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LIMIT POINTS FOR NORMALIZED LAPLACIAN EIGENVALUES*

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Abstract. Limit points for the positive eigenvalues of the normalized Laplacian matrix of a graph are considered. Specifically, it is shown that the set of limit points for the j-th smallest such eigenvalues is equal to $[0, 1]$, while the set of limit points for the j-th largest such eigenvalues is equal to $[1, 2]$. Limit points for certain functions of the eigenvalues, motivated by considerations for random walks, distances between vertex sets, and isoperimetric numbers, are also considered.

Key words. Normalized Laplacian Matrix, Limit Point, Eigenvalue.

AMS subject classifications. 05C50, 15A18.

1. Introduction. Suppose that we have a connected graph G on n vertices. There are a number of matrices associated with G, including the adjacency matrix, the Laplacian matrix (sometimes known as the combinatorial Laplacian matrix, see [1]) and the normalized Laplacian matrix. Each of these matrices furnishes one or more eigenvalues of interest, and there is a great deal of work that investigates the interplay between the graph-theoretic properties of G and eigenvalues of the matrices associated with G; see, for example, [1], [2] and [7].

One line of investigation regarding the eigenvalues of matrices associated with graphs arises from the work of Hoffman. In [4], Hoffman considers the spectral radius of the adjacency matrix of a graph G, $\rho(G)$, say, and defines a real number x to be a *limit point for the spectral radius* if there is a sequence of graphs G_k such that $\rho(G_k) \neq \rho(G_j)$ whenever $k \neq j$ and $\rho(G_k) \to x$ as $k \to \infty$. Evidently this is equivalent to x being a point of accumulation of the set $\{\rho(G)|G \text{ is a graph}\}$. In a similar vein, it is shown in [5] that any nonnegative real number is a limit point for algebraic connectivity (that notion of limit point being defined analogously with the definition of Hoffman), while [3] investigates limit points for the Laplacian spectral radius.

In this paper, we again consider limit points for eigenvalues of matrices associated with graphs, focusing our attention on the normalized Laplacian matrix. For a graph G on vertices $1, \ldots, n$, let d_i denote the degree of vertex i, and let $D = diag(d_1, \ldots, d_n)$; the normalized Laplacian for G is given by $\mathcal{L} = I - D^{\frac{-1}{2}} A D^{\frac{-1}{2}}$, where A is the adjacency matrix of G. It is straightforward to see that all the eigenvalues of $\mathcal{L}$ lie in the interval $[0, 2]$, and that for any graph, 0 is an eigenvalue of the corresponding normalized Laplacian matrix. We note in passing that 0 is a simple eigenvalue of $\mathcal{L}$ if and only if G is connected. The eigenvalues of $\mathcal{L}$ have attracted some attention over the last decade or so, in part because of their connections with isoperimetric numbers and diameters for graphs, and with convergence rates for certain random walks. A comprehensive introduction to the normalized Laplacian matrix

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can be found in [1].

Let G be a connected graph on n vertices, with normalized Laplacian matrix $\mathcal{L}$. Order the eigenvalues of $\mathcal{L}$ as $0 = \lambda_0(G) < \lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_{n-1}(G) \leq 2$. Fix an index $j \in I\!N$. By analogy with the definition of Hoffman in [4], we say that a real number x is a *limit point for* λ_j if there is a sequence of connected graphs $G_k, k \in I\!N$ such that:
i) $\lambda_j(G_{k_1}) \neq \lambda_j(G_{k_2})$ whenever $k_1 \neq k_2$, and
ii) $\lim_{k\to\infty} \lambda_j(G_k) = x$.
We denote the set of all limit points for λ_j by Λ_j.

