmatrix},$$

where

$$\begin{aligned} A_1 &= \mathcal{J}_{p_1}(\lambda) \oplus \cdots \oplus \mathcal{J}_{p_k}(\lambda), \\ H_1 &= \varepsilon_1 Z_{p_1} \oplus \cdots \oplus \varepsilon_k Z_{p_k}, \\ H_2 &= \tilde{\varepsilon}_1 Z_{p_1} \oplus \cdots \oplus \tilde{\varepsilon}_k Z_{p_k}, \end{aligned}$$

and $G_j \in \mathbb{C}^{m\times m}$ for $m = \frac{n}{2}$. Observing that $-A^*G = GA$, we obtain that $G_1 = G_3 = 0$, since $\lambda \neq 0$, and $A_1^*G_2 = G_2A_1$. Moreover, $H^{-1}GH^{-1}G = \gamma^2 I_n$ implies that

$$H_1^{-1}G_2H_2^{-1}G_2^* = \gamma^2 I_m = H_2^{-1}G_2^*H_1^{-1}G_2. \tag{21}$$

Setting

$$Q = \begin{bmatrix} I_m & 0 \\ 0 & \gamma^{-1} H_2^{-1} G_2^* \end{bmatrix},$$

then from (21), $A_1^* G_2 = G_2 A_1$, $Z_p^{-1} \mathcal{J}_p(0)^* Z_p = \mathcal{J}_p(0)$ (Lemma 2.1), and the block forms of H_2 and A_1, we obtain that

$$Q^{-1}AQ = \begin{bmatrix} A_1 & 0 \\ 0 & -G_2^{-*} H_2 A_1 H_2^{-1} G_2^* \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & -A_1 \end{bmatrix},$$

$$Q^* H Q = \begin{bmatrix} H_1 & 0 \\ 0 & \frac{1}{|\gamma^2|} G_2 H_2^{-1} H_2 H_2^{-1} G_2^* \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & \left(\frac{\gamma}{|\gamma|}\right)^2 H_1 \end{bmatrix}, \quad \text{and}$$

$$Q^* G Q = \begin{bmatrix} 0 & \gamma^{-1} G_2 H_2^{-1} G_2^* \\ \overline{\gamma}^{-1} G_2 H_2^{-1} G_2^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & \gamma H_1 \\ \overline{\gamma} H_1 & 0 \end{bmatrix}.$$

Thus, it follows by applying an appropriate block permutation that we may assume that

$$A = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & -\mathcal{J}_p(\lambda) \end{bmatrix}, \; H = \begin{bmatrix} \varepsilon Z_p & 0 \\ 0 & \varepsilon \left(\frac{\gamma}{|\gamma|}\right)^2 Z_p \end{bmatrix}, \; \text{and} \; G = \begin{bmatrix} 0 & \gamma \varepsilon Z_p \\ \overline{\gamma} \varepsilon Z_p & 0 \end{bmatrix}.$$

Hence, setting

$$\tilde{Q} = \begin{bmatrix} \varepsilon^{-1} I_m & 0 \\ 0 & I_m \end{bmatrix},$$

we find that $\tilde{Q}^{-1} A \tilde{Q}$, $\tilde{Q}^* H \tilde{Q}$, and $\tilde{Q}^* G \tilde{Q}$ have the desired forms.

Subcase (1c): The spectrum of A is $\{\lambda, -\lambda\}$, where $\lambda \in i\mathbb{R}$, $\mathrm{Im}(\lambda) > 0$.

The matrix $-iA$ is G-self-adjoint, H-skew-adjoint, and has only a pair of real eigenvalues. Noting that the spectrum of $\varrho G - H$ is contained in $\{\gamma^{-1}, -\gamma^{-1}\}$, we can reduce the problem to Case (1b), i.e., it is sufficient to consider the case that $-iA$, G, and H are as listed in Type (1b):

$$-iA = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & -\mathcal{J}_p(\lambda) \end{bmatrix}, G = \begin{bmatrix} \tilde{\varepsilon} Z_m & 0 \\ 0 & \tilde{\varepsilon}(\frac{|\gamma|}{\gamma})^2 Z_m \end{bmatrix}, H = \begin{bmatrix} 0 & \gamma^{-1} Z_m \\ \overline{\gamma}^{-1} Z_m & 0 \end{bmatrix},$$

where $\tilde{\varepsilon} \in \{+1, -1\}$. Setting

$$Q = \begin{bmatrix} \overline{\gamma}^{\frac{1}{2}} I_m & 0 \\ 0 & \gamma^{\frac{1}{2}} I_m \end{bmatrix},$$

we obtain that $Q^{-1}(-iA)Q = -iA$,

$$Q^* H Q = \begin{bmatrix} 0 & Z_m \\ Z_m & 0 \end{bmatrix}, \quad \text{and} \quad Q^* G Q = \tilde{\varepsilon} |\gamma| \begin{bmatrix} Z_m & 0 \\ 0 & \left(\frac{|\gamma|}{\gamma}\right)^2 Z_m \end{bmatrix}.$$

Subcase (1d): The spectrum of A is $\{0\}$.

It follows from Lemma 7.5 in the Appendix that the triple (A, H, G) can be reduced to blocks of Type (1d1) or of Type (1d2).

Case (2): $\mathrm{Re}(\gamma)\mathrm{Im}(\gamma) \neq 0$.

By Corollary 2.4, we may assume that the pencil $\varrho H - G$ is already in the form

$$\varrho H - G = \varrho \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} - \begin{bmatrix} 0 & \gamma\Sigma \\ \overline{\gamma}\Sigma & 0 \end{bmatrix},$$

where $m = \frac{n}{2} \in \mathbb{N}$ and $\Sigma = \mathrm{diag}(I_p, I_{m-p})$, $1 \leq p \leq m$, and, furthermore, we have (14). Then Lemma 4.9 implies that A has the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_1^* \end{bmatrix}.$$

Note that by Lemma 4.9, a similarity transformation on A with a corresponding block-diagonal matrix and simultaneous congruence transformations on H, G does not change the block structure of A and the identity (14), but it does change the block forms in H and G. Hence we can apply similarity transformations on A_1 and at the same time keep the relation (14). Again, we will consider the following four subcases.

Subcase (2a): The spectrum of A is $\{\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}\}$, where $\mathrm{Re}(\lambda)\mathrm{Im}(\lambda) > 0$.

Again, the eigenvalues $\lambda, -\lambda, \overline{\lambda}$, and $-\overline{\lambda}$ have the same Jordan structure. Moreover, there exists a nonsingular matrix

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_1^{-*} \end{bmatrix} \in \mathbb{C}^{n \times n}$$

such that

$$Q^{-1}AQ = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & A_{11}^* & 0 \\ 0 & 0 & 0 & A_{22}^* \end{bmatrix} \quad \text{and} \quad Q^*HQ = H,$$

where $A_{11} \in \mathbb{C}^{k \times k}$ has the eigenvalues λ and $-\lambda$ and $A_{22} \in \mathbb{C}^{(\frac{n}{2}-k)\times(\frac{n}{2}-k)}$ has the eigenvalues $\overline{\lambda}$ and $-\overline{\lambda}$. Partitioning Q^*GQ conformably, i.e.,

$$Q^*GQ = \begin{bmatrix} 0 & 0 & G_1 & G_2 \\ 0 & 0 & G_3 & G_4 \\ G_1^* & G_3^* & 0 & 0 \\ G_2^* & G_4^* & 0 & 0 \end{bmatrix},$$

we obtain from the equation $-A^*G = GA$ and the fact that A_{11} and $-A_{22}$ have no common eigenvalues that $G_2 = G_3 = 0$. Thus, after an appropriate block permutation, we may consider two smaller subproblems. The first one is

$$\tilde{A} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{11}^* \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} 0 & I_k \\ I_k & 0 \end{bmatrix}, \quad \text{and} \quad \tilde{G} = \begin{bmatrix} 0 & G_1 \\ G_1^* & 0 \end{bmatrix},$$

and (14) implies that

$$\tilde{H}^{-1}\tilde{G}\tilde{H}^{-1}\tilde{G} = \begin{bmatrix} \overline{\gamma}^2 I_k & 0 \\ 0 & \gamma^2 I_k \end{bmatrix}.$$

Hence, after applying a similarity transformation on A_{11}, we may assume that $\tilde{A}$, $\tilde{G}$, and $\tilde{H}$ are in the forms (17) and (18), where $(G_1)^2 = \gamma^2 I$. The remainder of the proof then proceeds analogously to Subcase (1a). The second subproblem with respect to A_{22} can be transformed in the same way.

Subcase (2b): The spectrum of A is $\{\lambda, -\lambda\}$, where $\lambda > 0$.

We obtain from Theorem 3.3 that the Jordan structures associated with λ and $-\lambda$ are the same. Hence, both A_1 and A_1^* must have the eigenvalues λ and $-\lambda$ with the same Jordan structure. Thus, there exists a nonsingular matrix

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_1^{-*} \end{bmatrix}$$

such that

$$Q^{-1}AQ = \begin{bmatrix} \mathcal{J}(\lambda) & 0 & 0 & 0 \\ 0 & -\mathcal{J}(\lambda) & 0 & 0 \\ 0 & 0 & \mathcal{J}(\lambda) & 0 \\ 0 & 0 & 0 & -\mathcal{J}(\lambda) \end{bmatrix} \quad \text{and} \quad Q^*HQ = H,$$

where $k = \frac{n}{4}$ and $\mathcal{J}(\lambda)$ is an $k \times k$ matrix in Jordan canonical form associated with only one eigenvalue λ. Partitioning Q^*GQ conformably, i.e.,

$$Q^*GQ = \begin{bmatrix} 0 & 0 & G_1 & G_2 \\ 0 & 0 & G_3 & G_4 \\ G_1^* & G_3^* & 0 & 0 \\ G_2^* & G_4^* & 0 & 0 \end{bmatrix},$$

we obtain from $-A^*G = GA$ and the fact that $\mathcal{J}(\lambda)$ and $-\mathcal{J}(\lambda)$ have no common eigenvalues that $G_1 = G_4 = 0$, and that $\mathcal{J}(\lambda)^*G_2 = G_2\mathcal{J}(\lambda)$, $\mathcal{J}(\lambda)^*G_3 = G_3\mathcal{J}(\lambda)$. Moreover, we still have (14), which implies that $G_3G_2 = \gamma^2 I$. Thus we may assume that A, G, and H are in the forms (17) and (18), where $G_3G_2 = \gamma^2 I$. The remainder of the proof then proceeds analogously to Subcase (1a).

