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STRUCTURED CONDITION NUMBERS AND BACKWARD ERRORS IN SCALAR PRODUCT SPACES*

FRANÇOISE TISSEUR† AND STEF GRAILLAT‡

Abstract. The effect of structure-preserving perturbations on the solution to a linear system, matrix inversion, and distance to singularity is investigated. Particular attention is paid to linear and nonlinear structures that form Lie algebras, Jordan algebras and automorphism groups of a scalar product. These include complex symmetric, pseudo-symmetric, persymmetric, skew-symmetric, Hamiltonian, unitary, complex orthogonal and symplectic matrices. Under reasonable assumptions on the scalar product, it is shown that there is little or no difference between structured and unstructured condition numbers and distance to singularity for matrices in Lie and Jordan algebras. Hence, for these classes of matrices, the usual unstructured perturbation analysis is sufficient. It is shown that this is not true in general for structures in automorphism groups. Bounds and computable expressions for the structured condition numbers for a linear system and matrix inversion are derived for these nonlinear structures.

Structured backward errors for the approximate solution of linear systems are also considered. Conditions are given for the structured backward error to be finite. For Lie and Jordan algebras it is proved that, whenever the structured backward error is finite, it is within a small factor of or equal to the unstructured one. The same conclusion holds for orthogonal and unitary structures but cannot easily be extended to other matrix groups.

This work extends and unifies earlier analyses.

Key words. Structured matrices, Normwise structured perturbations, Structured linear systems, Condition number, Backward error, Distance to singularity, Lie algebra, Jordan algebra, Automorphism group, Scalar product, Bilinear form, Sesquilinear form, Orthosymmetric.

AMS subject classifications. 15A12, 65F35, 65F15.

1. Motivations.

Condition numbers and backward errors play an important role in numerical linear algebra. Condition numbers measure the sensitivity of the solution of a problem to perturbation in the data whereas backward errors reveal the stability of a numerical method. Also, when combined with a backward error estimate, condition numbers provide an approximate upper bound on the error in a computed solution.

There is a growing interest in structured perturbation analysis (see for example [1], [6], [15] and the literature cited therein) due to the substantial development of algorithms for structured problems. For these problems it is often more appropriate to define condition numbers that measure the sensitivity to structured perturbations.

Structured perturbations are not always as easy to handle as unstructured ones and the backward error analysis of an algorithm is generally more difficult when structured perturbations are concerned. Our aim in this paper is to identify classes of

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†School of Mathematics, The University of Manchester, Sackville Street, Manchester, M60 1QD, UK (ftisseur@ma.man.ac.uk, http://www.ma.man.ac.uk/~ftisseur/). This work was supported by Engineering and Physical Sciences Research Council grant GR/S31693.

‡Laboratoire LP2A, Université de Perpignan Via Domitia, 52, avenue Paul Alduy, F-66860 Perpignan Cedex, France (graillat@univ-perp.fr, http://gala.univ-perp.fr/~graillat/).

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matrices for which the ratio between the structured and the unstructured condition number of a problem and the ratio between the structured and unstructured backward error are close to one. For example, it is well-known that for symmetric linear systems and for normwise distances it makes no difference to the backward error or condition number whether matrix perturbations are restricted to be symmetric or not [1], [5], [16]. Rump [15] recently showed that the same holds for skew-symmetric and persymmetric structures. We unify and extend these results to a large class of structured matrices.

The structured matrices we consider belong to Lie and Jordan algebras or group automorphisms associated with unitary and orthosymmetric scalar products. These include for example symmetric, complex symmetric and complex skew-symmetric matrices, pseudo-symmetric, persymmetric and perskew-symmetric matrices, Hamiltonian and skew-Hamiltonian matrices, Hermitian, skew-Hermitian and J-Hermitian matrices, but also nonlinear structures such as orthogonal, complex orthogonal, symplectic, unitary and conjugate symplectic structures. Structures not in the scope of our work include Toeplitz, Hankel and circulant structures, which are treated by Rump [15].

The paper is organized as follows. In Section 2, we set up the notation and recall some necessary background material. In Section 3, we consider the condition number for the matrix inverse and the condition number for linear systems as well as the distance to singularity. We show that for matrices belonging to Lie or Jordan algebras associated with a unitary and orthosymmetric scalar product and for perturbations measured in the Frobenius norm or 2-norm, the structured and unstructured condition numbers are equal or within a small factor of each other and that the distance to singularity is unchanged when the perturbations are forced to be structured. Hence, for these classes of matrices the usual unstructured perturbation analysis is sufficient. We show that this is false in general for nonlinear structures belonging to automorphism groups. We give bounds and derive computable expressions for the matrix inversion and linear system structured condition numbers. Structured backward errors for linear systems are considered in Section 4. Unlike in the unstructured case, the structured backward error may not exist and we say in this case that it is infinite. We give conditions on b and the approximate solution $\widehat{x}$ to $Ax = b$ for the structured backward error to be finite. We show that for Lie and Jordan algebras of orthosymmetric and unitary scalar products when the structured backward error is finite it is within a small factor of the unstructured one. Unfortunately, there is no general technique to derive explicit expressions for the structured backward error for structures in automorphism groups. For orthogonal and unitary matrices, we show that the Frobenius norm structured backward error when it exists differs from the unstructured one by at most a factor $\sqrt{2}$.

2. Preliminaries.

We begin with the basic definitions and properties of scalar products and the structured classes of matrices associated with them. Then we give auxiliary results for these structured matrices.

2.1. Structured matrices in scalar product spaces.

The term *scalar product* will be used to refer to any nondegenerate bilinear or sesquilinear form $\langle \cdot, \cdot \rangle$ on

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TABLE 2.1
A sampling of structured matrices associated with scalar products $\langle\cdot,\cdot\rangle_M$, where M is the matrix defining the scalar product.

Space	M	**Automorphism Group** $\mathbb{G} = \{G : G^\star = G^{-1}\}$	**Jordan Algebra** $\mathbb{J} = \{S : S^\star = S\}$	**Lie Algebra** $\mathbb{L} = \{K : K^\star = -K\}$
Symmetric bilinear forms				
$\mathbb{R}^n$	I	Real orthogonals	Symmetrics	Skew-symmetrics
$\mathbb{C}^n$	I	Complex orthogonals	Complex symmetrics	Cplx skew-symmetrics
$\mathbb{R}^n$	$\Sigma_{p,q}$	Pseudo-orthogonals	Pseudo symmetrics	Pseudo skew-symmetrics
$\mathbb{C}^n$	$\Sigma_{p,q}$	Cplx pseudo-orthogonals	Cplx pseudo-symm.	Cplx pseudo-skew-symm.
$\mathbb{R}^n$	R	Real perplectics	Persymmetrics	Perskew-symmetrics
Skew-symmetric bilinear forms				
$\mathbb{R}^{2n}$	J	Real symplectics	Skew-Hamiltonians	Hamiltonians
$\mathbb{C}^{2n}$	J	Complex symplectics	Cplx J-skew-symm.	Complex J-symmetrics
Hermitian sesquilinear forms				
$\mathbb{C}^n$	I	Unitaries	Hermitian	Skew-Hermitian
$\mathbb{C}^n$	$\Sigma_{p,q}$	Pseudo-unitaries	Pseudo Hermitian	Pseudo skew-Hermitian
Skew-Hermitian sesquilinear forms				
$\mathbb{C}^{2n}$	J	Conjugate symplectics	J-skew-Hermitian	J-Hermitian

Here, $R = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$ and $\Sigma_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \in \mathbb{R}^{n\times n}$ are symmetric and $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is skew-symmetric.

$\mathbb{K}^n$. Here $\mathbb{K}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$. It is well known that any real or complex bilinear form $\langle\cdot,\cdot\rangle$ has a unique matrix representation given by $\langle\cdot,\cdot\rangle = x^TMy$, while a sesquilinear form can be represented by $\langle\cdot,\cdot\rangle = x^*My$, where the matrix M is nonsingular. We will denote $\langle\cdot,\cdot\rangle$ by $\langle\cdot,\cdot\rangle_M$ as needed.

A bilinear form is symmetric if $\langle x, y\rangle = \langle y, x\rangle$, and skew-symmetric if $\langle x, y\rangle = -\langle y, x\rangle$. Hence for a symmetric form $M = M^T$ and for a skew-symmetric form $M = -M^T$. A sesquilinear form is Hermitian if $\langle x, y\rangle = \overline{\langle y, x\rangle}$ and skew-Hermitian if $\langle x, y\rangle = -\overline{\langle y, x\rangle}$. The matrices associated with such forms are Hermitian and skew-Hermitian, respectively.