It will also be convenient to think of the normalized Laplacian eigenvalues in nonincreasing order. So for a connected graph G on n vertices with normalized Laplacian matrix $\mathcal{L}$, we denote the nonincreasingly ordered eigenvalues of $\mathcal{L}$ by $2 \geq \gamma_1(G) \geq \gamma_2(G) \geq \ldots \geq \gamma_{n-1}(G) > \gamma_n(G) = 0$; evidently $\gamma_j(G) = \lambda_{n-j}(G)$ for each $j = 1, \ldots, n$. Fix an index $j \in I\!N$. We say that a real number y is a *limit point for* γ_j if there is a sequence of connected graphs $G_k, k \in I\!N$ such that:
i) $\gamma_j(G_{k_1}) \neq \gamma_j(G_{k_2})$ whenever $k_1 \neq k_2$, and
ii) $\lim_{k\to\infty} \gamma_j(G_k) = y$.
We denote the set of all limit points for γ_j by Γ_j.

Evidently Λ_j is the set of accumulation points of the set

$$\{\lambda_j(G)|G \text{ is a connected graph}\},$$

while Γ_j is the set of accumulation points of the set

$$\{\gamma_j(G)|G \text{ is a connected graph}\}.$$

In this paper, we show that for each $j \in I\!N$, $\Lambda_j = [0, 1]$ and $\Gamma_j = [1, 2]$. Our technique is straightforward, relying on a few simple observations and some suitably chosen classes of examples.

We also consider the set of limit points for three functions of $\lambda_1(G)$ and $\gamma_1(G)$; one function arises from an upper bound on the distance between subsets of vertices of G, another function is associated with the rate of convergence of a certain random walk associated with G while the last function arises from a bound on the isoperimetric number for G.

2. Limit points for eigenvalues.

LEMMA 2.1. *For each* $j \in I\!N, \Lambda_j \subseteq [0, 1]$.

Proof. Suppose that G is a connected graph on n vertices with normalized Laplacian matrix $\mathcal{L}$. Since $trace(\mathcal{L}) = n = \sum_{i=1}^{n-1} \lambda_i(G)$, we have $n \geq (n-j)\lambda_j(G)$, so that $0 \leq \lambda_j(G) \leq 1 + \frac{j}{n-j}$. Suppose that $x \in \Lambda_j$, and let G_k be a sequence of connected graphs such that $\lambda_j(G_k) \to x$ as $k \to \infty$. Letting n_k denote the number of vertices in G_k, we find that necessarily $n_k \to \infty$ as $k \to \infty$. Since $x = \lim_{k\to\infty} \lambda_j(G_k) \leq \lim_{k\to\infty} 1 + \frac{j}{n_k - j} = 1$, the conclusion follows. □

The following class of examples will allow us to complete our characterization of Λ_j. We use the notation $G_1 \vee G_2$ to denote the join of the graphs G_1 and G_2.

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EXAMPLE 2.2. Suppose that we have indices $p, q, j \in \mathbb{N}$. Consider the graph $G(p, q, j)$ on $p + (j + 1)q$ vertices defined by $G(p, q, j) = O_p \vee (K_q \cup \ldots \cup K_q)$, where there are $j + 1$ copies of K_q in the union and where O_p denotes the empty graph on p vertices. Note that $G(p, q, j)$ has p vertices of degree $(j + 1)q$ and $(j + 1)q$ vertices of degree $p + q - 1$. Let J denote an all ones matrix (whose order is to be taken from context). The corresponding normalized Laplacian matrix for $G(p, q, j)$ is $\mathcal{L} =$

$$\left[\begin{array}{c|c|c|c|c}
I & \frac{-1}{\sqrt{(j+1)q(p+q-1)}}J & \frac{-1}{\sqrt{(j+1)q(p+q-1)}}J & \cdots & \frac{-1}{\sqrt{(j+1)q(p+q-1)}}J \\
\hline
\frac{-1}{\sqrt{(j+1)q(p+q-1)}}J & \frac{p+q}{p+q-1}I - \frac{1}{p+q-1}J & 0 & \cdots & 0 \\
\hline
\vdots & & \ddots & & \vdots \\
\hline
\frac{-1}{\sqrt{(j+1)q(p+q-1)}}J & 0 & \cdots & 0 & \frac{p+q}{p+q-1}I - \frac{1}{p+q-1}J
\end{array}\right].$$

