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THE QUATERNIONIC DETERMINANT*

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Abstract. The determinant for complex matrices cannot be extended to quaternionic matrices. Instead, the Study determinant and the closely related q-determinant are widely used. In this paper it is shown that the Study determinant can be characterized as the unique functional extending the *absolute value* of the complex determinant, and its spectral and linear algebraic aspects are discussed.

Key words. Quaternions, Matrices, Determinant.

AMS subject classifications. 15A09, 15A33

1. Introduction. Quaternionic linear algebra is attracting growing interest in theoretical physics [1]–[5], mainly in the context of quantum mechanics and field theory [6]. Quaternionic mathematical structures have recently appeared in discussions of eigenvalue equations [7], [8], group theory [9], [10], and the grand unification model [11], [12] within a quaternionic formulation of quantum physics.

The question of extending the determinant from complex to quaternionic matrices has been considered in the physical literature [4]–[6]. The possibility of such an extension was contemplated by Cayley [13], without much success, as early as 1845. A canonical determinant functional was introduced by Study [14] and its properties axiomatized by Dieudonné [15]. The details can be found in the excellent survey paper of Aslaksen [16]. Study's determinant is denoted as Sdet, and, up to a trivial power factor, is identical to the q-determinant, $\det_q$, found in most of the literature [17], and to Dieudonné's determinant, denoted as Ddet. Specifically, $\det_q = \mathrm{Sdet}^2 = \mathrm{Ddet}^4$.

In these works, Sdet was considered as a *generalization* of the determinant, det, in the sense that the two functionals share a common set of axioms. Specifically, Sdet is the unique, up to a trivial power factor, functional $\mathcal{F} : \mathbb{H}^{n\times n}$ that satisfies the following three axioms [16]:

1. $\mathcal{F}(A) = 0$ if and only if A is singular;
2. $\mathcal{F}(AB) = \mathcal{F}(A)\mathcal{F}(B)$ (multiplicativity);
3. $\mathcal{F}(I + rE_{ij}) = 1$ for $i \neq j$ and $r \in \mathbb{H}$.

However, Sdet *does not truly extend* det. Indeed, the two functionals do not coincide on complex matrices, since the former is nonnegative while the latter is truly complex. In this paper we show that Sdet *does extend the nonnegative functional* $|\det|$, namely, the two functionals coincide for complex matrices. More precisely, we show the following:

1. There exists no multiplicative functional on quaternionic matrices that coincides with det on complex matrices.

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2. Sdet is the *only* nonconstant multiplicative functional that coincides with $|\det|$ on complex matrices. (We remark that just like $\det[M] \neq 0$, the inequality $|\det[M]| \neq 0$ characterizes nonsingular matrices over the complex numbers. The same central role in group theory over the quaternions will be played by $\mathrm{Sdet}[M] \neq 0$.)
3. The identities $|\det(M)| = \prod |\lambda_i|$, in terms of eigenvalues, and $|\det(M)| = \prod \sigma_i$, in terms of singular values, extend to M quaternionic. Thus, although Sdet is originally defined through complexification [18]–[21], it can be given concrete spectral and numerical-analytic interpretations that do not require complexification.
4. The Schur complements identity for complex matrices

$$\left|\det \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right| = |\det[A]|\,|\det[D - CA^{-1}B]|$$

extends to quaternionic matrices.

Also, we discuss formulas for $\det[H]$, $\mathrm{Adj}[H]$, and H^{-1} based on the classical permutation and minor calculation (some of this material can be found in [22]). It is interesting that this approach, pursued by Cayley without success in the context of *general* quaternionic matrices, is valid in the *Hermitian* case. The functional det, defined this way for Hermitian quaternionic matrices, is *not* multiplicative. Note that under the definition $\mathrm{Sdet}[M] := \det[M^+M]$, one can extend the Study determinant to nonsquare matrices.

In the last section we also discuss some open problems concerning the behavior of the determinant and the difficulties of extending the formula $M^{-1} = \mathrm{Adj}(M)/\det(M)$ to quaternions.