To each scalar product there corresponds a notion of *adjoint*, generalizing the idea of transpose ^{T} and conjugate transpose *, that is, for any matrix $A \in \mathbb{K}^{n\times n}$ there is a unique adjoint $A^\star$ with respect to the form defined by $\langle Ax, y\rangle_M = \langle x, A^\star y\rangle_M$ for all x and y in $\mathbb{K}^n$. In matrix terms the adjoint is given by

$$A^\star = \begin{cases} M^{-1}A^TM & \text{for bilinear forms,} \\ M^{-1}A^*M & \text{for sesquilinear forms.} \end{cases} \tag{2.1}$$

Associated with the scalar product $\langle\cdot,\cdot\rangle_M$ are the Lie algebra $\mathbb{L}$, the Jordan algebra

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$\mathbb{J}$ and the automorphism group $\mathbb{G}$ defined by

$$\begin{aligned}
\mathbb{L} &= \left\{L \in \mathbb{K}^{n\times n} : \langle Lx, y\rangle_{\mathrm{M}} = -\langle x, Ly\rangle_{\mathrm{M}}\right\} = \left\{L \in \mathbb{K}^{n\times n} : L^{\star} = -L\right\},\\
\mathbb{J} &= \left\{S \in \mathbb{K}^{n\times n} : \langle Sx, y\rangle_{\mathrm{M}} = \langle x, Sy\rangle_{\mathrm{M}}\right\} = \left\{S \in \mathbb{K}^{n\times n} : S^{\star} = S\right\},\\
\mathbb{G} &= \left\{G \in \mathbb{K}^{n\times n} : \langle Gx, Gy\rangle_{\mathrm{M}} = \langle x, y\rangle_{\mathrm{M}}\right\} = \{G \in \mathbb{K}^{n\times n} : \ G^{\star} = G^{-1}\}.
\end{aligned}$$

The sets $\mathbb{L}$ and $\mathbb{J}$ are linear subspaces. They are not closed under multiplication but are closed under inversion: if $A \in \mathbb{S}$ is nonsingular, where $\mathbb{S} = \mathbb{L}$ or $\mathbb{S} = \mathbb{J}$ then $A^{-1} \in \mathbb{S}$. Matrices in $\mathbb{G}$ form a Lie group under multiplication.

There are two important classes of scalar products termed unitary and orthosymmetric [13]. The scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ is *unitary* if αM is unitary for some $\alpha > 0$. A scalar product is *orthosymmetric* if

$$M = \begin{cases} \beta M^T, \ \beta = \pm 1, \ \text{(bilinear forms)},\\ \beta M^*, \ |\beta| = 1, \ \text{(sesquilinear forms)}.\end{cases}$$

(See [13, Definitions A.4 and A.6] for a list of equivalent properties.) One can show that up to a scalar multiple there are only three distinct types of orthosymmetric scalar products: symmetric and skew-symmetric bilinear, and Hermitian sesquilinear [14]. We will, however, continue to include separately stated results (without separate proofs) for skew-Hermitian forms for convenience, as this is a commonly occurring special case.

In the rest of this paper we concentrate on structured matrices in the Lie algebra, the Jordan algebra or the automorphism group of a scalar product which is both unitary and orthosymmetric. These include, but are not restricted to, the structured matrices listed in Table 2.1, all of which correspond to a unitary and orthosymmetric scalar product with $\alpha = 1$ and $\beta = \pm 1$.

2.2. Auxiliary results. The characterization of various structured condition numbers and backward errors for linear systems relies on the solution to the following problem: *Given a class of structured matrices $\mathbb{S}$, for which vectors x, b does there exist some $A \in \mathbb{S}$ such that $Ax = b$?* Mackey, Mackey and Tisseur [14], [12] give a solution for this problem when $\mathbb{S}$ is the Lie algebra, Jordan algebra or automorphism group of an orthosymmetric scalar product. Here and below we write $\imath = \sqrt{-1}$.

Theorem 2.1 ([14, Thm. 3.2] and [12]). *Let $\mathbb{S}$ be the Lie algebra $\mathbb{L}$, Jordan algebra $\mathbb{J}$ or automorphism group $\mathbb{G}$ of an orthosymmetric scalar product $\langle\cdot,\cdot\rangle_M$ on $\mathbb{K}^n$. Then for any given pair of vectors x, $b \in \mathbb{K}^n$ with $x \neq 0$, there exists $A \in \mathbb{S}$ such that $Ax = b$ if and only if the conditions given in the following table hold:*

$\mathbb{S}$	**Bilinear forms**		**Sesquilinear forms**	
	Symmetric	*Skew-symmetric*	*Hermitian*	*Skew-Hermitian*
$\mathbb{J}$	*always*	$b^TMx = 0$	$b^*Mx \in \mathbb{R}$	$b^*Mx \in \imath\mathbb{R}$
$\mathbb{L}$	$b^TMx = 0$	*always*	$b^*Mx \in \imath\mathbb{R}$	$b^*Mx \in \mathbb{R}$
$\mathbb{G}$	$x^TMx = b^TMb$		$x^*Mx = b^*Mb$	

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In what follows $\|\cdot\|_\nu$ denotes as the 2-norm $\|\cdot\|_2$ or the Frobenius norm $\|\cdot\|_F$. The next result will be useful.

THEOREM 2.2. *Let $\mathbb{S}$ be the Lie or Jordan algebra of a scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ which is both orthosymmetric and unitary and let $x, b \in \mathbb{K}^n$ of unit 2-norm be such that the relevant condition in Theorem 2.1 is satisfied. Then,*

$$\min\{\|A\|_\nu : A \in \mathbb{S},\ Ax = b\} = \begin{cases} 1 & \text{if } \nu = 2, \\ \sqrt{2 - \alpha^2 \langle b, x\rangle_{\mathrm{M}}^2} & \text{if } \nu = F, \end{cases}$$

where α is such that αM is unitary.

Proof. See [14, Thms 5.6 and 5.10]. ◻

Note that when it exists, the minimal Frobenius norm solution A_{opt} to $Ax = b$ with $A \in \mathbb{S}$ is unique and is given by

$$A_{\mathrm{opt}} = \frac{bx^*}{x^*x} + \epsilon \left(\frac{bx^*}{x^*x}\right)^\star \left(I - \frac{xx^*}{x^*x}\right), \quad \epsilon = \begin{cases} 1 & \text{if } \mathbb{S} = \mathbb{J}, \\ -1 & \text{if } \mathbb{S} = \mathbb{L}, \end{cases}$$

where $\star$ is the adjoint with respect to the scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ associated with $\mathbb{S}$.

The next lemma shows that when $\mathbb{S}$ is the Lie or Jordan algebra of an orthosymmetric scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ on $\mathbb{K}^n$, left multiplication by M maps $\mathbb{S}$ to the sets $\mathrm{Skew}(\mathbb{K})$ and $\mathrm{Sym}(\mathbb{K})$ for bilinear forms and, to a scalar multiple of $\mathrm{Herm}(\mathbb{C})$ for sesquilinear forms, where

$$\begin{aligned} \mathrm{Sym}(\mathbb{K}) &= \{A \in \mathbb{K}^{n\times n} : A^T = A\}, \\ \mathrm{Skew}(\mathbb{K}) &= \{A \in \mathbb{K}^{n\times n} : A^T = -A\}, \\ \mathrm{Herm}(\mathbb{C}) &= \{A \in \mathbb{C}^{n\times n} : A^* = A\} \end{aligned} \tag{2.2}$$

are the sets of symmetric and skew-symmetric matrices on $\mathbb{K}^{n\times n}$ and Hermitian matrices, respectively. This is a key result for our unified treatment of structured condition numbers and backward errors.

LEMMA 2.3 ([14, Lem. 5.9]). *Let $\mathbb{S}$ be the Lie or Jordan algebra of an orthosymmetric scalar product $\langle\cdot,\cdot\rangle_M$. Suppose $A \in \mathbb{S}$, so that $A^\star = \delta A$ where $\delta = \pm 1$.*

- *For bilinear forms on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) write, by orthosymmetry, $M = \beta M^T$ with $\beta = \pm 1$. Then*

$$M \cdot \mathbb{S} = \begin{cases} \mathrm{Sym}(\mathbb{K}) & \text{if } \delta = \beta, \\ \mathrm{Skew}(\mathbb{K}) & \text{if } \delta \neq \beta. \end{cases} \tag{2.3}$$

- *For sesquilinear forms on $\mathbb{C}^n$ write, by orthosymmetry, $M = \beta M^*$ with $|\beta| = 1$. Then*

$$M \cdot \mathbb{S} = \begin{cases} \beta^{1/2}\mathrm{Herm}(\mathbb{C}) & \text{if } \delta = +1, \\ \beta^{1/2}\,\imath\,\mathrm{Herm}(\mathbb{C}) & \text{if } \delta = -1. \end{cases} \tag{2.4}$$

In practice, when $A \in \mathbb{J}$ or $\mathbb{L}$, Lemma 2.3 suggests to rewrite $Ax = b$ as $\widetilde{A}x = Mb$, where $\widetilde{A} := MA$ belongs to one of the sets defined in (2.2). Then we can use a well understood structure preserving algorithm to numerically solve the modified system

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with $\widetilde{A}$. For instance, if A is pseudosymmetric then $M = \Sigma_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$ and $\widetilde{A} = \Sigma_{p,q}A$ is symmetric. Then we can use a block LDL^T factorization [8, Ch.11] to solve the modified linear system $\widetilde{A}x = \Sigma_{p,q}b$. However, since left multiplication of A by M changes its spectrum, mapping A to $\widetilde{A}$ may not be appropriate when solving large sparse systems of equations with an iterative method.