Using the notation $a^{(b)}$ to denote the fact that the number a appears with multiplicity b, we find that the eigenvalues of $\mathcal{L}$ are $1^{(p-1)}$ and $\left(\frac{p+q}{p+q-1}\right)^{(j+1)(q-1)}$, along with the eigenvalues of the $(j + 2) \times (j + 2)$ matrix

$$\left[\begin{array}{c|c}
1 & -\sqrt{\frac{q}{(j+1)(p+q-1)}}\mathbf{1}^T \\
\hline
-\frac{p}{\sqrt{(j+1)(p+q-1)}}\mathbf{1} & \frac{p}{p+q-1}I
\end{array}\right],$$

where $\mathbf{1}$ denotes an all ones vector. These last eigenvalues are $0, \left(\frac{p}{p+q-1}\right)^{(j)}$, and $\frac{2p+q-1}{p+q-1}$. In particular, it follows that $\lambda_j(p, q, j) = \frac{p}{p+q-1}$.

THEOREM 2.3. *For each* $j \in \mathbb{N}, \Lambda_j = [0, 1]$.

Proof. Fix $x \in [0, 1]$, and let a_k, b_k be sequences of natural numbers such that the sequence of rationals $\frac{a_k}{b_k}$ converges (strictly) monotonically to x as $k \to \infty$. From Example 2.2, we find that $\lambda_j(G(a_k, b_k - a_k + 1)) = \frac{a_k}{b_k} \to x$ as $k \to \infty$, and that the convergence is strictly monotonic. It follows that $[0, 1] \subseteq \Lambda_j$, and that fact, together with Lemma 2.1, yields the conclusion. □

Next, we turn our attention to Γ_j.

LEMMA 2.4. *For each* $j \in \mathbb{N}, \Gamma_j \subseteq [1, 2]$.

Proof. Let G be a connected graph on n vertices with normalized Laplacian matrix $\mathcal{L}$. Fix an index $j \in \mathbb{N}$. We have $trace(\mathcal{L}) = n = \sum_{i=1}^{n-1} \gamma_i(G) \leq 2(j-1)+(n-j)\gamma_j(G)$. Hence we have $2 \geq \gamma_j(G) \geq \frac{n+2-2j}{n-j} = 1 - \frac{j-2}{n-j}$. Suppose that $y \in \Gamma_j$, and let G_k be a sequence of graphs such that $\gamma_j(G_k) \to y$ as $k \to \infty$. Denoting the number of vertices of G_k by n_k, we have $\gamma_j(G_k) \geq 1 - \frac{j-2}{n_k-j}$, from which it follows that $y \geq 1$. □

THEOREM 2.5. $\Gamma_1 = [1, 2]$.

Proof. Referring to Example 2.2 we see that for any $p, q \in \mathbb{N}, \gamma_1(G(p, q, 1)) = \frac{2p+q-1}{p+q-1} = 2 - \frac{q-1}{p+q-1}$. Suppose that $y \in [1, 2]$, and set $r = 2 - y$. Let a_k, b_k be sequences of natural numbers so that $\frac{a_k}{b_k}$ converges monotonically to r. Then $\gamma_1(G(b_k - a_k, a_k + 1, 1)) = 2 - \frac{a_k}{b_k}$, which converges monotonically to $2 - r = y$. The conclusion now follows from Lemma 2.4. □

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The following two classes of examples will enable us to complete our discussion of Γ_j when $j \geq 2$.

EXAMPLE 2.6. Suppose that we have $p, q \in I\!N$, and let $H(p,q) = O_p \vee K_q$. The normalized Laplacian for $H(p,q)$ is given by

$$\left[\begin{array}{c|c} I & \frac{-1}{\sqrt{q(p+q-1)}}J \\ \hline \frac{-1}{\sqrt{q(p+q-1)}}J & \frac{p+q}{p+q-1}I - \frac{1}{p+q-1}J \end{array}\right].$$

The eigenvalues are readily seen to be $0, 1^{(p-1)}, \left(\frac{p+q}{p+q-1}\right)^{(q-1)}$ and $1 + \frac{p}{p+q-1}$.