Subcase (2c): The spectrum of A is $\{\lambda, -\lambda\}$, where $\lambda \in i\mathbb{R}$.

The proof proceeds analogously to the proof of Subcase (1c).

Subcase (2d): The spectrum of A is $\{0\}$.

This case follows from Lemma 7.8 in the Appendix and by applying the Z-trick.

Uniqueness: Analogously to the proof of Theorem 4.4, it is sufficient to prove uniqueness for the case that the spectrum of A is $\{\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}\}$ for some $\lambda \in \mathbb{C}$ and that the spectrum of $\varrho H - G$ is contained in $\{\gamma, -\gamma, \overline{\gamma}, -\overline{\gamma}\}$ for some $\gamma \in \mathbb{C}$. Again, the canonical form for the pair (A, H) is unique. In any case except for the case that $\lambda = 0$ and $\gamma^2 \notin \mathbb{R}$, the matrix G is then uniquely determined by the invariants γ with $\mathrm{Re}(\gamma), \mathrm{Im}(\gamma) \geq 0$ (and signs ε or $\tilde{\varepsilon}$ in some cases that are uniquely determined by the canonical form for the pair (A,G)). Only in the case $\lambda = 0$ and $\gamma^2 \notin \mathbb{R}$ do we have an additional invariant ε that is not an invariant of the canonical form for the pair (A,G). In this case, the uniqueness follows from Lemma 7.8 in the Appendix.

In all of Cases (1a)–(2d), it is easy to verify that the H- and G-structure indices of each block are as claimed in the theorem. □

Again, we obtain as an immediate consequence the result for the special case $H = I$, i.e., A is Hermitian and G-skew-adjoint.

COROLLARY 4.11. *Let G be nonsingular and Hermitian and let A be Hermitian and G-skew-adjoint. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$U^*AU = \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \lambda_m & 0 \\ 0 & -\lambda_m \end{bmatrix} \oplus [0],$$
$$U^*GU = \begin{bmatrix} 0 & \gamma_1 \\ \gamma_1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & \gamma_m \\ \gamma_m & 0 \end{bmatrix} \oplus G_{m+1},$$

where $\lambda_1,\ldots,\lambda_m \in \mathbb{R}$, $\gamma_1,\ldots,\gamma_m > 0$, and $G_{m+1} \in \mathbb{C}^{(n-2m)\times(n-2m)}$ is diagonal.

Proof. It follows from Sylvester's law of inertia (see, e.g., [5]) applied to $H = I$ that in the canonical form for the triple (A, I, G), only blocks may appear in which the matrix H_j has only positive eigenvalues. These are blocks of Type (1b) with parameters $\varepsilon = +1$ and $p = 1$ and blocks of Type (1d1) with parameters $\varepsilon = +1$ and $p = 1$. Noting, furthermore, that G is Hermitian and thus has only real eigenvalues, it follows also that the parameter γ must be real. It now follows from Theorem 4.10 that there exists a nonsingular matrix $U \in \mathbb{C}^{n\times n}$ such that $U^{-1}AU$ and U^*GU have the forms claimed in the corollary and such that $U^*HU = H = I$. The latter identity implies that U is unitary. □

REMARK 4.12. The result can be generalized to the case that H is positive definite. In this case, there exists a nonsingular matrix P such that $P^*HP = I$. Then we can apply Corollary 4.11 to $P^{-1}AP$ and P^*GP.

5. Singly and doubly structured pencils. In this section, we discuss canonical forms for matrix pencils $\varrho A - B$, where both A and B are matrices that are singly or doubly structured with respect to some indefinite inner product. It turns out that the case of structured pencils can be reduced to the matrix case. This is done in the following theorem.

THEOREM 5.1. *Let the matrices $G, H \in \mathbb{C}^{n\times n}$ be nonsingular and Hermitian or skew-Hermitian, i.e.,*

$$G^* = \eta_G G \quad \text{and} \quad H^* = \eta_H H,$$

where $\eta_G, \eta_H \in \{1,-1\}$. Furthermore, let $\varrho A - B \in \mathbb{C}^{n\times n}$ be a regular pencil such that

$$\begin{array}{ll} A^*H = \varepsilon_A HA, & A^*G = \delta_A GA, \\ B^*H = \varepsilon_B HB, & B^*G = \delta_B GB, \end{array} \tag{22}$$

where $\varepsilon_A, \varepsilon_B, \delta_A, \delta_B \in \{1, -1\}$*. Then there exist nonsingular matrices* $P, Q \in \mathbb{C}^{n\times n}$ *such that*

$$P^{-1}(\varrho A - B)Q = \varrho \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} M & 0 \\ 0 & I_{n_2} \end{bmatrix},$$
$$Q^*HP = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix},$$
$$Q^*GP = \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix},$$

where $M, H_{11}, G_{11} \in \mathbb{C}^{n_1\times n_1}$ *and* $N, H_{22}, G_{22} \in \mathbb{C}^{n_2\times n_2}$*. Moreover,* M *and* N *are in Jordan canonical form,* N *is nilpotent, and the following conditions are satisfied.*

$$\begin{array}{rlrl} H_{11}^* &= \eta_H \varepsilon_A H_{11}, & G_{11}^* &= \eta_G \delta_A G_{11}, \\ M^* H_{11} &= \varepsilon_A \varepsilon_B H_{11} M, & M^* G_{11} &= \delta_A \delta_B G_{11} M, \\ H_{22}^* &= \eta_H \varepsilon_B H_{22}, & G_{22}^* &= \eta_G \delta_B G_{22}, \\ N^* H_{22} &= \varepsilon_A \varepsilon_B H_{22} N, & N^* G_{22} &= \delta_A \delta_B G_{22} N. \end{array}$$

Proof. Let $P, Q \in \mathbb{C}^{n\times n}$ be nonsingular matrices such that the pencil

$$P^{-1}(\varrho A - B)Q = \varrho \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} M & 0 \\ 0 & I_{n_2} \end{bmatrix} \tag{23}$$

is in Kronecker canonical form (see [6]), where M, N are in Jordan canonical form and N is nilpotent. Then (22) and (23) imply, in particular, that

$$Q^*H(\varrho\varepsilon_A A - \varepsilon_B B) = Q^*(\varrho A^* - B^*)H = \left(\varrho \begin{bmatrix} I_{n_1} & 0 \\ 0 & N^* \end{bmatrix} - \begin{bmatrix} M^* & 0 \\ 0 & I_{n_2} \end{bmatrix}\right) P^*H.$$

From this and (23), we obtain that

$$\begin{array}{rcccl} Q^*HP \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} & = & Q^*HAQ & = & \varepsilon_A \begin{bmatrix} I_{n_1} & 0 \\ 0 & N^* \end{bmatrix} P^*HQ, \\ Q^*HP \begin{bmatrix} M & 0 \\ 0 & I_{n_2} \end{bmatrix} & = & Q^*HBQ & = & \varepsilon_B \begin{bmatrix} M^* & 0 \\ 0 & I_{n_2} \end{bmatrix} P^*HQ. \end{array}$$

Setting $Q^*HP = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ and noting that $P^*HQ = \eta_H(Q^*HP)^*$, we find that

$$\begin{bmatrix} H_{11} & H_{12}N \\ H_{21} & H_{22}N \end{bmatrix} = \eta_H \varepsilon_A \begin{bmatrix} H_{11}^* & H_{21}^* \\ N^*H_{12}^* & N^*H_{22}^* \end{bmatrix}$$

$$\text{and} \qquad \begin{bmatrix} H_{11}M & H_{12} \\ H_{21}M & H_{22} \end{bmatrix} = \eta_H \varepsilon_B \begin{bmatrix} M^*H_{11}^* & M^*H_{21}^* \\ H_{12}^* & H_{22}^* \end{bmatrix}.$$

This implies, in particular, that

$$H_{12} = \eta_H \varepsilon_B M^* H_{21}^* = \varepsilon_A \varepsilon_B M^* H_{12} N = (\varepsilon_A\varepsilon_B)^k (M^*)^k H_{12} N^k \quad \text{for every } k \in \mathbb{N}.$$

ELA

Since N is nilpotent, it follows that $H_{12} = 0$ and thus, also $H_{21} = \eta_H \varepsilon_A N^* H_{12}^* = 0$. Moreover, $H_{11} = \eta_H \varepsilon_A H_{11}^*$ and $H_{22} = \eta_H \varepsilon_B H_{22}^*$, and $H_{22}N = \eta_H \varepsilon_A N^* H_{22}^* = \varepsilon_A \varepsilon_B N^* H_{22}$, $H_{11}M = \eta_H \varepsilon_B M^* H_{11}^* = \varepsilon_A \varepsilon_B M^* H_{11}$. Analogously, we show that Q^*GP has the structure claimed in the theorem. This concludes the proof. □

We note that M is a doubly structured matrix with structures induced by H_{11} and G_{11} and that N is a nilpotent doubly structured matrix with structures induced by H_{22} and G_{22}, where H_{11}, G_{11}, H_{22}, and G_{22} are all Hermitian or skew-Hermitian. Therefore, Theorem 5.1 gives a general description of how to obtain the canonical forms for the pencil case from the canonical forms in the matrix case that are given in the previous sections. We only have to further reduce M and N by applying the results from Section 4. Note that Theorem 5.1 does not require the pencil $\varrho H - G$ to be nondefective. However, canonical forms for the matrix case are known for this case only.

Theorem 5.1 also describes the case of singly structured pencils. In this case one may choose $H = G$, $\varepsilon_A = \delta_A$, and $\varepsilon_B = \delta_B$. Thus, Theorem 5.1 gives a general description of how to obtain canonical forms for singly and doubly structured pencils from the canonical forms in the matrix case. For obvious reasons, we do not give a list of the canonical forms for all possible cases, but only one example to illustrate the effect of Theorem 5.1.