2.3. Nearest structured matrix. Suppose A has lost its structure, because of errors in its construction for example. An interesting optimization problem is to find the nearest structured matrix to A,

$$d_{\mathbb{S}}(A) := \min\{\|A - S\| : \ S \in \mathbb{S}\}. \tag{2.5}$$

This problem is easy to solve when $\mathbb{S}$ is the Jordan algebra $\mathbb{J}$ or Lie algebra $\mathbb{L}$ of an orthosymmetric and unitary scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ and $\|\cdot\|$ is any unitarily invariant norm. Indeed, if $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ is orthosymmetric then any $A \in \mathbb{K}^{n\times n}$ can be expressed in the form

$$A = \frac{A + A^\star}{2} + \frac{A - A^\star}{2} =: A_{\mathbb{J}} + A_{\mathbb{L}},$$

where $A_{\mathbb{J}} \in \mathbb{J}$ and $A_{\mathbb{L}} \in \mathbb{L}$. For any $S \in \mathbb{J}$, so that, $S = S^\star$ we have

$$\|A - A_{\mathbb{J}}\| = \|A_{\mathbb{L}}\| = \frac{1}{2}\|A - A^\star\| = \frac{1}{2}\|(A - S) + (S^\star - A^\star)\|.$$

Since $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ is unitary and the norm is unitarily invariant,

$$\|A - A_{\mathbb{J}}\| \le \frac{1}{2}\|A - S\| + \frac{1}{2}\|(S - A)^\star\| = \|A - S\| \quad \forall\, S \in \mathbb{J}.$$

We find similarly that for Lie algebras $\mathbb{L}$,

$$\|A - A_{\mathbb{L}}\| \le \|A - S\| \quad \forall\, S \in \mathbb{L}.$$

Hence, $S = A_{\mathbb{S}}$ with $\mathbb{S} = \mathbb{J}$ or $\mathbb{L}$ is the nearest structured matrix in $\mathbb{S}$ to A and

$$d_{\mathbb{J}}(A) = \frac{1}{2}\|A - A^\star\|, \qquad d_{\mathbb{L}}(A) = \frac{1}{2}\|A + A^\star\|.$$

Fan and Hoffman solved this problem for the class of symmetric matrices [3]. More recently it was solved by Cardoso, Kenney and Leite [2, Thm. 5.3] for Lie and Jordan algebras of bilinear scalar products $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ for which $MM^T = I$ and $M^2 = \pm I$; these scalar products are a subset of the unitary and orthosymmetric scalar products.

When $\mathbb{S}$ is the set of real orthogonal matrices or unitary matrices, the solution to the nearness problem (2.5) is well-known for the Frobenius norm and is given by

$$d_{\mathbb{S}}(A) = \|A - U\|_F,$$

where U is the orthogonal or unitary factor of the polar decomposition of A [4, p. 149], [7]. To our knowledge, (2.5) is an open problem for matrix groups other than the orthogonal and unitary groups.

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3. Structured condition numbers. We consider normwise relative structured condition numbers that measure the sensitivity of linear systems and of matrix inversion, and investigate the structured distance to singularity.

3.1. Linear systems. Let $\mathbb{S}$ be a class of structured matrices, not necessarily $\mathbb{J}$, $\mathbb{L}$ or $\mathbb{G}$. We define the structured normwise condition number for the linear system $Ax = b$ with $x \neq 0$ by

$$\operatorname{cond}(A, x; \mathbb{S}) = \lim_{\epsilon\to 0} \sup\Big\{ \frac{\|\varDelta x\|}{\epsilon\|x\|} : (A + \varDelta A)(x + \varDelta x) = b + \varDelta b, \quad A + \varDelta A \in \mathbb{S},\ \|\varDelta A\| \le \epsilon\|A\|,\ \|\varDelta b\| \le \epsilon\|b\| \Big\}, \tag{3.1}$$

where $\|\cdot\|$ is an arbitrary matrix norm. Let $\operatorname{cond}(A, x) \equiv \operatorname{cond}(A, x; \mathbb{K}^{n\times n})$ denote the unstructured condition number, where n is the dimension of A. Clearly,

$$\operatorname{cond}(A, x; \mathbb{S}) \le \operatorname{cond}(A, x).$$

If this inequality is not always close to being attained then $\operatorname{cond}(A, x)$ may severely overestimate the worst case effect of structured perturbations.

Let us define for nonsingular $A \in \mathbb{S}$ and nonzero $x \in \mathbb{K}^n$,

$$\phi(A, x; \mathbb{S}) = \lim_{\epsilon\to 0} \sup\left\{ \frac{\|A^{-1}\varDelta A x\|}{\epsilon\|A\|} :\ \|\varDelta A\| \le \epsilon\|A\|,\ A + \varDelta A \in \mathbb{S} \right\}. \tag{3.2}$$

We write $\phi(A, x) \equiv \phi(A, x; \mathbb{K}^{n\times n})$ when $A + \varDelta A$ is unstructured and $\phi_2(A, x)$ or $\phi_F(A, x)$ to specify that the 2-norm or Frobenius norm is used in (3.2). Similarly, cond_2 and cond_F mean that $\|\cdot\| = \|\cdot\|_2$ or $\|\cdot\| = \|\cdot\|_F$ in (3.1).

The next lemma is useful when comparing the structured and unstructured condition numbers for linear systems. The result is due to Rump [15] and holds for any class of structured matrices. The proof in [15, Thm. 3.2 and Thm. 33] for $\mathbb{K} = \mathbb{R}$ and the 2-norm extend trivially to $\mathbb{K} = \mathbb{C}$ and $\|\cdot\| = \|\cdot\|_F$.

LEMMA 3.1. *Let $A \in \mathbb{K}^{n\times n}$ be nonsingular and $x \in \mathbb{K}^n$ be nonzero. Then for $\nu = 2, F$,*

$$\frac{\phi_\nu(A, x; \mathbb{S})}{\|x\|_2}\Big(\|A\|_\nu + \frac{\|b\|_2}{\|x\|_2}\Big) \le \operatorname{cond}_\nu(A, x; \mathbb{S}) \le \|A^{-1}\|_2\Big(\|A\|_\nu + \frac{\|b\|_2}{\|x\|_2}\Big).$$

and

$$\operatorname{cond}_\nu(A, x, \mathbb{S}) = c\left(\phi_\nu(A, x; \mathbb{S})\frac{\|A\|_\nu}{\|x\|_2} + \frac{\|A^{-1}\|_2\|b\|_2}{\|x\|_2}\right), \quad \frac{1}{\sqrt{2}} \le c \le 1.$$

The difficulty in obtaining an explicit expression for $\phi(A, x; \mathbb{S})$ in (3.2) depends on the nature of $\mathbb{S}$ and the matrix norm $\|\cdot\|$. For example, for the Frobenius norm or the 2-norm and for unstructured perturbations (i.e., $\mathbb{S} = \mathbb{K}^{n\times n}$), the supremum

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in (3.2) is attained by $\Delta A = \epsilon \|A\|_\nu y x^*/\|x\|_2$, where y is such that $\|y\|_2 = 1$ and $\|A^{-1}y\|_2 = \|A^{-1}\|_2$. This implies the well known formula

$$\phi_2(A,x) = \phi_F(A,x) = \|A^{-1}\|_2\|x\|_2, \tag{3.3}$$

so that from the first part of Lemma 3.1,

$$\mathrm{cond}_\nu(A,x) = \|A^{-1}\|_2\|A\|_\nu + \frac{\|A^{-1}\|_2\|b\|_2}{\|x\|_2}, \quad \nu = 2, F. \tag{3.4}$$

The rest of this section is devoted to the study of $\phi(A,x;\mathbb{S})$ for the structured matrices in $\mathbb{L}$, $\mathbb{J}$ or $\mathbb{G}$ as defined in Section 2.1.

3.1.1. Lie and Jordan algebras. D. J. Higham [5] proves that for real symmetric structures,

$$\mathrm{cond}_\nu(A,x;\mathrm{Sym}(\mathbb{R})) = \mathrm{cond}_\nu(A,x), \quad \nu = 2, F$$

and Rump [15] shows that equality also holds in the 2-norm for persymmetric and skew-symmetric structures. These are examples of Lie and Jordan algebras (see Table 2.1). We extend these results to all Lie and Jordan algebras of orthosymmetric and unitary scalar products. Unlike the proofs in [15], our unifying proof does not need to consider each algebra individually.