EXAMPLE 2.7. Fix $p, q \in I\!N$, and suppose that $j \in I\!N$ with $j \geq 2$. For each $i = 1, \ldots, j$, let H_i be a copy of $H(p,q)$, and distinguish a vertex u_i and a vertex v_i of H_i having degrees q and $p+q-1$, respectively. Now construct a new connected graph $M(p,q,j)$ (on $j(p+q)$ vertices) from $\cup_{i=1}^{j} H_i$ by adding an edge between v_i and u_{i+1} for each $i = 1, \ldots, j-1$. Observe that the normalized Laplacian matrix for $M(p,q,j)$, $\mathcal{L}_1$ say, differs from that of $\cup_{i=1}^{j} H_i$, $\mathcal{L}_2$ say, only in the rows and columns corresponding to v_1, u_j and $u_i, v_i, i = 2, \ldots, j-1$. Note also that the eigenvalues of $\mathcal{L}_2$ are given by $0^{(j)}, 1^{(j(p-1))}, \left(\frac{p+q}{p+q-1}\right)^{(j(q-1))}$, and $\left(1 + \frac{p}{p+q-1}\right)^{(j)}$.

Let $N = \mathcal{L}_1 - \mathcal{L}_2$. It is straightforward to determine that the maximum absolute row sum for N is given by
$\max\{\frac{p+q-1}{\sqrt{q}}\left(\frac{1}{\sqrt{p+q-1}} - \frac{1}{\sqrt{p+q}}\right) + \frac{1}{\sqrt{(q+1)(p+q)}}, \frac{q-1}{\sqrt{p+q-1}}\left(\frac{1}{\sqrt{q}} - \frac{1}{\sqrt{q+1}}\right) + \frac{1}{\sqrt{q(p+q-1)}}\}$.
A couple of routine computations show that

$$\frac{p+q-1}{\sqrt{q}}\left(\frac{1}{\sqrt{p+q-1}} - \frac{1}{\sqrt{p+q}}\right) + \frac{1}{\sqrt{(q+1)(p+q)}} \leq \frac{3}{2\sqrt{q}}, \text{ and that}$$

$$\frac{q-1}{\sqrt{p+q-1}}\left(\frac{1}{\sqrt{q}} - \frac{1}{\sqrt{q+1}}\right) + \frac{1}{\sqrt{q(p+q-1)}} \leq \frac{3}{2\sqrt{q}},$$

so that the spectral radius of N is bounded above by $\frac{3}{2\sqrt{q}}$. In particular, it follows that

$$1 + \frac{p}{p+q-1} - \frac{3}{2\sqrt{q}} = \gamma_j(\cup_{i=1}^{j} H_i) - \frac{3}{2\sqrt{q}} \leq \gamma_j(M(p,q,j)) \leq$$

$$\gamma_j(\cup_{i=1}^{j} H_i) + \frac{3}{2\sqrt{q}} = 1 + \frac{p}{p+q-1} + \frac{3}{2\sqrt{q}}.$$

THEOREM 2.8. *For each* $j \in I\!N, \Gamma_j = [1,2]$.

Proof. From Lemma 2.4, we see that for each $j \in I\!N, \Gamma_j \subseteq [1,2]$.

To see the converse inclusion, fix $y \in (1,2)$ and let $x = y - 1$. For each $k \in I\!N$, select sequence of natural numbers p_k and q_k such that
i) $q_k \to \infty$ as $k \to \infty$, and

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ii) $x - \frac{1}{k} \leq \frac{p_k}{p_k+q_k-1} - \frac{3}{2\sqrt{q}} < \frac{p_k}{p_k+q_k-1} + \frac{3}{2\sqrt{q}} < x$.