2. Notation. Quaternions, introduced by Hamilton [23], [24] in 1843, can be represented by four real quantities

$$q = a + ib + jc + kd, \quad a, b, c, d \in \mathbb{R},$$

and three imaginary units i, j, k satisfying

$$i^2 = j^2 = k^2 = ijk = -1.$$

We will denote by

$$\mathrm{Re}[q] := a \quad \text{and} \quad \mathrm{Im}[q] := q - a = ib + jc + kd$$

the real and imaginary parts of q. The quaternion skew-field $\mathbb{H}$ is an associative but noncommutative algebra of rank 4 over $\mathbb{R}$ endowed with an involutory operation called quaternionic conjugation

$$\bar{q} = a - ib - jc - kd = \mathrm{Re}[q] - \mathrm{Im}[q]$$

satisfying $\overline{pq} = \overline{q}\,\overline{p}$ for all $q, p \in \mathbb{H}$. The quaternion norm $|q|$ is defined by

$$|q|^2 = q\bar{q} = a^2 + b^2 + c^2 + d^2.$$

Among the properties of the norm, to be used in subsequent sections, we mention here the following:

$$|pq| = |qp| = |q||p| \quad \text{and} \quad |1 - pq| = |1 - qp|.$$

Every nonzero quaternion q has a unique inverse

$$q^{-1} = \bar{q}/|q|^2.$$

Two quaternions p and q are called similar if

$$q = s^{-1}ps \quad \text{for some } s \in \mathbb{H}.$$

By replacing s with $u = s/|s|$, we may always assume s to be unitary. The usual complex conjugation in $\mathbb{C}$ may be obtained by choosing $s = j$ or $s = k$. A necessary and sufficient condition for the similarity of p and q is given by

$$\mathrm{Re}[q] = \mathrm{Re}[p] \quad \text{and} \quad |\mathrm{Im}[q]| = |\mathrm{Im}[p]|.$$

An equivalent condition is $Re[q] = Re[p]$ and $|q| = |p|$. Every similarity class contains a complex number, unique up to conjugation. Namely, every quaternion q is similar to $\mathrm{Re}[q] \pm i|\mathrm{Im}[q]|$. In particular, q and $\bar{q}$ are similar. It can be seen that $s \in \mathbb{H}$ conjugates q and $\bar{q}$ (i.e., $\bar{q} = s^{-1}qs$) if and only if $\mathrm{Im}[q] = 0$ or $\mathrm{Re}[qs] = \mathrm{Re}[s] = 0$. However, there exists no fixed $s \in \mathbb{H}$ that conjugates q and $\bar{q}$ for all $q \in \mathbb{H}$.

3. Spectral theory. Spectral theory for complex matrices admits several possible quaternionic extensions that do not necessarily respect the fundamental theorem of algebra [7], [8], [25]–[28]. We shall be interested in the extension usually described as "right eigenvalues" [7], [8], [29].

Every $n \times n$ quaternion matrix M is similar to an upper-triangular matrix. This can be shown just as in the complex case. Using elementary Gaussian operations, the general case can be reduced to the case of 2×2 matrices, where one wishes to find $\alpha \in \mathbb{H}$ so that

$$\begin{bmatrix} \star & \star \\ 0 & \star \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\alpha & 1 \end{bmatrix}$$

given $a, b, c, d \in \mathbb{H}$. By permutation similarity we may assume that $b \neq 0$. Solvability for α is expressed by the noncommuting quadratic equation

$$\alpha^2 b + \alpha(d - a) - c = 0,$$

which always has a solution [25], [26].

Note that in the complex case the similarity matrix obtained in this procedure is not in general unitary; however, a different procedure, Schur's lemma, triangularizes the matrix using unitary similarity. Schur's lemma has been extended to quaternionic matrices [30].

A modified version of the Jordan canonical form is valid for quaternionic matrices. Namely, every matrix $M \in \mathbb{H}^{n\times n}$ is similar, over the quaternions, to a *complex* Jordan matrix J, defining a set of n complex eigenvalues. However, the eigenvalues $\lambda_i \in \mathbb{C}$ are determined only up to complex conjugation [18].