THEOREM 3.2. *Let $\mathbb{S}$ be the Lie algebra or Jordan algebra of a unitary and orthosymmetric scalar product $\langle\cdot,\cdot\rangle_M$ on $\mathbb{K}^n$. For nonsingular $A \in \mathbb{S}$ and nonzero $x \in \mathbb{K}^n$, we have*

$$\phi_2(A,x;\mathbb{S}) = \phi_2(A,x),$$
$$\frac{1}{\sqrt{2}}\phi_F(A,x) \le \phi_F(A,x;\mathbb{S}) \le \phi_F(A,x).$$

Proof. Clearly $\phi_\nu(A,x;\mathbb{S}) \le \phi_\nu(A,x)$. Since Lie and Jordan algebras are linear subspaces of $\mathbb{K}^{n\times n}$, $\phi_\nu(A,x,\mathbb{S})$ can be rewritten as

$$\phi_\nu(A,x;\mathbb{S}) = \sup\left\{\|A^{-1}\Delta A x\|_2 : \ \|\Delta A\|_\nu \le 1, \ \Delta A \in \mathbb{S}\right\}.$$

Suppose that there exists $u \in \mathbb{K}^n$ of unit 2-norm such that $\|A^{-1}u\|_2 = \|A^{-1}\|_2$ and such that the pair $(x/\|x\|_2, u)$ satisfies the relevant condition in Theorem 2.1. Then Theorem 2.2 tells us that there exist $E_2, E_F \in \mathbb{S}$ such that $E_\nu x = \|x\|_2 u$, $\nu = 2, F$ and $\|E_2\|_2 = 1$, $\|E_F\|_F \le \sqrt{2}$. Thus

$$\phi_\nu(A,x) = \|A^{-1}\|_2\|x\|_2 = \|A^{-1}u\|_2\|x\|_2 = \|A^{-1}E_\nu x\|_2 \le c_\nu \phi_\nu(A,x;\mathbb{S}),$$

with $c_2 = 1$ and $c_F = \sqrt{2}$ and the statement of the Theorem follows.

To complete the proof we need to show that there is always a vector u such that $\|A^{-1}u\|_2 = \|A^{-1}\|_2$ and such that $x/\|x\|_2, u$ satisfy the relevant condition in Theorem 2.1. First we note that $A \in \mathbb{S} \ \Rightarrow \ A^\star = \delta A$, $\delta = \pm 1$.

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(i) For bilinear forms, orthosymmetry of the scalar product implies that $M^T = \beta M$, $\beta = \pm 1$. If $\delta = \beta$, Theorem 2.1 says that there is no condition on x and u for the existence of a structured matrix mapping x to $\|x\|_2 u$, so u can be any singular vector of A^{-1} associated with the largest singular value. If $\delta \neq \beta$ then x and u must satisfy $\langle u, x\rangle_{\mathrm{M}} = 0$. But [13, Thm. 8.4] says that the singular values of A and therefore A^{-1} all have even multiplicity. Hence there exits orthogonal vectors u_1 and u_2 in $\mathbb{K}^n$ such that $\|u_1\|_2 = \|u_2\|_2 = 1$ and $\|A^{-1}u_1\|_2 = \|A^{-1}u_2\|_2 = \|A^{-1}\|_2$. Let $u \in \mathrm{span}\{u_1, u_2\}$ such that $\langle u, x\rangle_{\mathrm{M}} = u^T M x = 0$ and $\|u\|_2 = 1$. Then necessarily, $\|A^{-1}u\|_2 = \|A^{-1}\|_2$.

(ii) For sesquilinear forms, since singular vectors u are defined up to a nonzero scalar multiple, we choose u to be the singular vector associated with the largest singular value of A^{-1} and such that the condition in Theorem 2.1 is satisfied. ∎

Lemma 3.1 and Theorem 3.2 together yield the following result.

Theorem 3.3. *Let $\mathbb{S}$ be the Lie or Jordan algebra of a unitary and orthosymmetric scalar product $\langle\cdot,\cdot\rangle_M$ on $\mathbb{K}^n$. Let $A \in \mathbb{S}$ be nonsingular and $x \in \mathbb{K}^n$ with $x \neq 0$ be given. Then*

$$\mathrm{cond}_2(A, x; \mathbb{S}) = \mathrm{cond}_2(A, x),$$
$$\frac{1}{\sqrt{2}}\mathrm{cond}_F(A, x) \leq \mathrm{cond}_F(A, x; \mathbb{S}) \leq \mathrm{cond}_F(A, x).$$

We conclude from Theorem 3.3 that for many Lie and Jordan algebras, the constraint $A + \Delta A \in \mathbb{S}$ has little or no effect on the condition number. This is certainly true for all the examples in Table 2.1.

3.1.2. Automorphism groups. When $\mathbb{S}$ is a smooth manifold the task of computing the supremum (3.2) simplifies to a linearly constrained optimization problem. This was already observed by Karow, Kressner and Tisseur in [10] for a different supremum.

Lemma 3.4. *Let $\mathbb{S}$ be a smooth real or complex manifold. Then for $A \in \mathbb{S}$ nonsingular we have*

$$\phi(A, x; \mathbb{S}) = \sup\left\{\|A^{-1}Ex\| : \ \|E\| = 1, \ E \in T_A\mathbb{S}\right\}, \tag{3.5}$$

where $T_A\mathbb{S}$ is the tangent space at A.

Proof. Let $E \in T_A\mathbb{S}$ with $\|E\| = 1$. Then there is a smooth curve $g_E : (-\epsilon, \epsilon) \to \mathbb{K}^{n\times n}$ satisfying $g_E(0) = 0$, $g'_E(0) = E$ and $A + g_E(t) \in \mathbb{S}$ for all t. We have

$$\lim_{t\to 0} \frac{g_E(t)}{\|g_E(t)\|} = \lim_{t\to 0} \frac{Et + O(|t|^2)}{\|Et + O(|t|^2)\|} = E.$$

Hence

$$\lim_{t\to 0} \frac{\|A^{-1}g_E(t)x\|}{\|g_E(t)\|} = \|A^{-1}Ex\|.$$

This implies $\phi(A, x; \mathbb{S}) \geq \sup\left\{\|A^{-1}Ex\| : \ \|E\| = 1, \ E \in T_A\mathbb{S}\right\}$. Equality holds since the union of the curves $A + g_E$ contains an open neighborhood in $\mathbb{S}$ of A (see [10] for details). ∎

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An automorphism group $\mathbb{G}$ forms a smooth manifold. Define $F(A) = A^\star A - I$ so that $A \in \mathbb{G}$ is equivalent to $F(A) = 0$. The tangent space $T_A\mathbb{G}$ at $A \in \mathbb{G}$ is given by the kernel of the Fréchet derivative J_A of F at A:

$$
\begin{aligned}
T_A\mathbb{G} &= \{X \in \mathbb{K}^{n\times n} : J_A(X) = 0\} \\
&= \{X \in \mathbb{K}^{n\times n} : A^\star X + XA = 0\} \\
&= \{AH \in \mathbb{K}^{n\times n} : H^\star = -H\} \\
&= A\cdot \mathbb{L},
\end{aligned} \tag{3.6}
$$

where $\mathbb{L}$ is the Lie algebra of $\langle\cdot,\cdot\rangle_{\mathrm{M}}$.

For $\mathbb{S} = \mathbb{G}$, the supremum (3.5) can then be rewritten as

$$\phi(A, x; \mathbb{G}) = \sup\{\|Hx\| : \ \|AH\| = 1,\ H \in \mathbb{L}\}. \tag{3.7}$$

In a similar way to [6] and [10], an explicit expression for $\phi(A, x; \mathbb{G})$ can be obtained for the Frobenius norm. Let us rewrite

$$Hx = (x^T \otimes I_n)\mathrm{vec}(H),$$

where $\otimes$ denotes the Kronecker product and vec is the operator that stacks the columns of a matrix into one long vector. We refer to Lancaster and Tismenetsky [11, Chap. 12] for properties of the vec operator and the Kronecker product. Since $\mathbb{L}$ is a linear vector space of dimension $m \le n^2$, there is an $n^2 \times m$ matrix B such that for every $H \in \mathbb{L}$ there exists a uniquely defined parameter vector q with

$$\mathrm{vec}(H) = Bq. \tag{3.8}$$

Let $(I \otimes A)B = QR$ be a reduced QR factorization of $(I \otimes A)B$, i.e., $Q \in \mathbb{K}^{n^2\times m}$, $R \in \mathbb{K}^{m\times m}$, and let $p = Rq$. Then,

$$\mathrm{vec}(AH) = (I \otimes A)\mathrm{vec}(H) = (I \otimes A)Bq = Qp$$

so that $\|AH\|_F = \|\mathrm{vec}(AH)\|_2 = \|p\|_2$ and

$$Hx = (x^T \otimes I_n)\mathrm{vec}(H) = (x^T \otimes I_n)(I \otimes A)^{-1}\mathrm{vec}(AH) = (x^T \otimes A^{-1})Qp.$$

Hence (3.7) becomes

$$\phi_F(A, x; \mathbb{G}) = \sup\left\{\|(x^T \otimes A^{-1})Qp\|_2 : \ \|p\|_2 = 1,\ p \in \mathbb{K}^m\right\} = \|(x^T \otimes A^{-1})Q\|_2.$$

Using the last part of Lemma 3.1 we then have, up to a small scalar multiple, a directly computable expression for the structured condition number.