THEOREM 5.2. *Let $H \in \mathbb{C}^{n\times n}$ be Hermitian and nonsingular and let $\varrho A - B \in \mathbb{C}^{n\times n}$ be a regular pencil such that A and B are H-self-adjoint. Then there exist nonsingular matrices $P, Q \in \mathbb{C}^{n\times n}$ such that*

$$P^{-1}(\varrho A - B)Q = \varrho \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix} - \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_k \end{bmatrix},$$

$$Q^*HP = \begin{bmatrix} H_1 & & 0 \\ & \ddots & \\ 0 & & H_k \end{bmatrix},$$

where the blocks A_j, B_j, and H_j have corresponding sizes and are of one and only one of the following forms:

1. blocks associated with real eigenvalues:

$$A_j = I_p, \quad B_j = \mathcal{J}_p(\lambda), \quad \text{and} \quad H_j = \varepsilon Z_p,$$

where $p \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and $\varepsilon \in \{1, -1\}$;

2. blocks associated with a pair of nonreal eigenvalues:

$$A_j = I_{2p}, \quad B_j = \begin{bmatrix} \mathcal{J}_p(\lambda) & 0 \\ 0 & \mathcal{J}_p(\overline{\lambda}) \end{bmatrix}, \quad \text{and} \quad H_j = Z_{2p},$$

where $p \in \mathbb{N}$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$;

3. blocks associated with the eigenvalue ∞:

$$A_j = \mathcal{J}_p(0), \quad B_j = I_p, \quad \text{and} \quad H_j = \varepsilon Z_p,$$

where $p \in \mathbb{N}$ and $\varepsilon \in \{1, -1\}$.

Moreover, this form is uniquely determined up to permutation of blocks.

Proof. This follows directly from Theorem 5.1 and Theorem 3.1. □

Note that with the assumptions and notation of Theorem 5.2, the pencil $H(\varrho A - B) = \varrho HA - HB$ is a Hermitian pencil. It turns out that Theorem 5.2 is a generalization of Theorem 2.3. Indeed, the pencil $Q^*HPP^{-1}(\varrho A - B)Q$ is a Hermitian pencil in canonical form.

6. Conclusions. We have derived canonical forms for matrices and matrix pencils that are doubly structured in the sense that they are H-self-adjoint (or H-skew-adjoint) and at the same time G-self-adjoint (or G-skew-adjoint), where we have assumed that G, H are nonsingular Hermitian (or skew Hermitian) and $\varrho G - H$ is a nondefective pencil. The general case that G or H is singular, or that the pencil $\varrho G - H$ is defective, is still an open problem. Also, the associated real canonical forms, which appear to be much more difficult, are open.

In view of the applications to eigenvalue computations, it is also important to restrict the transformation matrices to be unitary (or orthogonal in the real case). This case will be covered in a forthcoming paper, which will also address numerical methods, in particular for the classes of pencils arising in quantum chemistry that we discussed in the introduction.

Appendix. In the Appendix we derive some technical lemmas. Recall the Kronecker product; see, e.g., [9, 12].

DEFINITION 7.1. Let $A = [a_{jk}] \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{p \times q}$. Then

$$A \otimes B := \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{mp \times nq}.$$

This product has the following basic properties; see, e.g., [9, 12].

PROPOSITION 7.2. *Let $A, C \in \mathbb{C}^{p_1 \times p_2}$, $B, D \in \mathbb{C}^{q_1 \times q_2}$, $E \in \mathbb{C}^{p_2 \times p_3}$, and $F \in \mathbb{C}^{q_2 \times q_3}$. Then the following identities hold.*

1. $A \otimes (B + D) = A \otimes B + A \otimes D, \quad (A + C) \otimes B = A \otimes B + C \otimes B.$

2. $(A \otimes B)(E \otimes F) = (AE) \otimes (BF)$.

3. $A \otimes B$ *is invertible if and only if* A *and* B *are invertible. In this case we have that* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

4. $(A \otimes B)^T = A^T \otimes B^T, \quad (A \otimes B)^* = A^* \otimes B^*$.

5. $A \otimes B = 0$ *if and only if* $A = 0$ *or* $B = 0$.

We will frequently need the permutation matrix

$$\Omega_{m,n} = [e_1, e_{n+1}, \dots, e_{(m-1)n+1}; e_2, e_{n+2}, \dots, e_{(m-1)n+2}; e_n, e_{2n}, \dots, e_{mn}].$$

If A, B are $m \times n$ and $p \times q$, respectively, then

$$\Omega^*_{m,p}(A \otimes B)\Omega_{n,q} = B \otimes A.$$

In the following we derive the canonical forms for doubly structured matrices that are nilpotent. This case is the most complicated case, since we have least symmetry in the spectrum. Therefore, we have to use a very technical reduction procedure.

For the sake of brevity of notation, let $\mathcal{J}_p$ denote the nilpotent Jordan block $\mathcal{J}_p(0)$ of size p. $\mathcal{O}_{pq}$ is the $p \times q$ zero matrix.

LEMMA 7.3. *Let Z_p, D_p, and F_p be defined as in Section 2 and let $k, l, p, q \in \mathbb{N}$, $(p \geq q)$. Then*

$$Z_p \mathcal{J}_p^l = (\mathcal{J}_p^l)^* Z_p, \quad D_p \mathcal{J}_p^l D_p = (-1)^l \mathcal{J}_p^l, \quad F_p \mathcal{J}_p^l = (-1)^l (\mathcal{J}_p^l)^* F_p, \tag{24}$$

$$Z_p \mathcal{J}_p^k \begin{bmatrix} \mathcal{J}_q^l \\ \mathcal{O}_{p-q,q} \end{bmatrix} = \begin{bmatrix} \mathcal{O}_{p-q,q} \\ Z_q \mathcal{J}_q^{k+l} \end{bmatrix}, \tag{25}$$

$$F_p \mathcal{J}_p^k \begin{bmatrix} \mathcal{J}_q^l \\ \mathcal{O}_{p-q,q} \end{bmatrix} = (-1)^{p-q} \begin{bmatrix} \mathcal{O}_{p-q,q} \\ F_q \mathcal{J}_q^{k+l} \end{bmatrix}, \tag{26}$$

$$D_p \begin{bmatrix} \mathcal{O}_{p-q,q} \\ F_q \end{bmatrix} = (-1)^{p-q} \begin{bmatrix} \mathcal{O}_{p-q,q} \\ D_q F_q \end{bmatrix}.$$

DEFINITION 7.4. Let $A = (a_{jk})_{nn} \in \mathbb{C}^{n \times n}$. Then the lth lower antidiagonal of A or, in short, the lth antidiagonal of A, is defined by the elements a_{jk}, where $j + k = n + 1 + l$. Here, we allow $l = 0$. The 0th antidiagonal is also called the main antidiagonal. If

$$B = \begin{bmatrix} 0 & \tilde{B} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 \\ \tilde{C} \end{bmatrix},$$

where $\tilde{B}$ and $\tilde{C}$ are square matrices, then the lth antidiagonal of $\tilde{B}$ and $\tilde{C}$ is called the lth antidiagonal of B and C, respectively. Analogously, we define the lth block antidiagonal for square and nonsquare block matrices.

LEMMA 7.5. *Let $G, H \in \mathbb{C}^{n \times n}$ be Hermitian nonsingular such that the pencil $\varrho H - G$ is nondefective and such that its spectrum is contained in $\{\gamma, -\gamma\}$, where $\gamma^2 \in \mathbb{R}\backslash\{0\}$ and $\mathrm{Re}(\gamma), \mathrm{Im}(\gamma) \geq 0$. Furthermore, let $A \in \mathbb{C}^{n \times n}$ be nilpotent, H-self-adjoint, and G-skew-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that*

$$\begin{aligned} P^{-1} A P &= A_1 \oplus \cdots \oplus A_k, \\ P^* G P &= G_1 \oplus \cdots \oplus G_k, \\ P^* H P &= H_1 \oplus \cdots \oplus H_k, \end{aligned} \tag{27}$$

where the blocks A_j, G_j, H_j have corresponding sizes and, for each j, are of one and only one of the following forms:

Type (1d1):

$$A_j = \mathcal{J}_p(0), \quad H_j = \varepsilon Z_p, \quad \text{and} \quad G_j = \tilde{\varepsilon} \gamma F_p, \tag{28}$$

where $\varepsilon, \tilde{\varepsilon} \in \{-1, 1\}$ and $p \in \mathbb{N}$ is odd if $\gamma \in \mathbb{R}$ and even if $\gamma \in i\mathbb{R}$;

Type (1d2):

$$A_j = \begin{bmatrix} \mathcal{J}_p(0) & 0 \\ 0 & \mathcal{J}_p(0) \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix}, \quad G_j = \begin{bmatrix} 0 & \gamma F_p \\ -\gamma F_p & 0 \end{bmatrix}, \tag{29}$$

where $p \in \mathbb{N}$ is even if $\gamma \in \mathbb{R}$ and odd if $\gamma \in i\mathbb{R}$.

Moreover, the form (27) is unique up to permutation of blocks.