From Example 2.7, we find that for $k \in I\!N$,

$$1 + x - \frac{1}{k} \leq \gamma_j(M(p_k, q_k, j)) < 1 + x.$$

In particular, it follows that as $k \to \infty$, the sequence $\gamma_j(M(p_k, q_k, j))$ converges to $1+x$. As no term in that sequence is equal to $1+x$, it follows that $y = 1+x$ is a limit point for γ_j. We conclude that $(1, 2) \subseteq \Gamma_j$, and since Γ_j is closed, we have $\Gamma_j = [1, 2]$. □

3. Limit points for functions of eigenvalues. Let G be a connected graph on n vertices, and define $\phi(G)$ as $\phi(G) = \frac{\gamma_1+\lambda_1}{\gamma_1-\lambda_1}$. The function $\phi(G)$ is of interest in part because of the role that it plays in the following bound on the distance between subsets of the vertex set for G (see Chapter 3 of [1] for more details.) Here, for a subset of vertices X of a graph G, we denote its complement by $\overline{X}$. The *volume* of X, denoted $vol(X)$ is the sum of the degrees of the vertices in X. For vertices x and y of G, we let $d(x, y)$ denote the length of a shortest path from x to y. The following result, which is inspired by Theorem 3.1 of [1], appears in [6].

PROPOSITION 3.1. *Let G be a connected graph, and suppose that X and Y are nonempty subsets of its vertex set with $X \neq Y, \overline{Y}$. Then*

$$\min\{d(x, y)|x \in X, y \in Y\} \leq \max\{\left\lceil \frac{\log \sqrt{\frac{vol(\overline{X})vol(\overline{Y})}{vol(X)vol(Y)}}}{\log \frac{\gamma_1+\lambda_1}{\gamma_1-\lambda_1}} \right\rceil, 2\}.$$

We say that a number $x \in \mathbb{R}$ is a *limit point for* ϕ if there is a sequence of graphs G_k such that $\phi(G_k) \neq \phi(G_j)$ whenever $k \neq j$ and $\phi(G_k) \to x$ as $k \to \infty$. We denote the set of all limit points for ϕ by Φ.

THEOREM 3.2. *$\Phi = [1, \infty)$.*

Proof. Evidently $\phi(G) \geq 1$ for any graph G, so we need only show that each $x \geq 1$ is a limit point for ϕ. For any $p, q, j \in I\!N$, we have from Example 2.2 that $\lambda_1(G(p, q, j)) = \frac{p}{p+q-1}$, while $\gamma_1(G(p, q, j)) = \frac{2p+q-1}{p+q-1}$. Hence $\phi(G(p, q, j)) = \frac{3p+q-1}{p+q-1} = 1 + \frac{2p}{p+q-1}$. For each $y \in (0, 2]$, let p_k and q_k be sequences in $I\!N$ such that $\frac{q_k-1}{p_k}$ converges monotonically to $\frac{2-y}{y}$. We find that then $\phi(G(p_k, q_k, 1))$ converges monotonically to $1 + y$, from which we deduce that each $x \in [1, 3]$ is a limit point for ϕ.

Next, note that if $p, q \in I\!N$, we see from Example 2.6 that $\lambda_1(H(p, q+1)) = 1$ while $\gamma_1(H(p, q+1)) = 1 + \frac{p}{p+q}$. Hence, $\phi(H(p, q+1)) = 3 + \frac{2q}{p}$, and it now follows readily that each $x \geq 3$ is a limit point for ϕ. □

For a connected graph G on n vertices, let $\lambda'(G) = \min\{\lambda_1(G), 2 - \gamma_1(G)\}$. We note that the quantity $\lambda'(G)$ arises in a bound on the rate of convergence of a certain random walk associated with G; see Section 1.5 of [1].

We say that a number $x \in \mathbb{R}$ is a *limit point for* λ' if there is a sequence of graphs G_k such that $\lambda'(G_k) \neq \lambda'(G_j)$ whenever $k \neq j$ and $\lambda'(G_k) \to x$ as $k \to \infty$. Since $\lambda'(G) \leq \frac{\lambda_1(G)+2-\gamma_1(G)}{2} \leq 1$, we find that any limit point for λ' is an element of $[0, 1]$.

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The following class of examples will be useful in our discussion of limit points for λ'.