Theorem 3.5. *Let $\mathbb{G}$ be the automorphism group of any scalar product and let $\mathbb{L}$ be the associated Lie algebra. Then*

$$\mathrm{cond}_F(A, x, \mathbb{G}) = c\left(\|(x^T \otimes A^{-1})Q\|_2 \frac{\|A\|_F}{\|x\|_2} + \frac{\|A^{-1}\|_2\|b\|_2}{\|x\|_2}\right), \quad \frac{1}{\sqrt{2}} \le c \le 1,$$

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where $(I \otimes A)B = QR$ *is a reduced QR factorization of* $(I \otimes A)B$ *and* B *is a pattern matrix for* $\mathbb{L}$ *in the sense of* (3.8).

The expression for $\mathrm{cond}_F(A, x; \mathbb{G})$ in Theorem 3.5 has two disadvantages. First, it is expensive to compute, since it requires the QR factorization of an $n^2 \times m$ matrix. Note that since $A \in \mathbb{G}$, $A^{-1} = A^\star$ with $A^\star$ given in (2.1) so that we do not necessarily need to invert A. Second, it is difficult to compare with the unstructured condition number $\mathrm{cond}_F(A, x)$ in (3.4). However, when $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ is unitary and orthosymmetric, we can bound the ratio $\phi_\nu(A, x; \mathbb{G})/\phi_\nu(A, x)$ from below.

Lemma 3.6. *Let* $\mathbb{G}$ *be the automorphism group of a unitary and orthosymmetric scalar product on* $\mathbb{K}^n$. *Let* $A \in \mathbb{G}$ *and* $x \in \mathbb{K}^n$ *with* $x \neq 0$ *be given. Then*

$$c_\nu \frac{\phi_\nu(A, x)}{\|A\|_2 \|A^{-1}\|_2} \le \phi_\nu(A, x; \mathbb{G}) \le \phi_\nu(A, x), \qquad c_\nu = \begin{cases} 1 & \text{if } \nu = 2, \\ \dfrac{1}{\sqrt{2}} & \text{if } \nu = F. \end{cases}$$

Proof. We just need to prove the lower bound. Let $u \in \mathbb{K}^n$ be of unit 2-norm and such that x and u satisfy the relevant condition in Theorem 2.1. From Theorem 2.2, there exists $S_2 \in \mathbb{S}$ and $S_F \in \mathbb{S}$ such that $S_\nu(x/\|x\|_2) = u$, $\nu = 2, F$ and $\|S_2\|_2 = 1$, $\|S_F\|_F \le \sqrt{2}$. Let $H_\nu = \xi S_\nu \in \mathbb{S}$ with $\xi > 0$ such that $\|AH_\nu\|_\nu = 1$, $\nu = 2, F$. This implies $\xi \ge \frac{1}{\|A\|_2 \|S_\nu\|_\nu}$. Hence from (3.7), we have

$$\phi_\nu(A, x; \mathbb{G}) \ge \|H_\nu x\|_2 = \xi \|x\|_2 \ge \frac{\phi_\nu(A, x)}{\|A\|_2 \|A^{-1}\|_2 \|S_\nu\|_\nu}. \qquad \square$$

Lemma 3.6, together with Lemma 3.1 and the explicit expression for $\mathrm{cond}_\nu(A, x)$ in (3.4), yield bounds for $\mathrm{cond}(A, x; \mathbb{G})$.

Theorem 3.7. *Let* $\mathbb{G}$ *be the automorphism group of a unitary and orthosymmetric scalar product on* $\mathbb{K}^n$. *Let* $A \in \mathbb{G}$ *and* $x \in \mathbb{K}^n$ *with* $x \neq 0$ *be given. Then*

$$c_\nu \frac{\mathrm{cond}_\nu(A, x)}{\|A\|_2 \|A^{-1}\|_2} \le \mathrm{cond}_\nu(A, x; \mathbb{G}) \le \mathrm{cond}_\nu(A, x), \qquad c_\nu = \begin{cases} 1 & \text{if } \nu = 2, \\ \dfrac{1}{\sqrt{2}} & \text{if } \nu = F. \end{cases}$$

Theorem 3.7 shows that when A is well-conditioned (i.e., $\|A\|_2 \|A^{-1}\|_2 \approx 1$), the structured and unstructured condition numbers for $Ax = b$ are equal or nearly equal. This is the case for the real orthogonal group and the unitary group. For ill-conditioned A, the bounds may not be sharp, as illustrated by the following example. Suppose that $M = J$ and that $\langle\cdot,\cdot\rangle_J$ is a real bilinear form. Then $\mathbb{G}$ is the set of real symplectic matrices (see Table 2.1). Consider the symplectic matrix

$$A = \begin{bmatrix} D & D \\ 0 & D^{-1} \end{bmatrix}, \quad D = \mathrm{diag}(10^{-6}, 10^2, 2) \tag{3.9}$$

and define the ratio

$$\rho = \mathrm{cond}_F(A, x; \mathbb{G})/\mathrm{cond}_F(A, x)$$

between the structured and unstructured condition numbers. For this particular example, Theorem 3.7 provides the non-sharp bounds $7 \times 10^{-13} \le \rho \le 1$. We use the

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computable expression in Theorem 3.5 with $c_F = 1$ to approximate $\mathrm{cond}_F(A, x; \mathbb{G})$. For $x = [1, 0, 0.1, 0, 0.1, -1]^T$ we obtain $\rho \approx 8 \times 10^{-5}$ showing that $\mathrm{cond}_F(A, x; \mathbb{G}) \ll \mathrm{cond}_F(A, x)$ may happen.

3.2. Matrix inversion. Let $\mathbb{S}$ be a class of structured matrices. The structured condition number for the matrix inverse can be defined by

$$\kappa(A; \mathbb{S}) = \lim_{\epsilon \to 0} \sup \left\{ \frac{\|(A + \Delta A)^{-1} - A^{-1}\|}{\epsilon \|A^{-1}\|} : \ A + \Delta A \in \mathbb{S}, \ \|\Delta A\| \le \epsilon \|A\| \right\}. \tag{3.10}$$

When ΔA is unstructured we write $\kappa(A) \equiv \kappa(A; \mathbb{K}^{n \times n})$. It is well-known that for the 2- and Frobenius norms $\kappa(A)$ has the characterization

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2, \qquad \kappa_F(A) = \frac{\|A\|_F \|A^{-1}\|_2^2}{\|A^{-1}\|_F}. \tag{3.11}$$

See [8, Thm. 6.4] for κ_2 and [5] for κ_F. Again, when $\mathbb{S}$ is a smooth manifold (3.10) simplifies to a linearly constrained optimization problem.

LEMMA 3.8. *Let $\mathbb{S}$ be a smooth real or complex manifold. Then for $A \in \mathbb{S}$ nonsingular we have*

$$\kappa(A; \mathbb{S}) = \frac{\|A\|}{\|A^{-1}\|} \sup\{\|A^{-1} E A^{-1}\| : \ \|E\| = 1, \ E \in T_A\mathbb{S}\}, \tag{3.12}$$

where $T_A\mathbb{S}$ is the tangent space at A.

Proof. The proof is similar to that of Lemma 3.4 and makes use of the expansion

$$(A + \Delta A)^{-1} - A^{-1} = -A^{-1} \Delta A A^{-1} + O(\|\Delta A\|^2),$$

in the definition of $\kappa(A; \mathbb{S})$ in (3.10). □

When $\mathbb{S}$ is the Lie algebra or Jordan algebra of a scalar product, the tangent space at $\mathbb{S}$ is $\mathbb{S}$ itself: $T_A\mathbb{S} = \mathbb{S}$. If we restrict $\langle \cdot, \cdot \rangle_M$ to be orthosymmetric and unitary then there is equality between the structured and unstructured condition numbers for both the 2-norm and Frobenius norm.

THEOREM 3.9. *Let $\mathbb{S}$ be the Lie algebra or Jordan algebra of a unitary and orthosymmetric scalar product $\langle \cdot, \cdot \rangle_M$ on $\mathbb{K}^n$. For nonsingular $A \in \mathbb{S}$ we have*

$$\kappa_\nu(A; \mathbb{S}) = \kappa_\nu(A), \qquad \nu = 2, F.$$

Proof. Using the inequality $\|ABC\|_\nu \le \|A\|_2 \|B\|_\nu \|C\|_2$ in (3.12) gives

$$\kappa_\nu(A; \mathbb{S}) \le \|A\|_\nu \|A^{-1}\|_2^2 / \|A^{-1}\|_\nu = \kappa_\nu(A).$$

Since αM is unitary for some $\alpha > 0$, we have $\kappa_\nu(A; \mathbb{S}) = \kappa_\nu(\widetilde{A}; M \cdot \mathbb{S})$ with $\widetilde{A} = \alpha M A$. Now assume that there exists $E \in M \cdot \mathbb{S}$ such that

$$\|E\|_\nu = 1, \qquad \|\widetilde{A}^{-1} E \widetilde{A}^{-1}\|_\nu = \|A^{-1}\|_2^2. \tag{3.13}$$

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Then (3.12) implies

$$\kappa(\widetilde{A}, M\cdot\mathbb{S}) \geq \frac{\|\widetilde{A}\|_\nu}{\|\widetilde{A}^{-1}\|_\nu}\|\widetilde{A}^{-1}E\widetilde{A}^{-1}\|_\nu = \frac{\|A\|_\nu}{\|A^{-1}\|_\nu}\|A^{-1}\|_2^2 = \kappa_\nu(A).$$

Hence to complete the proof we just need to construct $E \in M\cdot\mathbb{S}$ satisfying (3.13). Since the scalar product is orthosymmetric, we have from Lemma 2.3 that $M\cdot\mathbb{S}$ is one of $\mathrm{Sym}(\mathbb{K})$, $\mathrm{Skew}(\mathbb{K})$ and $\mathrm{Herm}(\mathbb{C})$.