Proof. Applying Theorem 3.1, we may assume that (A, H) is in canonical form, i.e., collecting blocks of the same size and representing them by means of the Kronecker product, we may assume that

$$A = \begin{bmatrix} I_{m_1} \otimes \mathcal{J}_{p_1} & & 0 \\ & \ddots & \\ 0 & & I_{m_k} \otimes \mathcal{J}_{p_k} \end{bmatrix}, \quad H = \begin{bmatrix} \Sigma_{m_1} \otimes Z_{p_1} & & 0 \\ & \ddots & \\ 0 & & \Sigma_{m_k} \otimes Z_{p_k} \end{bmatrix},$$

where $p_1 > \cdots > p_k$ are the sizes of Jordan blocks and Σ_{m_j} are signature matrices for $j = 1, \ldots, k$. Setting

$$F = \begin{bmatrix} I_{m_1} \otimes F_{p_1} & & 0 \\ & \ddots & \\ 0 & & I_{m_k} \otimes F_{p_k} \end{bmatrix},$$

we obtain from $-A^*G = GA$ and (24) that A and FG commute. Thus, the structure of G is implicitly given by the well-known form for matrices that commute with matrices in Jordan canonical form; see [5]. For the sake of clarity, we will not work directly on A, H, and G, but first apply a permutation. Setting $\Omega = \Omega_{m_1,p_1} \oplus \cdots \oplus \Omega_{m_k,p_k}$ and updating A, H, G by $\Omega^{-1}A\Omega$, $\Omega^*H\Omega$, $\Omega^*G\Omega$, we are led to consider the following situation:

$$A = \begin{bmatrix} \mathcal{J}_{p_1} \otimes I_{m_1} & & 0 \\ & \ddots & \\ 0 & & \mathcal{J}_{p_k} \otimes I_{m_k} \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} H_{11} & & 0 \\ & \ddots & \\ 0 & & H_{kk} \end{bmatrix},$$

where $H_{jj} := Z_{p_j} \otimes \Sigma_{m_j}$. Partitioning

$$G = \begin{bmatrix} G_{11} & \ldots & G_{1k} \\ \vdots & \ddots & \vdots \\ G_{1k}^* & \ldots & G_{kk} \end{bmatrix} \tag{30}$$

conformably and using the structures of matrices that commute with matrices in

Jordan canonical form [5], we obtain that

$$
(31)\qquad \begin{aligned}
G_{qq} &= \textstyle\sum_{l=0}^{p_q-1}(F_{p_q}\mathcal{J}_{p_q}^l)\otimes G_{q,q}^{(l)} \\
&= \begin{bmatrix}
0 & \dots & \dots & 0 & G_{q,q}^{(0)} \\
\vdots & & \iddots & -G_{q,q}^{(0)} & -G_{q,q}^{(1)} \\
\vdots & \iddots & G_{q,q}^{(0)} & G_{q,q}^{(1)} & \vdots \\
0 & \iddots & \iddots & & \vdots \\
(-1)^{p_q+1}G_{q,q}^{(0)} & \dots & \dots & \dots & (-1)^{p_q+1}G_{q,q}^{(p_q-1)}
\end{bmatrix}
\end{aligned}
$$

for $q = 1, \dots, k$, where $G_{q,q}^{(l)} \in \mathbb{C}^{m_q\times m_q}$ and

$$
(32)\qquad G_{qr} = \sum_{l=0}^{p_r-1}\begin{bmatrix} \mathcal{O}_{p_q-p_r,p_r} \\ F_{p_r}\mathcal{J}_{p_r}^l \end{bmatrix}\otimes G_{q,r}^{(l)}
$$

for $1 \le q < r \le k$, with $G_{q,r}^{(l)} \in \mathbb{C}^{m_q\times m_r}$.

We will stepwise reduce the matrix G, while keeping the forms of A and H.

Step (1): Show that $G_{j,j}^{(0)}$ is nonsingular for $j = 1, \dots, k$.

Since the pencil $\varrho H - G$ is nondefective and has only the eigenvalues $\gamma, -\gamma$, where $\gamma^2 \in \mathbb{R}\backslash\{0\}$, we obtain from Lemma 4.9 that

$$GH^{-1}G = \gamma^2 H.$$

Comparing the jth diagonal blocks on both sides, this implies in particular that

$$
(33)\qquad \gamma^2 H_{jj} = G_{1j}^* H_{11} G_{1j} + \dots + G_{jj}H_{jj}G_{jj} + \dots + G_{jk}H_{kk}G_{jk}^*.
$$

Because of the structure of the blocks G_{qr}, it follows that all the block antidiagonals of $G_{qr}^* H_{qq} G_{qr}$ and $G_{qr}H_{rr}G_{qr}^*$ are zero for $q < r$, and hence, comparing the main block antidiagonals on both sides of (33), we obtain that

$$
\begin{aligned}
\gamma^2 Z_{p_j}\otimes\Sigma_{m_j} &= (F_{p_j}\otimes G_{j,j}^{(0)})(Z_{p_j}\otimes\Sigma_{m_j})(F_{p_j}\otimes G_{j,j}^{(0)}) \\
&= (F_{p_j}Z_{p_j}F_{p_j})\otimes(G_{j,j}^{(0)}\Sigma_{m_j}G_{j,j}^{(0)}).
\end{aligned}
$$

Since $F_{p_j}Z_{p_j}F_{p_j} = Z_{p_j}$, this implies that

$$
(34)\qquad G_{j,j}^{(0)}\Sigma_{m_j}G_{j,j}^{(0)} = \frac{1}{\gamma^2}\Sigma_{m_j}
$$

and thus, $G_{j,j}^{(0)}$ is nonsingular.

Step (2): Eliminate $G_{12}, \dots, G_{1k}$.

Assume that we already have $G_{1,j}^{(s)} = 0$ for all $j = 2, \dots, k$ and all $s = 0, \dots, l-1$, and $G_{1,j}^{(l)} = 0$ for $j = 2, \dots, r-1$, where $l \ge 0$ and $r \ge 2$. We then show how to eliminate $G^{(l)}_{1,r}$ while keeping the forms of A and H. Let

$$X = \begin{array}{c} \\ \\ \\ r \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & & r & & \end{array} \\ \begin{bmatrix} I & & X_{1r} & & \\ & \ddots & & & \\ & & & & \\ X_{r1} & & \ddots & & \\ & & & & I \end{bmatrix} \end{array}$$

have a block form analogous to G, where zero blocks of the matrix are indicated by blanks and, moreover,

$$X_{1r} = \begin{bmatrix} \mathcal{J}^l_{p_r} \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \otimes \left(\frac{1}{2}(-1)^{p_1-p_r+1}(G^{(0)}_{1,1})^{-1} G^{(l)}_{1,r} \right) \quad \left(\hat{=} \left[\begin{array}{cccc} 0 & * & & 0 \\ & & \ddots & \\ 0 & & & * \\ \hline 0 & & \dots & 0 \end{array} \right] \right),$$

$$X_{r1} = \begin{bmatrix} \mathcal{O}_{p_r,p_1-p_r} & \mathcal{J}^l_{p_r} \end{bmatrix} \otimes \left(\frac{1}{2}(-1)^{l+1}(G^{(0)}_{r,r})^{-*}(G^{(l)}_{1,r})^* \right).$$

Substep (2a): Note that X_{1r} and X_{r1} are chosen so that X commutes with A.

Substep (2b): In the updated matrix $\tilde{G} := X^*GX$, we have $\tilde{G}^{(l)}_{1,r} = 0$.

Indeed, it is easy to see that $\tilde{G}$ is again a matrix of the form (30), (31), and (32). The $(1,r)$-block of $\tilde{G}$ satisfies

$$\tilde{G}_{1r} = G_{11}X_{1r} + X^*_{r1}G^*_{1r}X_{1r} + G_{1r} + X^*_{r1}G_{rr}. \tag{35}$$

From the structure of G and X, we immediately find that the first $l-1$ block antidiagonals of all the summands of the right-hand side of (35) are zero. Furthermore, the lth block antidiagonal of $\tilde{G}_{1r}$ has the form

$$\begin{aligned} & (F_{p_1} \otimes G^{(0)}_{1,1}) \left(\begin{bmatrix} \mathcal{J}^l_{p_r} \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \otimes \left(\frac{1}{2}(-1)^{p_1-p_r+1}(G^{(0)}_{1,1})^{-1} G^{(l)}_{1,r} \right) \right) \\ & + \begin{bmatrix} \mathcal{O}_{p_1-p_r,p_r} \\ F_{p_r}\mathcal{J}^l_{p_r} \end{bmatrix} \otimes G^{(l)}_{1,r} \\ & + \left(\begin{bmatrix} \mathcal{O}_{p_r,p_1-p_r} \\ (\mathcal{J}^l_{p_r})^* \end{bmatrix} \otimes \left(\frac{1}{2}(-1)^{l+1} G^{(l)}_{1,r}(G^{(0)}_{r,r})^{-1} \right) \right) (F_{p_r} \otimes G^{(0)}_{r,r}) \\ = {} & \frac{1}{2}(-1)^{p_1-p_r+1} \left(F_{p_1} \begin{bmatrix} \mathcal{J}^l_{p_r} \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \right) \otimes G^{(l)}_{1,r} + \begin{bmatrix} \mathcal{O}_{p_1-p_r,p_r} \\ F_{p_r}\mathcal{J}^l_{p_r} \end{bmatrix} \otimes G^{(l)}_{1,r} \\ & + \frac{1}{2}(-1)^{l+1} \left(\begin{bmatrix} \mathcal{O}_{p_r,p_1-p_r} \\ (\mathcal{J}^l_{p_r})^* \end{bmatrix} F_{p_r} \right) \otimes G^{(l)}_{1,r} \\ = {} & -\frac{1}{2} \begin{bmatrix} \mathcal{O}_{p_1-p_r,p_r} \\ F_{p_r}\mathcal{J}^l_{p_r} \end{bmatrix} \otimes G^{(l)}_{1,r} + \begin{bmatrix} \mathcal{O}_{p_1-p_r,p_r} \\ F_{p_r}\mathcal{J}^l_{p_r} \end{bmatrix} \otimes G^{(l)}_{1,r} \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\left[\begin{array}{c}\mathcal{O}_{p_1-p_r,p_r}\\ (-1)^l(\mathcal{J}_{p_r}^l)^*F_{p_r}\end{array}\right]\otimes G_{1,r}^{(l)} \qquad \text{(using (26))}\\
&= 0 \qquad \text{(using (24))}.
\end{aligned}
$$

Substep (2c): In the updated matrix $\tilde{G} := X^*GX$, we still have $\tilde{G}_{1,j}^{(s)} = 0$ for all $j = 2,\ldots,k$ and all $s = 0,\ldots,l-1$, and $\tilde{G}_{1,j}^{(l)} = 0$ for $j = 2,\ldots,r-1$.

Indeed, the elements of the first block row of $\tilde{G}$ have the form

$$
\begin{aligned}
&G_{1q} + X_{r1}^*G_{qr}^* \quad \text{for } 1 < q < r \quad \text{and}\\
&G_{1q} + X_{r1}^*G_{rq} \quad \text{for } r < q.
\end{aligned}
$$

From the block structure of G_{1q}, G_{rq}, G_{qr}, and X_{r1}, we obtain that the first p_q-p_r+2l block antidiagonals in $X_{r1}^*G_{qr}^*$ and the first p_r-p_q+2l-1 block antidiagonals in $X_{r1}^*G_{rq}$ are zero.

Substep (2d): Show that the matrix $\tilde{H} := X^*HX$ is block diagonal.