The Schur and Jordan canonical forms are associated with *right eigenvalues* $M\psi = \psi q$, $\psi \in \mathbb{H}^{n\times 1}$, $q \in \mathbb{H}$, which are determined only up to quaternionic similarity. This is further discussed in [7], [8], [19], [29].

Let us denote by $\mathcal{Z}[M]$ the complexification [18], [20], [21], [31] of the quaternionic matrix M, i.e.,

$$\mathcal{Z}[M] := \begin{bmatrix} M_1 & -M_2^* \\ M_2 & M_1^* \end{bmatrix}, \quad M = M_1 + jM_2, \quad M_{1,2} \in \mathbb{C}^{n\times n}. \tag{1}$$

It has been shown in [18] that if J is the complex Jordan form of M, then $J \oplus J^*$ is the Jordan form of $\mathcal{Z}[M]$. Consequently, the spectrum of $\mathcal{Z}[M]$ is $\{\lambda_1, \lambda_1^*, \ldots, \lambda_n, \lambda_n^*\}$.

4. On extending the determinant. The difficulty in extending the determinant to quaternions results from the lack of commutativity. Starting with Cayley's itself, expressions for the determinant have failed. Let us consider the case of 2×2 matrices. Here, one may consider four different generalizations:

$$ad - cb, \quad ad - bc, \quad da - cb, \quad da - bc. \tag{2}$$

It is easy to see that none of these expressions, alone or jointly, has relevance to the invertibility of the matrix. Consider, for example, the two matrices

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ j & k \end{bmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{bmatrix} i & j \\ j & i \end{bmatrix}. \tag{3}$$

In the case of A, exactly two expressions in (2) vanish; in the case of B, all four expressions are zero. However, both A and B are unitary.

In a different line of attack, one may look for multiplicative functionals $\mathcal{F}$ from $\mathbb{H}^{n\times n}$ to $\mathbb{H}$ that coincide with the determinant on complex matrices. Again, the result is negative:

There is no multiplicative functional

$$\mathcal{F} : \mathbb{H}^{n\times n} \to \mathbb{H}$$

that coincides with det *on complex diagonal matrices.*

Proof. It is enough to obtain one counterexample for $n = 2$. Consider the 2×2 matrices

$$M = \begin{bmatrix} 1+i & 0 \\ 0 & i \end{bmatrix}, \quad N = \begin{bmatrix} 1+i & 0 \\ 0 & -i \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}.$$

Since S is invertible, $\mathcal{F}[S] \neq 0$; see Lemma 5.2. Since $SM = NS$, we conclude that $\mathcal{F}[S]\mathcal{F}[M] = \mathcal{F}[N]\mathcal{F}[S]$, hence $\mathcal{F}[M]$ and $\mathcal{F}[N]$ should be similar. This is a contradiction because obviously $\mathrm{Re}\,\{\mathcal{F}[M]\} \neq \mathrm{Re}\,\{\mathcal{F}[N]\}$. $\square$

5. On extending the absolute value of the determinant. A more positive result is obtained with respect to the functional $|\det|$.

THEOREM 5.1. *Sdet is the unique functional*

$$\mathcal{D} : \mathbb{H}^{n\times n} \to \mathbb{R}_+ \tag{4}$$

that is multiplicative, i.e.,

$$\mathcal{D}[MN] = \mathcal{D}[NM] = \mathcal{D}[M]\mathcal{D}[N], \tag{5}$$

and satisfies the scaling condition

$$\mathcal{D}[qI] = |q|^n \quad \forall q \in \mathbb{H}. \tag{6}$$

Before proving this theorem, we make some observations concerning nonnegative multiplicative functionals. The only nontrivial part here is the last property, which has been proved elsewhere.