(i) $M\cdot\mathbb{S} = \mathrm{Sym}(\mathbb{K})$. Let $\widetilde{A}^{-1} = U\Sigma U^T$ be the Takagi factorization of $\widetilde{A}^{-1} \in \mathrm{Sym}(\mathbb{K})$, where U is unitary and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$ with $\|A^{-1}\|_2 = \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ [9, Cor. 4.4.4]. When $\mathbb{K} = \mathbb{R}$, U is orthogonal and $\widetilde{A}^{-1} = U\Sigma U^T$ is the singular value decomposition of $\widetilde{A}^{-1}$. Take $E = \bar{U}e_1e_1^TU^*$.

(ii) $M\cdot\mathbb{S} = \mathrm{Skew}(\mathbb{K})$. We consider the skew-symmetric analog of the Takagi factorization (Problems 25 and 26 in [9, Sec. 4.4]). Since $\widetilde{A}^{-1} \in \mathrm{Skew}(\mathbb{K})$, there exists a unitary matrix U such that $\widetilde{A}^{-1} = UDU^T$ where $D = D_1 \oplus \cdots \oplus D_{n/2}$, with $D_j = \begin{bmatrix} 0 & +z_j \\ -z_j & 0 \end{bmatrix}$, $0 \neq z_j \in \mathbb{C}$, $j = 1{:}\,n/2$. When $\mathbb{K} = \mathbb{R}$, U is orthogonal. Assume that the z_j are ordered such that $\|A^{-1}\|_2 = |z_1| \geq |z_2| \geq \cdots \geq |z_k|$. Take $E = c_\nu\bar{U}(e_1e_2^T - e_2e_1^T)U^*$, $\nu = 2, F$ with $c_2 = 1$ and $c_F = 1/\sqrt{2}$.

(iii) $M\cdot\mathbb{S} = \mathrm{Herm}(\mathbb{C})$. Take $E = Ue_1e_1^*U^*$ where U is the unitary factor in the singular value decomposition of $\widetilde{A}^{-1}$. ∎

When $\mathbb{S}$ is the automorphism group $\mathbb{G}$ of a scalar product there is a directly computable expression for $\kappa_F(A;\mathbb{G})$. Its derivation is similar to that of $\mathrm{cond}(A, x;\mathbb{G})$ described in section 3.1.2.

THEOREM 3.10. *Let $\mathbb{G}$ be the automorphism group of any scalar product and let $\mathbb{L}$ be the associated Lie algebra. Then*

$$\kappa_F(A;\mathbb{G}) = \frac{\|A\|_F}{\|A^{-1}\|_F}\|(A^T \otimes A)^{-1}Q\|_2.$$

where $(I \otimes A)B = QR$ is a reduced QR factorization of $(I \otimes A)B$ and B is a pattern matrix for $\mathbb{L}$ in the sense of (3.8).

It is well-known that $\kappa_2(A) \geq 1$. This is also true for $\kappa_2(A;\mathbb{G})$ when $\mathbb{G}$ is the automorphism group of a unitary and orthosymmetric scalar product. Indeed let $u, v \in \mathbb{K}^n$ be such that $A^{-1}u = \|A^{-1}\|_2 v$ with $\|u\|_2 = \|v\|_2 = 1$. Then, from (3.12), (3.6) and (3.7),

$$\begin{aligned}\kappa_2(A;\mathbb{G}) &= \frac{\|A\|_2}{\|A^{-1}\|_2}\sup\{\|HA^{-1}\|_2 : \|AH\|_2 = 1,\ H \in \mathbb{L}\}\\ &\geq \frac{\|A\|_2}{\|A^{-1}\|_2}\|A^{-1}\|_2\sup\{\|Hv\|_2 : \|AH\|_2 = 1,\ H \in \mathbb{L}\}\\ &= \|A\|_2\phi_2(A, v;\mathbb{G}).\end{aligned}$$

But from Lemma 3.6, $\phi_2(A, v;\mathbb{G}) \geq 1/\|A\|_2$ so that

$$1 \leq \kappa_2(A;\mathbb{G}) \leq \kappa_2(A).$$

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Clearly, if A is well conditioned then $\kappa_2(A;\mathbb{G}) \approx \kappa_2(A)$. This is certainly true for the orthogonal and unitary groups. For the symplectic matrix A defined in (3.9), we find that $\kappa_{\mathrm{F}}(A;\mathbb{G}) \approx \kappa_2(A;\mathbb{G}) = 1.0001$ but that $\kappa_2(A) = 1 \times 10^{12}$, showing that $\kappa_2(A;\mathbb{G}) \ll \kappa_2(A)$ may happen.

3.3. Nearness to singularity. The *structured distance to singularity* of a matrix is defined by

$$\delta(A;\mathbb{S}) = \min\Big\{\epsilon : \ \|\Delta A\| \le \epsilon\|A\|, \ A + \Delta A \text{ singular}, \ A + \Delta A \in \mathbb{S}\Big\}. \tag{3.14}$$

D. J. Higham [5] showed that for symmetric perturbations,

$$\delta_\nu(A;\mathrm{Sym}(\mathbb{R})) = \delta_\nu(A), \quad \nu = 2, F,$$

so that the constraint $\Delta A = \Delta A^T$ has no effect on the distance. This is a special case of the following more general result.

Theorem 3.11. *Let $\mathbb{S}$ be the Lie or Jordan algebra of a unitary and orthosymmetric scalar product. Let nonsingular $A \in \mathbb{S}$ be given. Then*

$$\delta_2(A;\mathbb{S}) = \delta_2(A),$$
$$\frac{1}{\sqrt{2}}\delta_F(A) \le \delta_F(A;\mathbb{S}) \le \delta_F(A).$$

Proof. It is well-known that the relative distance to singularity is the reciprocal of the condition number $\kappa(A)$ or a scalar multiple of it [8, p.111]. Clearly, $\delta_\nu(A;\mathbb{S}) \ge \delta_\nu(A)$.

Since the scalar product $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ is unitary, αM is unitary for some $\alpha > 0$ and we have $\delta_\nu(A;\mathbb{S}) = \delta_\nu(\widetilde{A},\widetilde{\mathbb{S}})$, where $\widetilde{A} = \alpha M A$ and $\widetilde{\mathbb{S}} = M \cdot \mathbb{S}$. Since the scalar product is orthosymmetric we have from Lemma 2.3 that $\widetilde{\mathbb{S}}$ is one of $\mathrm{Sym}(\mathbb{K})$, $\mathrm{Skew}(\mathbb{K})$ and $\beta\mathrm{Herm}(\mathbb{C})$ with $|\beta| = 1$. Without loss of generality, assume that $\|\widetilde{A}\|_\nu = 1$. Since $(\widetilde{A} + \Delta\widetilde{A})^{-1} = (I + \widetilde{A}^{-1}\Delta\widetilde{A})^{-1}\widetilde{A}^{-1}$ we just need to find $\Delta\widetilde{A} \in M\cdot\mathbb{S}$ such that $I + \widetilde{A}^{-1}\Delta\widetilde{A}$ is singular and $\|\Delta\widetilde{A}\|_\nu \le 1/\|A^{-1}\|_2$. This is achieved by taking $\Delta\widetilde{A} = -E/\|A^{-1}\|_2$, where E is the perturbation used in the proof of Theorem 3.9. ☐

Matrices in automorphism groups are nonsingular so $\delta(A;\mathbb{G}) = \infty$. This ends our treatment of various structured condition numbers.

4. Normwise structured backward errors. When solving a linear system $Ax = b$ with $A \in \mathbb{S} \subset \mathbb{K}^{n\times n}$, one is interested in whether a computed solution $\widehat{x}$ solves a nearby structured system: for example whether the particular structured backward error

$$\mu(\widehat{x};\mathbb{S}) = \min\{\|\Delta A\| : \ (A + \Delta A)\widehat{x} = b, \ A + \Delta A \in \mathbb{S}\} \tag{4.1}$$

is relatively small. Note that in (4.1), only the coefficient matrix A is perturbed. Let ΔA_{opt} be an optimal solution defined by $\|\Delta A_{\mathrm{opt}}\| = \mu(\widehat{x};\mathbb{S})$ and let $r = b - A\widehat{x}$ be the residual vector. It is well known that for unstructured perturbations,

$$\mu_\nu(\widehat{x}) \equiv \mu_\nu(\widehat{x};\mathbb{K}^{n\times n}) = \|r\|_2/\|\widehat{x}\|_2, \quad \nu = 2, F$$

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and this is achieved by $\Delta A_{\text{opt}} = r\widehat{x}^*/(\widehat{x}^*\widehat{x})$. Unlike the unstructured case, there may not be any solution ΔA to $(A + \Delta A)\widehat{x} = b$ with $A + \Delta A \in \mathbb{S}$. In this case we write that $\mu(\widehat{x}; \mathbb{S}) = \infty$. If one uses a structure preserving algorithm to solve the linear system then $A + \Delta A$ is structured and the backward error is guaranteed to be finite. This may not be the case if the system is solved using a linear solver that does not take advantage of the structure even if it is backward stable.