The only changes outside the block diagonal can have happened to the $(1,r)$-block $\tilde{H}_{1r}$ and the $(r,1)$-block $\tilde{H}_{r1} = \tilde{H}_{1r}^*$. The $(1,r)$-block has the form

$$
\begin{aligned}
\tilde{H}_{1r} &= (Z_{p_1}\otimes\Sigma_{m_1})X_{1r} + X_{r1}^*(Z_{p_r}\otimes\Sigma_{m_r})\\
&= \frac{1}{2}\left[\begin{array}{c}\mathcal{O}_{p_1-p_r,p_r}\\ Z_{p_r}\mathcal{J}_{p_r}^l\end{array}\right]\\
&\quad\otimes((-1)^{p_1-p_r+1}\Sigma_{m_1}(G_{1,1}^{(0)})^{-1}G_{1,r}^{(l)} + (-1)^{l+1}G_{1,r}^{(l)}(G_{r,r}^{(0)})^{-1}\Sigma_{m_r}),
\end{aligned} \tag{36}
$$

using (24) and (25). On the other hand, we have $GH^{-1}G = \gamma^2H$. Noting that $H^{-1} = H$ and comparing the $(1,r)$-blocks of both sides, we obtain that

$$
0 = G_{11}H_{11}G_{1r} + \left(\sum_{q=2}^{r-1}G_{1q}H_{qq}G_{qr}\right) + G_{1r}H_{rr}G_{rr} + \left(\sum_{q=r+1}^{k}G_{1q}H_{qq}G_{rq}^*\right). \tag{37}
$$

Clearly, the first $l-1$ antidiagonals of all the summands in (37) are zero. We now consider the lth block antidiagonal. We note that $G_{11}H_{11}G_{1r}$ and $G_{1r}H_{rr}G_{rr}$ are the only summands that have a nonzero lth block antidiagonal. For the terms $G_{1q}H_{qq}G_{qr}$, $1 < q < r$, this follows from the fact that the lth block antidiagonal of G_{1q} is already zero. For $G_{1q}H_{qq}G_{rq}^*$, $q > r$, this can be seen as follows. If we write the jth block antidiagonal of $G_{1q}H_{qq}G_{rq}^*$ in the form $S_j\otimes T_j$, then we obtain

$$
\begin{aligned}
S_j &= \begin{array}{c} \\ p_1-p_q \\ p_q\end{array}\begin{array}{c} p_q \\ \left[\begin{array}{c}0\\ F_{p_q}\mathcal{J}_{p_q}^j\end{array}\right]\end{array} Z_{p_q}\left(\begin{array}{cc} & \begin{array}{cc} p_r-p_q & p_q\end{array}\\ p_q & \left[\begin{array}{cc}0 & F_{p_q}^*\end{array}\right]\end{array}\right)\\
&= \begin{array}{c} \\ p_1-p_r\\ p_r-p_q\\ p_q\end{array}\begin{array}{c}\begin{array}{cc} p_r-p_q & \qquad p_q\end{array}\\ \left[\begin{array}{cc}0 & 0\\ 0 & 0\\ 0 & F_{p_q}\mathcal{J}_{p_q}^jZ_{p_q}F_{p_q}^*\end{array}\right]\end{array}.
\end{aligned}
$$

Having in mind that the first $l-1$ block antidiagonals of G_{1q} are zero, we find that the first $p_r - p_q + l - 1$ block antidiagonals of $G_{1q}H_{qq}G_{rq}^*$ are zero.

Finally, comparing the lth block antidiagonals in (37), we obtain

$$\begin{aligned} 0 &= (F_{p_1} \otimes G_{1,1,0})(Z_{p_1} \otimes \Sigma_{m_1}) \left(\begin{bmatrix} \mathcal{O}_{p_1-p_r,p_r} \\ F_{p_r}\mathcal{J}_{p_r}^l \end{bmatrix} \otimes G_{1,r}^{(l)} \right) \\ &\quad + \left(\begin{bmatrix} \mathcal{O}_{p_1-p_r,p_r} \\ F_{p_r}\mathcal{J}_{p_r}^l \end{bmatrix} \otimes G_{1,r}^{(l)} \right) (Z_{p_r} \otimes \Sigma_{m_r})(F_{p_r} \otimes G_{r,r}^{(0)}) \\ &= \begin{bmatrix} \mathcal{O}_{p_1-p_r,p_r} \\ Z_{p_r}\mathcal{J}_{p_r}^l \end{bmatrix} \otimes ((-1)^{p_1-p_r} G_{1,1}^{(0)}\Sigma_{m_1}G_{1,r}^{(l)} + (-1)^l G_{1,r}^{(l)}\Sigma_{m_r}G_{r,r}^{(0)}), \end{aligned}$$

using (24), (25), and (26). Using (34) and (36), this implies $\tilde{H}_{1r} = 0$.

Substep (2e): Retrieve H.

Although $\tilde{H}$ is block diagonal, the $(1,1)$- and (r,r)-blocks may differ from those of H. We now show how to retrieve H from $\tilde{H}$ while keeping the zero block antidiagonals of $\tilde{G}$. It follows from Theorem 3.1 that there exists a nonsingular matrix $T \in \mathbb{C}^{p_1m_1 \times p_1m_1}$ such that

$$T^{-1}(\mathcal{J}_{p_1} \otimes I_{m_1})T = \mathcal{J}_{p_1} \otimes I_{m_1} \quad \text{and} \quad T^*\tilde{H}_{11}T = Z_{p_1} \otimes \Sigma_{m_1}.$$

Since T commutes with $\mathcal{J}_{p_1} \otimes I_{m_1}$, it has the block structure

$$T = \begin{bmatrix} T_1 & \dots & T_{m_1} \\ & \ddots & \vdots \\ 0 & & T_1 \end{bmatrix}$$

with $T_j \in \mathbb{C}^{p_1 \times p_1}$, $j = 1, \dots, m_1$. Analogously, we find a matrix $T' \in \mathbb{C}^{p_rm_r \times p_rm_r}$ of similar structure such that

$$T'^{-1}(\mathcal{J}_{p_r} \otimes I_{m_r})T' = \mathcal{J}_{p_r} \otimes I_{m_r} \quad \text{and} \quad T'^*\tilde{H}_{rr}T' = Z_{p_r} \otimes \Sigma_{m_r}.$$

Setting

$$\tilde{T} := T \oplus I_{p_2m_2} \oplus \dots \oplus I_{p_{r-1}m_{r-1}} \oplus T' \oplus I_{p_{r+1}m_{r+1}} \oplus \dots \oplus I_{p_km_k},$$

we obtain

$$\tilde{T}^{-1}A\tilde{T} = A \quad \text{and} \quad \tilde{T}^*\tilde{H}\tilde{T} = H.$$

Moreover, let us look at the $(1,q)$-block of $\tilde{T}^*\tilde{G}\tilde{T}$. Note that, because of the block-triangularity of T, the multiplication of $\tilde{G}$ from the left by $\tilde{T}^*$ neither changes the first $l-1$ zero block antidiagonals of $\tilde{G}_{1q}$ for $q = 2, \dots, k$ nor the lth zero block antidiagonals of $\tilde{G}_{1q}$ for $q = 2, \dots, r$. The same argument holds for the multiplication from the right by $\tilde{T}$, because of the block-triangularity of T'.

Substep (2f): By consecutively repeating Substeps (2a)–(2e) several times, we can eliminate $G_{1,j}^{(0)}$ for all $j = 2, \dots, k$, then $G_{1,j}^{(1)}$ for all $j = 2, \dots, k$, and so on.

After having also eliminated $G_{1,j}^{(p_1-1)}$, we finally obtain that there exists a nonsingular matrix S such that

$$S^{-1}AS = \begin{bmatrix} \mathcal{J}_{p_1}\otimes I_{m_1} & 0 \\ 0 & A_2 \end{bmatrix}, \quad S^*HS = \begin{bmatrix} Z_{p_1}\otimes \Sigma_{m_1} & 0 \\ 0 & H_2 \end{bmatrix},$$

$$\text{and} \quad S^*GS = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$

where $G_1 \in \mathbb{C}^{p_1m_1\times p_1m_1}$. Hence, it is sufficient to assume that we are in the following situation:

$$A = \mathcal{J}_p \otimes I_m, \quad H = Z_p \otimes \Sigma, \quad \text{and}$$

$$G = \sum_{k=0}^{p-1}(F_p\mathcal{J}_p^k)\otimes G_k = \begin{bmatrix} 0 & & & G_0 \\ & & -G_0 & -G_1 \\ & \iddots & \iddots & \vdots \\ (-1)^{p+1}G_0 & \dots & \dots & (-1)^{p+1}G_{p-1} \end{bmatrix},$$

where $k,m,p\in\mathbb{N}$, Σ is a signature matrix, and $G_j\in\mathbb{C}^{m\times m}$ for $j=0,\dots,p-1$.

Step (3): Reduce G to block antidiagonal form.

Assume that we already have $G_1=\cdots=G_{l-1}=0$ for some $l\le p-1$. We then eliminate G_l while keeping the structure of A and H.

Substep (3a): Eliminate G_l.

Since G is Hermitian and F_p is Hermitian for odd p and skew-Hermitian for even p, we obtain that

$$G_k^* = (-1)^{p+k+1}G_k. \tag{38}$$

This implies, in particular, that

$$(G_0^{-1}G_l)^* = (-1)^l G_l G_0^{-1}. \tag{39}$$

Setting

$$X := I_p\otimes I_m - \frac{1}{2}\mathcal{J}_p^l\otimes(G_0^{-1}G_l),$$

it follows that X commutes with A. Moreover, we obtain that the first $l-1$ block antidiagonals in $\tilde{G} := X^*GX$ are still zero. Then, using (24), it follows that the lth block antidiagonal has the form

$$(I_p\otimes I_m)((F_p\mathcal{J}_p^l)\otimes G_l)(I_p\otimes I_m) - \frac{1}{2}(-1)^l((\mathcal{J}_p^l)^*\otimes(G_lG_0^{-1}))(F_p\otimes G_0)(I_p\otimes I_m)$$
$$-\frac{1}{2}(I_p\otimes I_m)(F_p\otimes G_0)(\mathcal{J}_p^l\otimes(G_0^{-1}G_l)) = 0.$$

Substep (3b): Retrieve H.