EXAMPLE 3.3. Suppose that $p, q \in I\!N$, with $p, q \geq 2$, and let $M(p,q) = K_p \vee (K_q \cup K_q)$. The corresponding normalized Laplacian matrix is given by

$$\mathcal{L} = \left[\begin{array}{c|c|c} \frac{p+2q}{p+2q-1}I - \frac{1}{p+2q-1}J & \frac{-1}{\sqrt{(p+q-1)(p+2q-1)}}J & \frac{-1}{\sqrt{(p+q-1)(p+2q-1)}}J \\ \hline \frac{-1}{\sqrt{(p+q-1)(p+2q-1)}}J & \frac{p+q}{p+q-1}I - \frac{1}{p+q-1}J & 0 \\ \hline \frac{-1}{\sqrt{(p+q-1)(p+2q-1)}}J & 0 & \frac{p+q}{p+q-1}I - \frac{1}{p+q-1}J \end{array}\right].$$

We find that the eigenvalues for $\mathcal{L}$ are $\left(\frac{p+2q}{p+2q-1}\right)^{(p-1)}$ and $\left(\frac{p+q}{p+q-1}\right)^{(2q-2)}$, along with the eigenvalues of the matrix

$$\left[\begin{array}{ccc} \frac{2q}{p+2q-1} & \frac{-q}{\sqrt{(p+q-1)(p+2q-1)}} & \frac{-q}{\sqrt{(p+q-1)(p+2q-1)}} \\ \frac{-p}{\sqrt{(p+q-1)(p+2q-1)}} & \frac{p}{p+q-1} & 0 \\ \frac{-p}{\sqrt{(p+q-1)(p+2q-1)}} & 0 & \frac{p}{p+q-1} \end{array}\right],$$

which are $0, \frac{p}{p+q-1}$, and $\frac{2q}{p+2q-1} + \frac{p}{p+q-1}$. In particular, we have $\lambda_1 = \frac{p}{p+q-1}$, while the largest eigenvalue is $\frac{2q}{p+2q-1} + \frac{p}{p+q-1}$. It now follows that if $p \leq (q-1)^2$, then $\lambda_1(M(p,q)) = \lambda'(M(p,q)) = \frac{p}{p+q-1}$.

THEOREM 3.4. *Suppose that $x \in [0,1]$. Then x is a limit point for λ'. In fact:*
a) there is a sequence of graphs G_k such that $\lambda'(G_k) = \lambda_1(G_k)$ and $\lambda'(G_k)$ converges monotonically to x; and
b) there is a sequence of graphs G_k such that $\lambda'(G_k) = 2 - \gamma_1(G_k)$ and $\lambda'(G_k)$ converges monotonically to x.

Proof. a) Fix $x \in [0,1]$. From Example 2.6, it follows that if $p, q \in I\!N$, then $\lambda'(H(p,q)) = \min\{1, \frac{q}{p+q}\} = 2 - \gamma_1(H(p,q))$. Selecting sequences $p_k, q_k \in I\!N$ such that $\frac{q_k}{p_k+q_k}$ converges monotonically to x, the conclusion follows.
b) Fix $x \in (0,1)$, and select sequences $p_k, q_k \in I\!N$, both diverging to ∞ such that the sequence $\frac{p_k}{q_k-1}$ converges monotonically to $\frac{x}{1-x}$. Observe that asymptotically, we have $p_k \approx \frac{x}{1-x}(q_k - 1) < (q_k-1)^2$. So for all sufficiently large k, we have $\lambda'(M(p_k, q_k)) = \lambda_1(M(p_k, q_k)) = \frac{p_k}{p_k+q_k-1}$, which converges monotonically to x. The conclusion now follows. $\square$

Suppose that we have a connected graph G; partition its vertex as $S \cup \overline{S}$, where neither S nor $\overline{S}$ is empty. Let $E(S, \overline{S})$ denote the number of edges in G having one end point in S and the other in $\overline{S}$. The *isoperimetric number* of G is given by

$$h(G) = \min\{\frac{E(S,\overline{S})}{\min\{vol(s), vol(\overline{S})\}} | S \cup \overline{S} \text{ is a partitioning of the vertex set of } G\}.$$

A standard inequality (see Lemma 2.1 in [1]) asserts that for any graph G,

$$2h(G) \geq \lambda_1(G). \tag{3.1}$$

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In particular, for any connected graph G, we have $\frac{h(G)}{\lambda_1(G)} \geq \frac{1}{2}$. In this last collection of results, we consider small limit points for the function $\frac{h(G)}{\lambda_1(G)}$, or equivalently, small points of accumulation for the set $\{\frac{h(G)}{\lambda_1(G)} | \text{G is a graph}\}$.