When both A and b are perturbed we define the corresponding structured backward error by

$$\eta(\widehat{x}; \mathbb{S}) = \min\{\epsilon : \ (A + \Delta A)\widehat{x} = b + \Delta b, \ A + \Delta A \in \mathbb{S}, \\ \|\Delta A\| \le \epsilon \|A\|, \ \|\Delta b\| \le \epsilon \|b\|\}. \tag{4.2}$$

Again, for unstructured perturbations we have the well-known explicit expression [8, Thm. 7.1],

$$\eta_\nu(\widehat{x}) \equiv \eta(\widehat{x}; \mathbb{K}^{n\times n}) = \frac{\|r\|_2}{\|A\|_\nu \|\widehat{x}\|_2 + \|b\|_2}, \quad \nu = 2, F. \tag{4.3}$$

The structured backward error $\eta(\widehat{x}; \mathbb{S})$ is harder to analyze than $\mu(\widehat{x}; \mathbb{S})$. The constraint $A + \Delta A \in \mathbb{S}$ generally implies an extra constraint on the perturbation Δb. For example, if $\mathbb{S} = \text{Skew}(\mathbb{R})$ then for there to exist ΔA skew-symmetric such that $(A+\Delta A)\widehat{x} = b+\Delta b$ one needs $\widehat{x}^T(b+\Delta b) = 0$ (see Theorem 2.1). On the other hand one may have $\eta(\widehat{x}; \mathbb{S})$ finite but $\mu(\widehat{x}; \mathbb{S}) = \infty$. The next theorem shows that when the particular backward error $\mu(\widehat{x}; \mathbb{S})$ exists, its study provides useful information on $\eta(\widehat{x}; \mathbb{S})$.

Theorem 4.1. *Let $A \in \mathbb{S}$, where $\mathbb{S}$ is a class of structured matrices and let $\widehat{x} \ne 0$ be an approximate solution to $Ax = b$. If $\mu_\nu(\widehat{x}; \mathbb{S})$ is finite with $\mu_\nu(\widehat{x}; \mathbb{S}) \le c_\nu \mu_\nu(\widehat{x})$ then for $\eta_\nu(\widehat{x}) < 1$,*

$$\eta_\nu(\widehat{x}) \le \eta_\nu(\widehat{x}; \mathbb{S}) \le \frac{2c_\nu \, \eta_\nu(\widehat{x})}{1 - \eta_\nu(\widehat{x})}, \qquad \nu = 2, F. \tag{4.4}$$

Proof. The proof is modelled from the solution to [8, Exercise 7.7] concerning the unstructured case. Clearly, from the definitions of $\eta_\nu(\widehat{x}; \mathbb{S})$ and $\mu_\nu(\widehat{x}; \mathbb{S})$,

$$\eta_\nu(\widehat{x}) \le \eta_\nu(\widehat{x}; \mathbb{S}) \le \frac{\mu_\nu(\widehat{x}; \mathbb{S})}{\|A\|_\nu} \le c_\nu \frac{\mu_\nu(\widehat{x})}{\|A\|_\nu} = c_\nu \frac{\|r\|_2}{\|A\|_\nu \|\widehat{x}\|_2}. \tag{4.5}$$

Let $\epsilon = \eta_\nu(\widehat{x})$. Then from (4.3) we have that $\|r\|_2 \le \epsilon(\|A\|_\nu\|\widehat{x}\|_2 + \|b\|_2)$ with $\|b\|_2 = \|(A+\Delta A)\widehat{x} - \Delta b\|_2 \le (1+\epsilon)\|A\|_\nu\|\widehat{x}\|_2 + \epsilon\|b\|_2$ yielding $\|b\|_2 \le (1-\epsilon)^{-1}(1+\epsilon)\|A\|_\nu\|\widehat{x}\|_2$. Thus

$$\|r\|_2 \le \frac{2\epsilon}{1-\epsilon}\|A\|_\nu\|\widehat{x}\|_2$$

and (4.5) yields the required bounds. ☐

Theorem 4.1 shows that when $\eta_\nu(\widehat{x})$ and c_ν are small, the structured relative backward error is small. In the rest of this section, we concentrate on the study of $\mu(\widehat{x}; \mathbb{S})$ and particular structures $\mathbb{S}$.

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4.1. Lie and Jordan algebras. For Lie or Jordan algebras ($\mathbb{S} = \mathbb{L}$ or $\mathbb{J}$) of a scalar product $\langle\cdot,\cdot\rangle_{\rm M}$ the structured backward error $\mu(\widehat{x};\mathbb{S})$ can be rewritten as

$$\mu(\widehat{x};\mathbb{S}) = \min\{\|\varDelta A\| : \ \varDelta A\widehat{x} = r, \ \varDelta A \in \mathbb{S}\},$$

where $r = b - A\widehat{x}$. When the scalar product is orthosymmetric and unitary (i.e., αM is unitary for some $\alpha > 0$), Theorem 2.2 implies that for $\widehat{x}$, b satisfying the relevant condition in Theorem 2.1,

$$\mu_\nu(\widehat{x};\mathbb{S}) = \begin{cases} \dfrac{\|r\|_2}{\|\widehat{x}\|_2} & \text{if } \nu = 2, \\ \sqrt{2\dfrac{\|r\|_2^2}{\|\widehat{x}\|_2^2} - \alpha^2\dfrac{|\langle r,\widehat{x}\rangle_{\rm M}|^2}{\|\widehat{x}\|_2^4}} & \text{if } \nu = F. \end{cases}$$

Since $\alpha|\langle r,\widehat{x}\rangle_{\rm M}| \le \|r\|_2\|\alpha M\widehat{x}\|_2 = \|\widehat{x}\|_2\|r\|_2$ the next result follows.

THEOREM 4.2. *Let $\mathbb{S}$ be the Lie algebra or Jordan algebra of an orthosymmetric and unitary scalar product. Let $A \in \mathbb{S}$ be nonsingular and $\widehat{x} \in \mathbb{K}^n$ be an approximate nonzero solution to $Ax = b$. If $\widehat{x}$ and b satisfy the conditions in Theorem* 2.1 *then, for $\nu = 2, F$,*

$$\mu_2(\widehat{x};\mathbb{S}) = \mu_2(\widehat{x}),$$
$$\mu_F(\widehat{x}) \le \mu_F(\widehat{x};\mathbb{S}) \le \sqrt{2}\mu_F(\widehat{x}),$$

otherwise $\mu_\nu(\widehat{x};\mathbb{S}) = \infty$, $\nu = 2, F$.

Theorem 4.2 shows that forcing the backward error matrix to have the structure of $\mathbb{S}$ has either little effect or a drastic effect on its norm. Not all structures require conditions on $\widehat{x}$ and b for $\mu(\widehat{x};\mathbb{S})$ to be finite. For example, for symmetric, complex symmetric, pseudo-symmetric, persymmetric and Hamiltonian structures the structured backward error always exists and is equal to or within a small factor of the unstructured one.

4.2. Automorphism groups. Structured backward errors are difficult to analyze for nonlinear structures. Some progress is made in this section for specific automorphism groups. For that we need a characterization of the set of all matrices in the automorphism group $\mathbb{G}$ of an orthosymmetric scalar product mapping x to b.

THEOREM 4.3 ([12]). *Suppose $\mathbb{G}$ is the automorphism group of an orthosymmetric scalar product $\langle\cdot,\cdot\rangle_{\rm M}$, and let $x, b, v \in \mathbb{K}^n$ such that $\langle x,x\rangle_{\rm M} = \langle b,b\rangle_{\rm M} = \langle v,v\rangle_{\rm M}$. Let G_x, G_b be any fixed elements of $\mathbb{G}$ such that $G_x x = v$ and $G_b b = v$, and let $\mathbb{G}_v = \{\, Q \in \mathbb{G} : Qv = v \,\}$. Then*

$$\{\, A \in \mathbb{G} : Ax = b \,\} = \{\, G_b^\star Q G_x : \ Q \in \mathbb{G}_v \,\}. \tag{4.6}$$

The proof in [12] is short and we recall it.

Proof. Let $\mathcal{S}_1 = \{\, A \in \mathbb{G} : Ax = b \,\}$ and $\mathcal{S}_2 = \{\, A = G_b^\star Q G_x : Q \in \mathbb{G}_v \,\}$. Suppose $A \in \mathcal{S}_1$, and consider $Q := G_b A G_x^{-1} \in \mathbb{G}$. Clearly $G_b A G_x^{-1} v = v$, so $Q \in \mathbb{G}_v$. Hence $A = G_b^{-1} Q G_x = G_b^\star Q G_x$ for some $Q \in \mathbb{G}_v$, so $A \in \mathcal{S}_2$.