Comparing the lth block antidiagonals on both sides of $GH^{-1}G = \gamma^2 H$ and using that $G_1 = \cdots = G_{l-1} = 0$, by applying (24) and Lemma 2.1, we obtain that

$$\begin{aligned} 0 &= (F_p \otimes G_0)(Z_p \otimes \Sigma)((F_p \mathcal{J}_p^l) \otimes G_l) + ((F_p \mathcal{J}_p^l) \otimes G_l)(Z_p \otimes \Sigma)(F_p \otimes G_0) \\ &= (F_p Z_p F_p \mathcal{J}_p^l) \otimes (G_0 \Sigma G_l + (-1)^l G_l \Sigma G_0). \end{aligned}$$

This implies, in particular, that for $l \geq p-1$,

$$G_l G_0^{-1} \Sigma + (-1)^l \Sigma G_0^{-1} G_l = 0.$$

Here we have used the identity $G_0 \Sigma G_0 = \gamma^2 \Sigma$, which follows from comparing the diagonal blocks in $GH^{-1}G = \gamma^2 H$. Therefore, with this relation and (39) we obtain that

$$\begin{aligned} &X^* H X \\ &= Z_p \otimes \Sigma - \frac{1}{2} Z \mathcal{J}_p^l \otimes (G_l^* G_0^{-*} \Sigma + \Sigma G_0^{-1} G_l) + \frac{1}{4}((\mathcal{J}_p^l)^* Z_p \mathcal{J}_p^l) \otimes (G_l^* G_0^{-*} \Sigma G_0^{-1} G_l) \\ &= Z_p \otimes \Sigma - \frac{1}{4}(Z_p \mathcal{J}_p^{2l}) \otimes (\Sigma (G_0^{-1} G_l)^2). \end{aligned}$$

The $(2l)$th block antidiagonal of $X^* H X$ can then be eliminated by a congruence transformation with

$$Y = I_r \otimes I_m + \frac{1}{8} \mathcal{J}_p^{2l} \otimes (G_0^{-1} G_l)^2.$$

This transformation does not change the first l block antidiagonals of $\tilde{G}$ but may change the jth block antidiagonal of $X^* H X$ for some $j > 2l$. However, repeating the procedure described above a finite number of times, we can finally retrieve H while keeping the property that the first l block antidiagonals in $\tilde{G}$ are zero.

Substep (3c): By consecutively applying Substeps (3a) and (3b), we finally obtain that there exists a nonsingular matrix S such that

$$S^{-1} A S = \mathcal{J}_p \otimes I_m, \quad S^* H S = Z_p \otimes \Sigma = \begin{bmatrix} 0 & & \Sigma \\ & \iddots & \\ \Sigma & & 0 \end{bmatrix},$$

$$\text{and} \quad S^* G S = F_p \otimes G_0 = \begin{bmatrix} 0 & & G_0 \\ & \iddots & \\ (-1)^{p+1} G_0 & & 0 \end{bmatrix}.$$

Step (4): Complete the final reduction of G.

Since the pencil $\varrho H - G$ is nondefective and its spectrum is contained in $\{\gamma, -\gamma\}$, this also holds for each subpencil $\varrho \Sigma - (\pm G_0)$. We will distinguish four cases.

Case (a): $\gamma \in \mathbb{R}$ and p is even.

Identity (38) implies that G_0 is skew-Hermitian. Since the pencil $\varrho\Sigma - (\pm G_0)$ has only real eigenvalues γ and/or $-\gamma$, it follows that $\varrho\Sigma - (\pm G_0)$ has both eigenvalues with equal algebraic multiplicity. This implies, in particular, that m is even and that there exists a nonsingular matrix $R \in \mathbb{C}^{m\times m}$ such that

$$R^*\Sigma R = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad \text{and} \quad R^*G_0R = \begin{bmatrix} 0 & \gamma I \\ -\gamma I & 0 \end{bmatrix}.$$

Set $\mathcal{R} = I_p \otimes R$. Then

$$\mathcal{R}^{-1}A\mathcal{R} = A, \quad \mathcal{R}^*H\mathcal{R} = Z_p \otimes \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \text{and} \quad \mathcal{R}^*G\mathcal{R} = F_p \otimes \begin{bmatrix} 0 & \gamma I \\ -\gamma I & 0 \end{bmatrix}.$$

Applying a transformation with $\Omega_{p,m}$ the form stated in (29) for the case that p is even follows from an appropriate block permutation.

Case (b): $\gamma \in \mathbb{R}$ and p is odd.

In this case, (38) implies that G_0 is Hermitian. Considering the Hermitian pencil $\varrho\Sigma - (\pm G_0)$, there exists a nonsingular matrix $R \in \mathbb{C}^{m\times m}$ such that

$$R^*\Sigma R = \Sigma \quad \text{and} \quad R^*G_0R = \gamma\tilde{\Sigma},$$

where $\tilde{\Sigma}$ is another signature matrix. Setting $\mathcal{R} := I_p \otimes R$ and applying transformations with $\mathcal{R}$ and $\Omega_{p,m}$, the form stated in (28) for the case that p is odd follows from an appropriate block permutation.

Case (c): $\gamma \in i\mathbb{R}$ and p is even.

In this case, (38) implies that G_0 is Hermitian. The rest follows as in Case (b).

Case (d): $\gamma \in i\mathbb{R}$ and p is odd.

This case follows analogously to Case (a). This concludes the reduction to the canonical form.

Uniqueness: The canonical form for the pair (A, H) is unique. The matrix G is then uniquely determined by the invariants $\tilde{\varepsilon}$ and γ. □

DEFINITION 7.6. Let $A = (a_{jk})_{nn} \in \mathbb{C}^{n\times n}$. Then the lth upper diagonal of A or, in short, the lth diagonal of A, is defined by the elements a_{jk}, where $k = j + l$. Here, we allow $l = 0$. If

$$B = \begin{bmatrix} 0 & \tilde{B} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} \tilde{C} \\ 0 \end{bmatrix},$$

where $\tilde{B}$ and $\tilde{C}$ are square matrices, then the lth diagonals of $\tilde{B}$ and $\tilde{C}$ are called the lth diagonals of B and C, respectively. Analogously, we define the lth block diagonal for square and nonsquare block matrices.

LEMMA 7.7. *Suppose that $A_0, G_0 \in \mathbb{C}^{n\times n}$ anticommute, i.e., $A_0G_0 = -G_0A_0$. Furthermore, let A_0 be nilpotent and G_0 be diagonalizable and nonsingular. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}A_0P &= A_1 \oplus \cdots \oplus A_k, \\ P^{-1}G_0P &= G_1 \oplus \cdots \oplus G_k, \end{aligned} \tag{40}$$

where the blocks A_j*,* G_j *have corresponding sizes and, for each* j*, are of the following form:*

$$A_j = \mathcal{J}_p(0) \quad \textit{and} \quad G_j = \varepsilon_j \gamma D_p,$$

where $p \in \mathbb{N}$*,* $\gamma \in \mathbb{C}$ *with* $\mathrm{Re}(\gamma) \geq 0$ *and* $\mathrm{Im}(\gamma) > 0$ *if* $\mathrm{Re}(\gamma) = 0$*, and* $\varepsilon_j \in \{+1, -1\}$*. Moreover, the form (40) is unique up to the permutation of blocks.*

Proof. Let $Q \in \mathbb{C}^{n \times n}$ be nonsingular such that

$$Q^{-1} A_0 Q = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad Q^{-1} G_0 Q = \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix},$$

where the spectrum of G_{11} is contained in $\{\gamma, -\gamma\}$ and the spectrum of G_{22} is disjoint from $\{\gamma, -\gamma\}$. Then $-A_0 G_0 = G_0 A_0$ implies $A_{12} = A_{21} = 0$. Hence, we may assume without loss of generality that G_0 has at most the eigenvalues $\gamma, -\gamma$, where $\gamma \in \mathbb{C} \backslash \mathbb{R}$ with $\mathrm{Re}(\gamma) \geq 0$ and $\mathrm{Im}(\gamma) > 0$ if $\mathrm{Re}(\gamma) = 0$. Since G_0 is diagonalizable, this implies in particular that $G_0^2 = \gamma^2 I_n$. Furthermore, we may assume that A_0 is in Jordan canonical form. Thus, we obtain that

$$A_0 = \begin{bmatrix} \mathcal{J}_{p_1} \otimes I_{m_1} & & 0 \\ & \ddots & \\ 0 & & \mathcal{J}_{p_k} \otimes I_{m_k} \end{bmatrix} \quad \text{and} \quad G_0 = \begin{bmatrix} G_{11} & \dots & G_{1k} \\ \vdots & \ddots & \vdots \\ G_{k1} & \dots & G_{kk} \end{bmatrix}$$

for integers $p_1 > \cdots > p_k$, $m_1, \ldots, m_k$ and $G_{qr} \in \mathbb{C}^{m_q \times m_r}$. Setting

$$D := \begin{bmatrix} D_{p_1} \otimes I_{m_1} & & 0 \\ & \ddots & \\ 0 & & D_{p_k} \otimes I_{m_k} \end{bmatrix}$$

and using (24), the fact that A_0 and G_0 anticommute is equivalent to $A_0(DG_0) = (DG_0)A_0$. Therefore, we obtain the following structures for the blocks of G_0:

$$G_{qq} = \sum_{j=0}^{p_q - 1} (D_{p_q} \mathcal{J}_{p_q}^j) \otimes G_{q,q}^{(j)}, \quad G_{qr} = \sum_{j=0}^{p_r - 1} \begin{bmatrix} D_{p_r} \mathcal{J}_{p_r}^j \\ \mathcal{O}_{p_q - p_r, p_r} \end{bmatrix} \otimes G_{q,r}^{(j)} \quad \text{for } q < r,$$

and

$$G_{qr} = \sum_{j=0}^{p_q - 1} \begin{bmatrix} \mathcal{O}_{p_q, p_r - p_q} & D_{p_q} \mathcal{J}_{p_q}^j \end{bmatrix} \otimes G_{q,r}^{(j)} \quad \text{for } q > r,$$

where $G_{q,q}^{(j)}$ and $G_{q,r}^{(j)}$ are matrices of suitable dimensions. We will now reduce G_0 stepwise to canonical form.