LEMMA 5.2. *If $\mathcal{F} : \mathbb{H}^{n\times n} \to \mathbb{R}_+$ is a nonconstant multiplicative functional, then*

1. *$\mathcal{F}[S] = 1$ if $S^2 = I$;*
2. *$\mathcal{F}[S]\mathcal{F}[S^{-1}] = 1$ and $\mathcal{F}[S^{-1}MS] = \mathcal{F}[M]$ if S is invertible;*
3. *$\mathcal{F}[P] = 1$ for all permutation matrices P;*
4. *$\mathcal{F}[M] = 0$ if and only if M is singular.*

Proof. Multiplicativity and nontriviality imply that $\mathcal{F}[I] = 1$. Now items 1–2 become trivial consequences of multiplicativity. Item 3 follows from the fact that every permutation matrix is a product of elementary permutation matrices P_i with $P_i^2 = I$. As for item 4, if M is not singular, then applying $\mathcal{F}$ to $MM^{-1} = I$ implies that $\mathcal{F}[M] \neq 0$. If M is singular, $\mathcal{F} \neq 0$ by a result of [32]; see [16, p. 58]. □

Proof of Theorem 5.1. Let $\mathcal{D}$ be a functional satisfying (4), (5), (6). Let $\{E_{ij}\}_{i,j=1}^n$ be the usual canonical basis over $\mathbb{H}$ in $\mathbb{H}^{n\times n}$. Let $q \in \mathbb{H}$ be nonzero. Consider the n diagonal elementary matrices

$$M_i(q) := I + (q-1)E_{ii}$$

and the $n(n-1)/2$ upper-triangular elementary matrices

$$M_{ij}(q) := I + qE_{ij}, \quad i < j.$$

First we show that

$$\mathcal{F}[M_i(q)] = |q|. \tag{7}$$

Indeed, by permutation similarity we see that $\mathcal{F}[M_i(q)]$ is independent of $1 \leq i \leq n$. So set $f(q) := \mathcal{F}[M_i(q)]$. We have $qI = \prod_{i=1}^n M_i(q)$, hence

$$|q|^n = \mathcal{F}[qI] = \prod_{i=1}^n \mathcal{F}[M_i(q)] = f^n(q).$$

Hence $f(q) = |q|$, as claimed. Next we show that

$$\mathcal{F}[M_{ij}(q)] = 1. \tag{8}$$

Indeed, it is easy to see that

$$M_{ij}^{-1}(q) = M_{ij}(-q) = M_i(-1)M_{ij}(q)M_i(-1),$$

hence $\mathcal{F}[M_{ij}^{-1}(q)] = \mathcal{F}[M_{ij}(q)]$. Now (8) follows by multiplicativity.

We have therefore established that $\mathcal{F} : \mathbb{H}^{n\times n} \to \mathbb{H}$ satisfies the three Dieudonné conditions (5), (8), and item 4 in Lemma 5.2. Therefore, according to Dieudonné's result [16], $\mathcal{F} = \mathrm{Ddet}^{2r} = \mathrm{Sdet}^r$ for some $r \in \mathbb{R}$. Due to (6), it is readily seen that $r = 1$. ◻

Note that, in general, if $\mathcal{F}$ is multiplicative and $r \in \mathbb{R}$, then $\mathcal{F}^r$ is also multiplicative. Therefore, we have a one-parameter group of nonnegative multiplicative functionals $\{\mathrm{Sdet}^r : r \in \mathbb{R}\}$. (The case $r = 0$ is interesting: it leads to the functional whose value is 1 on all the invertible matrices and 0 otherwise.) In view of Theorem 5.1 we conclude that this one-parameter family and the two constant functionals $\mathcal{F}_0[M] \equiv 0$ and $\mathcal{F}_1[M] \equiv 1$ are the only nonnegative multiplicative functionals on quaternionic matrices.

6. Concrete description of the Study determinant. Theorem 5.1 has the following main corollaries.

Corollary 6.1. *If M is upper triangular, then Sdet$(M) = \prod_{i=1}^n |M_{ii}|$.*

Proof. This follows easily by writing M explicitly as a product of elementary matrices using (7), (8). ◻

Corollary 6.2. *For all matrices M, Sdet$(M) = \prod_{i=1}^n |\lambda_i|$, where λ_i are the eigenvalues of M.*

Proof. By Lemma 5.2, item 2, it is enough to consider the Jordan form, or the Schur form, of M, which is of the type considered by Corollary 6.1. ◻

Since the eigenvalue identity just exhibited, restricted to complex matrices, is also valid for $|\det|$, we get immediately the following corollary.