The following example discusses $h(H(p, q))$.

EXAMPLE 3.5. Suppose that we have $p, q \in I\!N$. In this example, we consider $h(H(p, q))$ in the case that p and q are both even. Let $S(p_1, q_1)$ denote the subset of vertices consisting of p_1 vertices of degree q and q_1 vertices of degree $p + q - 1$; here we take $1 \leq p_1 + q_1 \leq p + q - 1$. We have $vol(S(p_1, q_1)) = p_1 q + q_1(p + q - 1)$ and $vol(\overline{S(p_1, q_1)}) = (p - p_1)q + (q - q_1)(p + q - 1)$, while $E(S(p_1, q_1), \overline{S(p_1, q_1)}) = p_1(q-2q_1)+q_1(p+q-q_1)$. Without loss of generality, we assume that $vol(S(p_1, q_1)) \leq vol(\overline{S(p_1, q_1)})$, or equivalently, that $p_1 \leq \frac{p}{2} + \frac{q-2q_1}{2q}(p + q - 1)$.

From the above considerations, it follows that

$$h(H(p, q)) = \min f(p_1, q_1), \tag{3.2}$$

where

$$f(p_1, q_1) = \frac{p_1(q - 2q_1) + q_1(p + q - q_1)}{p_1 q + q_1(p + q - 1)}, \tag{3.3}$$

and where the minimum in (3.2) is taken over the set of integers p_1, q_1 such that $0 \leq q_1 \leq q, 0 \leq p_1 \leq \min\{p, \frac{p}{2} + \frac{q-2q_1}{2q}(p + q - 1)\}$, and $1 \leq p_1 + q_1 \leq p + q - 1$.

Considered as a function of p_1, it is straightforward to see that $f(p_1, q_1)$ is decreasing in p_1, so that for fixed q_1, the minimum for $f(p_1, q_1)$ is taken at $p_1 = \min\{p, \frac{p}{2} + \frac{q-2q_1}{2q}(p + q - 1)\}$. Observe that $p \leq \frac{p}{2} + \frac{q-2q_1}{2q}(p + q - 1)$ if and only if $q_1 \leq \frac{q(q-1)}{2(p+q-1)}$, and so we consider f for the cases $q_1 \leq \frac{q(q-1)}{2(p+q-1)}$ and $q_1 \geq \frac{q(q-1)}{2(p+q-1)}$ separately.

If $q_1 \leq \frac{q(q-1)}{2(p+q-1)}$, then $f(p_1, q_1)$ is minimized for $p_1 = p$. We have $f(p, q_1) = \frac{(p+q_1)(q-q_1)}{pq+q_1(p+q-1)}$; note that considered as a function of q_1, the derivative of $f(p, q_1)$ is negative for all admissible q_1. It follows that when $q_1 \in [0, \frac{q(q+1)}{2(p+q-1)}]$, the minimum value for $f(p, q_1)$ in this case is taken at $q_1 = \frac{q(q-1)}{2(p+q-1)}$ (observe that this may not be an integer value for q_1). We find readily that $f(p, \frac{q(q-1)}{2(p+q-1)}) = \frac{p}{p+q-1} + \frac{q(q-1)}{2(p+q-1)^2}$. Thus we conclude that if q_1 is an integer and $0 \leq q_1 \leq \frac{q(q-1)}{2(p+q-1)}$, then $f(p_1, q_1) \geq \frac{p}{p+q-1} + \frac{q(q-1)}{2(p+q-1)^2}$.