Step (1): Since $G_0^2 = \gamma^2 I$, as in Step (1) in the proof of Lemma 7.5, it follows that $G_{q,q}^{(l)}$ is nonsingular.

Step (2): Eliminate $G_{12}, \ldots, G_{1k}$ and $G_{21}, \ldots, G_{k1}$.

Assume that we already have $G_{1,j}^{(s)} = 0$ for all $j = 2, \ldots, k$ and all $s = 0, \ldots, l-1$, and $G_{1,j}^{(l)} = 0$ for $j = 2, \ldots, r-1$, where $l \geq 0$, $r > 1$. We then eliminate $G_{1,r}^{(l)}$. Note that $G_0^2 = \gamma^2 I$ implies that

$$G_{11}G_{1r} + \cdots + G_{1k}G_{kr} = 0$$

for $r > 1$. From this and using an argument similar to the argument in Step (1) in the proof of Lemma 7.5, we obtain that only the blocks $G_{11}G_{1r}$ and $G_{1r}G_{rr}$ contribute to the lth diagonal of the left-hand side. Using (24), this implies that

$$\begin{aligned} 0 &= (D_{p_1} \otimes G_{1,1}^{(0)}) \left(\begin{bmatrix} D_{p_r}\mathcal{J}_{p_r}^l \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \otimes G_{1,r}^{(l)} \right) \\ &\quad + \left(\begin{bmatrix} D_{p_r}\mathcal{J}_{p_r}^l \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \otimes G_{1,r}^{(l)} \right) (D_{p_r} \otimes G_{r,r}^{(0)}) \\ &= \begin{bmatrix} \mathcal{J}_{p_r}^l \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \otimes (G_{1,1}^{(0)}G_{1,r}^{(l)} + (-1)^l G_{1,r}^{(l)}G_{r,r}^{(0)}). \end{aligned} \tag{41}$$

Setting

$$X_0 := \begin{array}{c} \begin{array}{ccc} & \quad r & \end{array} \\ \begin{bmatrix} I & & X_{1r} & \\ & \ddots & & \\ & & \ddots & \\ & & & I \end{bmatrix} \end{array},$$

where

$$X_{1r} = -\frac{1}{2} \begin{bmatrix} \mathcal{J}_{p_r}^l \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \otimes (G_{1,1}^{(0)})^{-1} G_{1,r}^{(l)},$$

we obtain that X_0 commutes with A_0. Furthermore, partitioning $\tilde{G}_0 := X_0^{-1}G_0X_0$ conformably to G_0, for the $(1,r)$-block $\tilde{G}_{1r}$ we obtain that

$$\tilde{G}_{1r} = G_{1r} - X_{1r}G_{rr} + G_{11}X_{1r} - X_{1r}G_{r1}X_{1r}.$$

From this and using (24) and (41), we obtain that the lth diagonal of $\tilde{G}_{1r}$ has the form

$$\begin{aligned} \begin{bmatrix} D_{p_r}\mathcal{J}_{p_r}^l \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \otimes G_{1,r}^{(l)} + \frac{1}{2} \left(\begin{bmatrix} \mathcal{J}_{p_r}^l \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} D_{p_r} \right) \otimes ((G_{1,1}^{(0)})^{-1} G_{1,r}^{(l)} G_{r,r}^{(0)}) \\ -\frac{1}{2} \left(D_{p_1} \begin{bmatrix} \mathcal{J}_{p_r}^l \\ \mathcal{O}_{p_1-p_r,p_r} \end{bmatrix} \right) \otimes G_{1,r}^{(l)} = 0. \end{aligned}$$

Analogously to the proof of Lemma 7.5, we can show that we still have $\tilde{G}_{1,j}^{(l-1)} = 0$ for all $j = 2, \ldots, k$ and $\tilde{G}_{1,j}^{(l)} = 0$ for $j = 2, \ldots, r-1$.

By consecutively repeating Step (2), we can first eliminate $G_{1,j}^{(0)}$ for $j = 2, \ldots, k$, then $G_{1,j}^{(1)}$, and so on. Finally, we can eliminate $G_{12}, \ldots, G_{1k}$. Moreover, we can eliminate $G_{21}, \ldots, G_{k1}$ using transformations of the form

$$X_0 := \begin{array}{c} \\ \\ r \\ \\ \end{array}\begin{bmatrix} I & & & \\ & \ddots & & \\ X_{r1} & & \ddots & \\ & & & I \end{bmatrix},$$

where

$$X_{r1} = -\frac{1}{2} \left[\begin{array}{cc} \mathcal{O}_{p_r, p_1 - p_r} & \mathcal{J}_{p_r}^l \end{array} \right] \otimes (G_{1,r}^{(l)} (G_{1,1}^{(0)})^{-1}).$$

Note that these transformations do not change $G_{12}, \ldots, G_{1k}$.

To complete Step (2), we may finally assume that

$$A = \mathcal{J}_p \otimes I_m, \quad G_0 = \sum_{j=k}^{p-1} (D_p \mathcal{J}_p^k) \otimes G_{0k} = \begin{bmatrix} G_{00} & \cdots & G_{0,p-1} \\ & \ddots & \vdots \\ 0 & & (-1)^{p+1} G_{00} \end{bmatrix},$$

where $p, m \in \mathbb{N}$.

Step (3): Reduce G_0 to block-diagonal form.

Assume that we have $G_{01} = \cdots = G_{0,l-1} = 0$ for some $0 < l \leq p-1$. We then show how to eliminate G_{0l}. The lth block diagonal of G_0^2 has the form

$$\begin{aligned} 0 &= ((D_p \mathcal{J}_p^l) \otimes G_{0l})(D_p \otimes G_{00}) + (D_p \otimes G_{00})((D_p \mathcal{J}_p^l) \otimes G_{0l}) \\ &= \mathcal{J}_p^l \otimes ((-1)^l G_{0l} G_{00} + G_{00} G_{0l}). \end{aligned}$$

Hence

$$G_{0l} G_{00} = (-1)^{l+1} G_{00} G_{01}. \tag{42}$$

The matrix $X_0 := I_p \otimes I_m - \frac{1}{2} \mathcal{J}_p^l \otimes (G_{00}^{-1} G_{0l})$ commutes with A_0. Moreover, X_0^{-1} has the structure

$$X_0^{-1} = I_p \otimes I_m + \frac{1}{2} \mathcal{J}_p^l \otimes (G_{00}^{-1} G_{0l}) + \sum_{k=2}^{\infty} \mathcal{J}_p^{kl} \otimes X_{0k}$$

for some matrices X_{0k}. Hence, setting $\tilde{G}_0 := X_0^{-1} G_0 X_0$, we obtain that the first $l-1$ block diagonals are still zero and that the lth block diagonal has the form

$$\begin{aligned} &\left(\frac{1}{2} \mathcal{J}_p^l \otimes (G_{00}^{-1} G_{0l}) \right) (D_p \otimes G_{00})(I_p \otimes I_m) + (D_p \mathcal{J}_p^l) \otimes G_{0l} \\ &+ (I_p \otimes I_m)(D_p \otimes G_{00}) \left(-\frac{1}{2} \mathcal{J}_p^l \otimes (G_{00}^{-1} G_{0l}) \right) \\ &= 0, \end{aligned}$$

using (24) and (42).

By consecutively repeating this procedure and then applying $\Omega_{m,p}$, we may finally assume that

$$A = I_m \otimes J_p \quad \text{and} \quad G_0 = G_{00} \otimes D_p.$$

Since G_0 is diagonalizable, this also holds for the matrix G_{00}. Moreover, G_{00} has at most the eigenvalues γ and $-\gamma$. Hence, there exists a nonsingular matrix R such that

$$R^{-1}G_{00}R = \begin{bmatrix} \gamma I_q & 0 \\ 0 & -\gamma I_{m-q} \end{bmatrix}$$

for some $q \in \mathbb{N}$. Setting $\mathcal{R} := R \otimes I_p$, we obtain that $\mathcal{R}^{-1}A_0\mathcal{R} = A_0$ and

$$\mathcal{R}^{-1}G\mathcal{R} = \begin{bmatrix} \gamma I_q & 0 \\ 0 & -\gamma I_{m-q} \end{bmatrix} \otimes D_p.$$

The assertion then follows by an appropriate block permutation.

Uniqueness: Analogously to the argument in the proofs of Theorems 4.4 and 4.10, it is sufficient to consider uniqueness for the case that G_0 has at most the eigenvalues $\gamma, -\gamma$ with $\mathrm{Re}(\gamma) \geq 0$ and $\mathrm{Im}(\gamma) > 0$ if $\mathrm{Re}(\gamma) = 0$. Assume that

$$A_0 = \begin{bmatrix} I_{m_p} \otimes \mathcal{J}_p & & \\ & \ddots & \\ & & I_{m_1} \otimes \mathcal{J}_1 \end{bmatrix}, \quad G_0 = \gamma \begin{bmatrix} \Sigma_{m_p} \otimes D_p & & \\ & \ddots & \\ & & \Sigma_{m_1} \otimes D_1 \end{bmatrix},$$

$$\text{and} \quad \tilde{G}_0 = \gamma \begin{bmatrix} \tilde{\Sigma}_{m_p} \otimes D_p & & \\ & \ddots & \\ & & \tilde{\Sigma}_{m_1} \otimes D_1 \end{bmatrix},$$

where we allow $m_j = 0$ for some $j = 1, \ldots, p$ and where Σ_{m_j} and $\tilde{\Sigma}_{m_j}$ are signature matrices. To prove the uniqueness of the form (40), we have to show that if $S \in \mathbb{C}^{n\times n}$ is nonsingular such that $S^{-1}A_0S = A_0$ and $S^{-1}G_0S = \tilde{G}_0$, then Σ_{m_j} and $\tilde{\Sigma}_{m_j}$ are similar for $j = 1, \ldots, p$.