Corollary 6.3. *For complex matrices we have Sdet$(M) = |\det(M)|$.*

Let us define the adjoint of M by $(M^+)_{ij} = \overline{M_{ji}}$. A matrix $U \in \mathbb{H}^{n\times n}$ is called unitary if $U^+U = I$. According to the quaternionic Schur lemma [30], every $n \times n$ quaternionic matrix M can be written as $M = U^+TU$, where U is unitary and T is triangular. Since, in addition, (8) is obviously valid for lower- as well as upper-triangular matrices, we get the following result.

Corollary 6.4. *Sdet(M^+) = Sdet(M). In particular, Sdet$(U) = 1$ if U is unitary.*

The identity $\mathrm{Sdet}(M) = 1$ may be taken as a basis to define the group of *unimodular* matrices.

Next, we calculate $\mathcal{F}$ in terms of singular values. The singular value decomposition (SVD) for complex matrices extends to quaternionic matrices in a straightforward way. Every $n \times n$ quaternionic matrix M has the SVD $M = U\Sigma V$, where U and V are unitary, $\Sigma = \Sigma_1 \oplus 0$, $\Sigma_1 = \sigma_1 \oplus \cdots \oplus \sigma_k$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq 0$ are the singular values of M [18], [30], [33], [34]. In these terms the following holds.

Corollary 6.5. $\mathcal{F}[M] = \prod_{i=1}^{n} \sigma_i$.

7. Hermitian matrices. A quaternionic matrix H is called Hermitian if $H^+ = H$. As we saw in Section 4, the common determinant cannot be extended to quaternionic matrices. However, it can be extended to *Hermitian* quaternionic matrices. The usual definition of the determinant in terms of permutations was generalized in the Chinese literature; see for example [22]. Another possible definition is analogous to Corollaries 6.2 and 6.5:

$$|H|_r = \prod_{i=1}^{n} \lambda_i.$$

Note that, for Hermitian matrices, the eigenvalues are uniquely determined and real. This follows from the fact that $\mathcal{Z}[M]$ is also Hermitian. Note that the set of Hermitian matrices is not closed under products, and the functional det : $H \to |H|_r$ is not multiplicative. However, it is invariant under congruence.

It is easy to show that, for Hermitian matrices, the following are equivalent:

1. H is positive definite, i.e., $x^+Hx > 0$ for all nonzero $x \in \mathbb{H}^{n\times 1}$.
2. All the eigenvalues λ_i are positive.
3. All the (signed) *real determinants* of the principal minors are positive.

We conclude this section by comparing the functional Sdet$[M]$, the functional $|H|_r$ just defined, and the *q-determinant* [17]

$$|M|_q = \det\left\{\mathcal{Z}\left[M\right]\right\},$$

when $\mathcal{Z}\left[M\right]$ is defined in (1). From previous considerations, we have

$$|M|_q = \prod_{i=1}^{n} |\lambda_i|^2 = \mathrm{Sdet}[M^+]\mathrm{Sdet}[M] = \mathrm{Sdet}^2[M] = \left|M^+M\right|_r.$$

Using this equation, one can extend the definition of Sdet from square to nonsquare matrices. This approach is found in [22], where the resulting functional is called the *double determinant*.

8. Schur complements. Let $\mathcal{R}$ be an associative ring. A matrix $M \in \mathcal{R}^{n\times n}$ is called invertible if $MN = NM = I_n$ for some $N \in \mathcal{R}^{n\times n}$, which is necessarily unique. It is shown in [17] that in the case $\mathcal{R} = \mathbb{H}$, $MN = I_n$ implies $NM = I_n$.