If $q \geq q_1 \geq \frac{q(q-1)}{2(p+q-1)}$, then the minimum for $f(p_1, q_1)$ in this case is taken at $p_1 = \frac{p}{2} + \frac{q-2q_1}{2q}(p + q - 1)$ (observe that this value may not be an integer). We find that

$$f(\frac{p}{2} + \frac{q - 2q_1}{2q}(p + q - 1), q_1) = \frac{p(q - 2q_1) + \frac{p+q-1}{q}(q - 2q_1)^2 + 2q_1(p + q - q_1)}{q(2p + q - 1)},$$

which is a quadratic in q_1 that is uniquely minimized when $q_1 = \frac{q}{2}$. It now follows that if q_1 is an integer and $q \geq q_1 \geq \frac{q(q-1)}{2(p+q-1)}$, then $f(p_1, q_1) \geq f(\frac{p}{2}, \frac{q}{2}) = \frac{2p+q}{2(2p+q-1)}$.

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Observe that since p and q are even, $f(p_1, q_1)$ attains the value $\frac{2p+q}{2(2p+q-1)}$ at the integers $p_1 = \frac{p}{2}, q_1 = \frac{q}{2}$.

In particular, note that if $\frac{q}{p} \leq \frac{2p}{p+q-1}$, then $\frac{2p+q}{2(2p+q-1)} \leq \frac{p}{p+q-1} \leq \frac{p}{p+q-1} + \frac{q(q-1)}{2(p+q-1)^2}$. Thus we see that if $\frac{q}{p} \leq \frac{2p}{p+q-1}$, $h(H(p,q)) = f(\frac{p}{2}, \frac{q}{2}) = \frac{2p+q}{2(2p+q-1)}$.

THEOREM 3.6. *If $x \in [\frac{1}{2}, 1]$ then x is a limit point for $\frac{h}{\lambda_1}$.*

Proof. Suppose that we have $p, q \in I\!N$ with both p and q even, and such that $\frac{q}{p} \leq \frac{2p}{p+q-1}$. From Example 3.5, we see that $h(G(p,q)) = \frac{2p+q}{2(2p+q-1)}$, while from Example 2.6, we have $\lambda_1(G(p,q)) = \frac{p}{p+q-1}$. Hence $\frac{h(G(p,q))}{\lambda_1(G(p,q))} = \frac{(2p+q)(p+q-1)}{2p(2p+q-1)} = \frac{1}{2}(1 + \frac{1}{2p+q-1})(1 + \frac{q-1}{p})$.

Suppose now that $x \in (\frac{1}{2}, 1)$, and let $z = 2x - 1$, so that $0 < z < 1$. Select sequences of even natural numbers p_k, q_k such that $\frac{q_k-1}{p_k}$ decreases monotonically to z, and such that $2p_k + q_k - 1$ is an increasing sequence. Observe that since $0 < z < 1$, we have $z < \frac{2}{1+z}$; it now follows that for all sufficiently large k, $\frac{q_k}{p_k} \leq \frac{2p_k}{p_k+q_k-1}$.

Hence we see that for all sufficiently large k, we have $\frac{h(G(p_k,q_k))}{\lambda_1(G(p_k,q_k))} = \frac{1}{2}(1 + \frac{1}{2p_k+q_k-1})(1 + \frac{q_k-1}{p_k})$, which decreases to its limit of $\frac{1+z}{2} = x$ as $k \to \infty$. Thus each element of $(\frac{1}{2}, 1)$ is a limit point for $\frac{h}{\lambda_1}$, and the conclusion follows. $\square$

REMARK 3.7. In Example 2.6 of [1], it is observed that for the $n-$cube Q_n, $h(Q_n) = \frac{2}{n} = \lambda_1(Q_n)$, so that the inequality (3.1) is sharp to within a constant factor. Theorem 3.6 provides further insight into the sharpness of (3.1) by showing that in fact the function $\frac{h(G)}{\lambda_1(G)}$ is dense in the interval $[\frac{1}{2}, 1]$.

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