Note that for each Jordan block there exists a Jordan chain $\{x_{\alpha\beta}^{(1)}, \ldots, x_{\alpha\beta}^{(\alpha)}\}$, where $\alpha = p, \ldots, 1$ and $\beta = 1, \ldots, m_p$. Let P be the permutation matrix that reorders these chains in the following way. First, we collect $x_{\alpha\beta}^{(1)}$ for $\alpha = p, \ldots, 1$, $\beta = 1, \ldots, m_p$,

ELA

then $x^{(2)}_{\alpha\beta}$ for $\alpha = p, \ldots, 2$, $\beta = 1, \ldots, m_p$, and so on. Setting $q_r = \sum_{j=1}^{r} m_j$, we have

$$\hat{A}_0 := P^{-1}A_0P = \begin{array}{cc} & \begin{array}{ccccc} q_p & q_{p-1} & q_{p-2} & \cdots & q_1 \end{array} \\ \begin{array}{c} q_p \\ q_{p_1} \\ q_{p-2} \\ \vdots \\ q_1 \end{array} & \left[\begin{array}{ccccc} 0 & \begin{bmatrix} I_{q_{p-1}} \\ 0 \end{bmatrix} & 0 & & \\ & 0 & \begin{bmatrix} I_{q_{p-2}} \\ 0 \end{bmatrix} & \ddots & \\ & & 0 & \ddots & 0 \\ & & & \ddots & \begin{bmatrix} I_{q_1} \\ 0 \end{bmatrix} \\ & & & & 0 \end{array}\right] \end{array}.$$

Moreover,

$$\hat{G}_0 := P^{-1}G_0P = \gamma \begin{bmatrix} G_{11} & & 0 \\ & \ddots & \\ 0 & & G_{pp} \end{bmatrix}$$

$$\text{and} \quad \hat{\tilde{G}}_0 := P^{-1}\tilde{G}_0P = \gamma \begin{bmatrix} \tilde{G}_{11} & & 0 \\ & \ddots & \\ 0 & & \tilde{G}_{pp} \end{bmatrix},$$

where

$$G_{jj} = (-1)^{j+1} \begin{bmatrix} \Sigma_{m_p} & & 0 \\ & \ddots & \\ 0 & & \Sigma_{m_j} \end{bmatrix} \quad \text{and} \quad \tilde{G}_{jj} = (-1)^{j+1} \begin{bmatrix} \tilde{\Sigma}_{m_p} & & 0 \\ & \ddots & \\ 0 & & \tilde{\Sigma}_{m_j} \end{bmatrix}.$$

Assume that there exists a nonsingular matrix T such that $T^{-1}\hat{A}_0T = \hat{A}_0$ and $T^{-1}\hat{G}_0T = \hat{\tilde{G}}_0$. Then the structure of $\hat{A}_0$ implies that T is block upper triangular with a block structure corresponding to $\hat{A}_0$. But then we obtain, in particular, that G_{jj} and $\tilde{G}_{jj}$ are similar for each j. This implies that Σ_{m_j} and $\tilde{\Sigma}_{m_j}$ are similar for each j. $\square$

Lemma 7.8. *Let $G, H \in \mathbb{C}^{n\times n}$ be Hermitian nonsingular such that the pencil $\varrho H - G$ is nondefective and such that its spectrum is contained in $\{\gamma, -\gamma, \overline{\gamma}, -\overline{\gamma}\}$, where $\gamma^2 \in \mathbb{C}\backslash\mathbb{R}$ and $\mathrm{Re}(\gamma)\mathrm{Im}(\gamma) \geq 0$. Furthermore, let $A \in \mathbb{C}^{n\times n}$ be nilpotent, H-self-adjoint, and G-skew-adjoint. Then there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} P^{-1}AP &= A_1 \oplus \cdots \oplus A_k, \\ P^*GP &= G_1 \oplus \cdots \oplus G_k, \\ P^*HP &= H_1 \oplus \cdots \oplus H_k, \end{aligned} \tag{43}$$

where, for each j, the blocks A_j, G_j, H_j have corresponding sizes and are of the following form:

Type (2d):

$$A_j = \begin{bmatrix} \mathcal{J}_p(0) & 0 \\ 0 & \mathcal{J}_p(0)^* \end{bmatrix}, \quad H_j = \begin{bmatrix} 0 & Z_p \\ Z_p & 0 \end{bmatrix},$$

$$\text{and} \quad G_j = \begin{bmatrix} 0 & \varepsilon\gamma F_p \\ \varepsilon(-1)^{p+1}\overline{\gamma} F_p & 0 \end{bmatrix},$$

where $p \in \mathbb{N}$ and $\varepsilon \in \{+1, -1\}$.

Moreover, the form (43) is unique up to the permutation of blocks.

Proof. Using the same argument as in Case (2) of the proof of Theorem 4.10, we may assume that A, H, and G have the following forms:

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & A_0^* \end{bmatrix}, \quad H = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \text{and} \quad G = \begin{bmatrix} 0 & G_0^* \\ G_0 & 0 \end{bmatrix},$$

where

$$H^{-1}GH^{-1}G = \begin{bmatrix} \overline{\gamma}^2 I & 0 \\ 0 & \gamma^2 I \end{bmatrix}.$$

This implies, in particular, that $G_0^2 = \overline{\gamma}^2 I$. From $-A^*G = GA$, we obtain that A_0 and G_0 anticommute. We will now reduce G by congruence transformations with matrices of the form

$$X = \begin{bmatrix} X_0 & 0 \\ 0 & X_0^{-*} \end{bmatrix}.$$

Then

$$X^{-1}AX = \begin{bmatrix} X_0^{-1}A_0X_0 & 0 \\ 0 & (X_0^{-1}A_0X_0)^* \end{bmatrix}, \quad X^*HX = H, \quad \text{and}$$

$$X^*GX = \begin{bmatrix} 0 & (X_0^{-1}G_0X_0)^* \\ X_0^{-1}G_0X_0 & 0 \end{bmatrix}.$$

Thus, the problem of reducing G, while keeping the forms of A and H, reduces to the problem of finding a canonical form for A_0 and G_0 under simultaneous similarity. This is done in Lemma 7.7. Hence, the result follows from noting that the spectrum of G_0 is contained in $\{\overline{\gamma}, -\overline{\gamma}\}$ and applying the Z-trick.

Uniqueness: Assume that

$$A = \begin{bmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J} \end{bmatrix}, H = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, G_1 = \begin{bmatrix} 0 & G_{11}^* \\ G_{11} & 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & G_{22}^* \\ G_{22} & 0 \end{bmatrix},$$

where $\mathcal{J}$ is a nilpotent matrix in Jordan canonical form, G_1, G_2 are Hermitian, and $\sigma(G_{11}) = \sigma(G_{22}) \subseteq \{\overline{\gamma}, -\overline{\gamma}\}$. Furthermore, assume that $T^{-1}AT = A$, $T^*HT = H$, and $T^*G_1T = G_2$ for some nonsingular matrix T. Partitioning

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \quad \text{and} \quad T^{-*} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

conformably with A, H, and G, we obtain that

$$T_{12} = S_{21} \quad \text{and} \quad G_{11}T_{12} = S_{21}G_{22}^* = T_{12}G_{22}^*.$$

This implies that $T_{12} = 0$. Analogously, we show that $T_{21} = 0$ and hence, we obtain by symmetry $T_{22} = T_{11}^{-*}$. Hence, the uniqueness of the form (43) follows from the uniqueness property in Lemma 7.7. □

REFERENCES

[1] G. Ammar, C. Mehl, and V. Mehrmann. Schur-like forms for matrix Lie groups, Lie algebras and Jordan algebras. *Linear Algebra Appl.*, 287:11–39, 1999.
[2] D.Z. Djokovic, J. Patera, P. Winternitz, and H. Zassenhaus. Normal forms of elements of classical real and complex Lie and Jordan algebras, *J. Math. Phys.*, 24:1363–1374, 1983.
[3] U. Flaschka. *Eine Variante des Lanczos-Algorithmus für große, dünn besetzte symmetrische Matrizen mit Blockstruktur.* Dissertation, Universität Bielefeld, Bielefeld, Germany, 1992.
[4] U. Flaschka, W.-W. Lin, and J.-L. Wu. A kqz algorithm for solving linear-response eigenvalue equations. *Linear Algebra Appl.*, 165:93–123, 1992.
[5] F. Gantmacher. *Theory of Matrices*, Volume 1. Chelsea, New York, 1959.
[6] F. Gantmacher. *Theory of Matrices*, Volume 2. Chelsea, New York, 1959.
[7] I. Gohberg, P. Lancaster, and L. Rodman. *Matrices and Indefinite Scalar Products.* Birkhäuser Verlag, Basel, Boston, Stuttgart, 1983.
[8] A. Hansen, B. Voigt, and S. Rettrup. Large-scale RPA calculations of chiroptical properties of organic molecules: Program RPAC. *Internat. J. Quantum Chem.*, XXIII:595–611, 1983.
[9] R. Horn and C. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, Cambridge, 1991.
[10] L. Kronecker. Algebraische Reduction von Scharen bilinearer Formen, 1890. In *Collected Works III (second part)*, pages 141–155, Chelsea, New York, 1968.
[11] P. Lancaster and L. Rodman. *Algebraic Riccati Equations.* Clarendon Press, Oxford, 1995.
[12] P. Lancaster and M. Tismenetsky. *The Theory of Matrices*, 2nd edition. Academic Press, Orlando, 1985.
[13] W.-W. Lin, V. Mehrmann, and H. Xu. Canonical forms for Hamiltonian and symplectic matrices and pencils. *Linear Algebra Appl.*, 301–303:469–533, 1999.
[14] C. Mehl. *Compatible Lie and Jordan algebras and applications to structured matrices and pencils.* Dissertation, Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany, 1998.
[15] C. Mehl. Condensed forms for skew-Hamiltonian/Hamiltonian pencils. *SIAM J. Matrix Anal. Appl.*, 21:454–476, 1999.
[16] V. Mehrmann and H. Xu. Structured Jordan canonical forms for structured matrices that are Hermitian, skew Hermitian or unitary with respect to indefinite inner products. *Electron. J. Linear Algebra*, 5:67–103, 1999.
[17] J. Olson, H. Jensen, and P. Jørgensen. Solution of large matrix equations which occur in response theory. *J. Comput. Phys.*, 74:265–282, 1988.
[18] R. Thompson. The characteristic polynomial of a principal subpencil of a Hermitian matrix pencil. *Linear Algebra Appl.*, 14:135–177, 1976.
[19] K. Weierstrass. Zur Theorie der bilinearen und quadratischen Formen. *Monatsb. Akad. d. Wiss. Berlin*, pages 310–338, 1867.