The Schur complements procedure [35] is a powerful tool in calculating inverses of matrices over rings. Let us write a generic n-dimensional matrix M in block form:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Assuming that $A \in \mathcal{R}^{k\times k}$ is invertible, one has

$$M = \begin{bmatrix} I_k & 0 \\ CA^{-1} & I_{n-k} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & A_S \end{bmatrix} \begin{bmatrix} I_k & A^{-1}B \\ 0 & I_{n-k} \end{bmatrix}, \tag{9}$$

with

$$A_S := D - CA^{-1}B.$$

We shall call A_S the *Schur complement* of A in M.

The invertibility of A ensures that the matrix M is invertible if and only if A_S is invertible, and the inverse is given by

$$M^{-1} = \begin{bmatrix} I_k & -A^{-1}B \\ 0 & I_{n-k} \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & {A_S}^{-1} \end{bmatrix} \begin{bmatrix} I_k & 0 \\ -CA^{-1} & I_{n-k} \end{bmatrix}. \tag{10}$$

The inversion of an n-dimensional matrix is thus reduced to the inversion of two smaller matrices, $A \in \mathcal{R}^{k\times k}$ and $A_S \in \mathcal{R}^{(n-k)\times(n-k)}$ (plus some multiplications); repeated use of this size reduction can be used to invert the matrix efficiently. It is not as efficient as Gaussian elimination, but the latter may not be available in general rings.

Corollary 8.1.

$$Sdet \begin{bmatrix} A & B \\ C & D \end{bmatrix} = Sdet[A]Sdet[D - CA^{-1}B]$$

as long as A^{-1} exists.

Proof. Indeed, from the construction of Sdet in the last section, we see that its value on each of the two block-triangular matrices in (9) is 1; since the eigenvalues of a direct sum are the union of the eigenvalues of the summands, we get that $\mathrm{Sdet}[A \oplus A_S] = \mathrm{Sdet}[A]\mathrm{Sdet}[A_S]$. This plus multiplicativity implies the result. □

As a result of the Schur complements determinant formula just exhibited, we get the following commutation formula for Sdet, which generalizes a well-known property of det (actually, of $|\det|$).

Corollary 8.2. *$Sdet[I + MN] = Sdet[I + NM]$ for all $M \in \mathbb{H}^{n\times m}$ and $N \in \mathbb{H}^{m\times n}$.*

Proof. Indeed, consider the matrix

$$\begin{bmatrix} I_1 & N \\ M & I_2 \end{bmatrix}$$

and apply to it Schur complements with respect to both I_1 and I_2, respectively. We get

$$\mathrm{Sdet} \begin{bmatrix} I_1 & N \\ M & I_2 \end{bmatrix} = \mathrm{Sdet}[I_1]\mathrm{Sdet}[I_2 - MI_1^{-1}N]$$

and

$$\mathrm{Sdet} \begin{bmatrix} I_1 & N \\ M & I_2 \end{bmatrix} = \mathrm{Sdet}[I_2]\mathrm{Sdet}[I_2 - NI_1^{-1}M],$$

implying the identity. □

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9. The case of 2 × 2 matrices. In this last section, we discuss inversion, adjoint, and determinant for 2×2 quaternionic matrices.

9.1. Inversion. Let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be an invertible 2×2 matrix with quaternionic entries. When a, b, c, d are all nonzero, four parallel applications of the Schur complements formula (10) lead to a concrete description of the inverse:

$$M^{-1} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix}, \tag{11}$$

where

$$\begin{aligned} \tilde{a} &= (a - bd^{-1}c)^{-1}, \quad & \tilde{b} &= (c - db^{-1}a)^{-1}, \\ \tilde{c} &= (b - ac^{-1}d)^{-1}, \quad & \tilde{d} &= (d - ca^{-1}b)^{-1}; \end{aligned} \tag{12}$$

see Gürsey and Tze [36, p. 115]. The invertibility of M guarantees that these four values are well-defined nonzero quaternions. What happens if some of the entries of M vanish? Assume for example that $a = 0$. The invertibility of M implies that $b, c \neq 0$. Consequently, the element $d - ca^{-1}b$ has *infinite* modulus. In this case, we define

$$\tilde{d} := \lim_{a \to 0} (d - ca^{-1}b)^{-1}.$$

A simple calculation,

$$\begin{aligned} |\tilde{d}| &:= \lim_{a \to 0} \frac{1}{|d - ca^{-1}b|} = \lim_{a \to 0} \frac{1}{|c||c^{-1}d - a^{-1}b|} \\ &= \lim_{a \to 0} \frac{|a|}{|c||ac^{-1}d - b|} = \lim_{a \to 0} \frac{|a|}{|c||b|} = 0, \end{aligned}$$

shows that $\tilde{d} = 0$. Thus,

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & 0 \end{bmatrix}, \quad \tilde{a} = -c^{-1}db^{-1}, \quad \tilde{b} = c^{-1}, \quad \tilde{c} = b^{-1}.$$

We conclude that (12) remains valid under appropriate conventions, when some entries in M are zero. We do not have a clear generalization of this phenomenon for $n > 2$.

9.2. Adjoint. Equations (11), (12) are valid in every associative ring $\mathcal{R}$. In the case that $\mathcal{R}$ is also commutative, (11), (12) reduce to the well-known formula

$$M^{-1} = \frac{\mathrm{Adj}[M]}{\det[M]}. \tag{13}$$

In calculating the inverse of real and complex matrices, (13) is of great theoretical importance. So far, we have failed to generalize this formula to quaternion matrices. At first sight, it might make sense to conjecture a noncommuting expression of the general form

$$M^{-1} = P\mathrm{Adj}[M]Q \tag{14}$$

with quaternionic diagonal matrices $P = \mathrm{diag}\{p_1, p_2\}$ and $Q = \mathrm{diag}\{q_1, q_2\}$. Nevertheless, the resulting constraints

$$\begin{array}{ll} p_1 = \tilde{a}q_1 d^{-1}, & p_2 = -\tilde{c}q_1 c^{-1}, \\ p_1 = -\tilde{b}q_2 b^{-1}, & p_2 = \tilde{d}q_1 a^{-1}, \end{array}$$

which, for commutative fields, are satisfied if $P = \det^{-1}[M]I$ and $Q = I$, are not always solvable. For example, the first matrix in (3) cannot be written in the form (14). Whether a further weakening, beyond (14), of (13) is valid for quaternionic matrices remains an open problem. The mere definition of $\mathrm{Adj}[M]$, $M \in \mathbb{H}^{n\times n}$, $n > 2$, preserving (13), is not clear.

A different generalization of (13) for 2×2 quaternionic matrices may be obtained using a Hadamard product between a nonnegative matrix and a termwise-unitary quaternionic matrix:

$$M^{-1} = \frac{1}{\mathrm{Sdet}[M]} \begin{bmatrix} |d| & |b| \\ |c| & |a| \end{bmatrix} \circ \begin{bmatrix} \frac{\tilde{a}}{|\tilde{a}|} & \frac{\tilde{b}}{|\tilde{b}|} \\ \frac{\tilde{c}}{|\tilde{c}|} & \frac{\tilde{d}}{|\tilde{d}|} \end{bmatrix}.$$

Another description of the inverse matrix is offered in Equation (37) of Chen [22].

9.3. Determinant. For $n = 2$, it is noteworthy that the following four quaternion expressions are equal:

$$|a||d - ca^{-1}b| = |b||c - db^{-1}a| = |c||b - ac^{-1}d| = |d||a - bd^{-1}c|.$$

From the Schur complements formula, Corollary 8.1, it follows that each of these expressions, properly extended in each case a, b, c, or d is zero, expresses the value of

$$\mathrm{Sdet} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Applying this formula on the two unitary matrices in (3), one obtains the expected result (these matrices are unitary, hence unimodular). For Hermitian quaternionic matrices, the real determinant is given by

$$\left|\begin{bmatrix} \alpha & q \\ \bar{q} & \delta \end{bmatrix}\right|_r = \lambda_1\lambda_2 = \alpha\delta - |q|^2, \quad \alpha, \delta \in \mathbb{R}, \quad q \in \mathbb{H}.$$

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