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Conversely, suppose $A \in \mathcal{S}_2$, so $A = G_b^{-1} Q G_x$ where $Q \in \mathbb{G}_v$. Because $\mathbb{G}$ is a group, $A \in \mathbb{G}$. Also $Ax = G_b^{-1} Q G_x x = G_b^{-1} Q v = G_b^{-1} v = b$, so $A \in \mathcal{S}_1$. $\square$

As a consequence of Theorem 4.3, if $\langle \widehat{x}, \widehat{x} \rangle_{\mathrm{M}} = \langle b, b \rangle_{\mathrm{M}}$ we can rewrite the structured backward error in (4.1) as

$$\mu(\widehat{x}; \mathbb{G}) = \min_{Q \in \mathbb{G}_v} \| G_b^{\star} Q G_{\widehat{x}} - A \|, \tag{4.7}$$

where $G_{\widehat{x}}, G_b$ are fixed elements of $\mathbb{G}$ satisfying $G_{\widehat{x}} \widehat{x} = G_b b = v$ for some vector $v \in \mathbb{K}^n$, and $\mathbb{G}_v = \{ Q \in \mathbb{G} : Qv = v \}$. For many of the groups listed in Table 2.1 it is possible to determine explicitly the set $\mathbb{G}_v$ for some suitably chosen vectors v. For example when $\mathbb{G}$ is the real orthogonal or unitary group, there exist $G_{\widehat{x}}, G_b$ unitary such that $G_{\widehat{x}} \widehat{x} = \|\widehat{x}\|_2 e_1$ and $G_b b = \|b\|_2 e_1$. Since $\|\widehat{x}\|_2 = \|b\|_2$ we can choose $v = \|x\|_2 e_1$ giving

$$\mathbb{G}_v = \{ Q \in \mathbb{G} : Q e_1 = e_1 \} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & V \end{bmatrix} : \ V^* = V^{-1} \in \mathbb{K}^{(n-1)\times(n-1)} \right\}.$$

For these two groups we derive an explicit expression for the structured backward error and show that it is within a small factor of the unstructured one.

In what follows $\mathbb{G}$ denotes either the real orthogonal group or the unitary group. Without loss of generality we assume that $\|x\|_2 = \|b\|_2 = 1$. Let $G_{\widehat{x}}$ be any element of $\mathbb{G}$ such that $G_{\widehat{x}} \widehat{x} = e_1$ and write $G_{\widehat{x}}^* = \begin{bmatrix} \widehat{x} & \widetilde{X} \end{bmatrix}$. Let $G \in \mathbb{G}$ be the unitary Householder reflector mapping $\widehat{x}$ to b. Then $G_b := G_{\widehat{x}} G^* \in \mathbb{G}$ satisfies $G_b b = e_1$ and $G_b^* = \begin{bmatrix} b & G\widetilde{X} \end{bmatrix}$. For any $Q \in \mathbb{G}_v$ with $v = e_1$ the quantity whose norm needs to be minimized in (4.7) can be rewritten as

$$\begin{aligned} G_b^{\star} Q G_{\widehat{x}} - A &= G_b^* \begin{bmatrix} 1 & \\ & V \end{bmatrix} G_{\widehat{x}} - A G_{\widehat{x}}^* G_{\widehat{x}}, \qquad V^* = V^{-1} \in \mathbb{K}^{(n-1)\times(n-1)} \\ &= b\widehat{x}^* + G\widetilde{X} V \widetilde{X}^* - A \begin{bmatrix} \widehat{x} & \widetilde{X} \end{bmatrix} G_{\widehat{x}} \\ &= \begin{bmatrix} b - Ax & G\widetilde{X}V - A\widetilde{X} \end{bmatrix} G_{\widehat{x}} \end{aligned}$$

so that, for the Frobenius norm, $\mu(\widehat{x}; \mathbb{G})$ in (4.7) becomes

$$\mu_F(\widehat{x}; \mathbb{G})^2 = \|r\|_2^2 + \min_{V^*V = I} \|G\widetilde{X}V - A\widetilde{X}\|_F^2.$$

The minimization problem above is a well-known Procrustes problem whose solution is given by $\|G\widetilde{X}U - A\widetilde{X}\|_F$, where U is the unitary factor of the polar decomposition of $\widetilde{X}^* G^* A \widetilde{X}$ [4, p. 601]. Hence, since G is unitary, we obtain an explicit expression for the structured backward error,

$$\mu_F(\widehat{x}; \mathbb{G})^2 = \|r\|_2^2 + \|\widetilde{X}U - G^* A \widetilde{X}\|_F^2. \tag{4.8}$$

We now compare the size of $\|\widetilde{X}U - G^* A\widetilde{X}\|_F$ to that of $\|r\|_2$ using a technique similar to that of Sun in [17]. Because $\begin{bmatrix} \widehat{x} & \widetilde{X} \end{bmatrix}$ is unitary, $\widehat{x}\widehat{x}^* + \widetilde{X}\widetilde{X}^* = I$ and

$$\widetilde{X}U - G^* A \widetilde{X} = X \begin{bmatrix} -\widehat{x}^* G^* A \widetilde{X} \\ U - \widetilde{X}^* G^* A \widetilde{X} \end{bmatrix}.$$

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Hence

$$\begin{aligned}\|\widetilde{X}U - G^*A\widetilde{X}\|_F^2 &= \|U - \widetilde{X}^*G^*A\widetilde{X}\|_F^2 + \|\widehat{x}^*G^*A\widetilde{X}\|_2^2 \\ &= \|I - H_2\|_F^2 + \|\widehat{x}^*G^*A\widetilde{X}\|_2^2,\end{aligned} \tag{4.9}$$

where H_2 is the Hermitian polar factor of $\widetilde{X}^*G^*A\widetilde{X}$. We now need the following Lemma.

LEMMA 4.4 ([17, Lem. 2.4] and [18, Lem. 2.2]). *Let $A \in \mathbb{C}^{m\times m}$ be unitary, $X_1 \in \mathbb{C}^{m\times k}$ with $2k \le m$, $X = [X_1, X_2]$ be unitary, and let H_1 and H_2 be the Hermitian polar factors of $X_1^*AX_1$ and $X_2^*AX_2$ respectively. Then, for any unitarily invariant norm,*

$$\|I - H_1\| = \|I - H_2\| \quad \textit{and} \quad \|X_1^*AX_2\| = \|X_2^*AX_1\|.$$

Applying Lemma 4.4 to (4.9) with $X_1 = \widehat{x}$, $X_2 = \widetilde{X}$ and G^*A in place of A yields

$$\|I - H_2\|_F^2 + \|\widehat{x}^*G^*A\widetilde{X}\|_2^2 = \|e^{\imath\theta} - \widehat{x}^*G^*A\widehat{x}\|_2^2 + \|\widetilde{X}^*G^*A\widehat{x}\|_2^2, \tag{4.10}$$

where $\theta = \arg(\widehat{x}^*B^*A\widehat{x})$. Finally, since $\begin{bmatrix}\widehat{x} & \widetilde{X}\end{bmatrix}$ is unitary,

$$\begin{bmatrix}\widehat{x} & \widetilde{X}\end{bmatrix}\begin{bmatrix} e^{\imath\theta} - \widehat{x}^*G^*A\widehat{x} \\ -\widetilde{X}^*G^*A\widehat{x}\end{bmatrix} = \widehat{x}e^{\imath\theta} - G^*A\widehat{x}$$

so that

$$\|e^{\imath\theta} - \widehat{x}^*G^*A\widehat{x}\|_2^2 + \|\widetilde{X}^*G^*A\widehat{x}\|_2^2 = \|\widehat{x}e^{\imath\theta} - G^*A\widehat{x}\|_2^2 \tag{4.11}$$

Hence combining (4.8)–(4.11) gives $\|\widetilde{B}U - A\widetilde{X}\|_F = \|\widehat{x}e^{\imath\theta} - G^*A\widehat{x}\|_2$. But

$$\|\widehat{x}e^{\imath\theta} - G^*A\widehat{x}\|_2 = \min_{|\rho|=1}\|\widehat{x}\rho - B^*A\widehat{x}\|_F \le \|\widehat{x} - B^*A\widehat{x}\|_F = \|b - A\widehat{x}\|_2 = \|r\|_2.$$

We have just proved the following result.

THEOREM 4.5. *Let $\mathbb{G}$ be the real orthogonal or unitary matrix group. Let $A \in \mathbb{G}$ and $\widehat{x} \in \mathbb{K}^n$ be a nonzero approximate solution to $Ax = b$. If $\|\widehat{x}\|_2 = \|b\|_2$ then*

$$\mu_F(\widehat{x}) \le \mu_F(\widehat{x};\mathbb{G}) \le \sqrt{2}\mu_F(\widehat{x}),$$

otherwise $\mu_F(\widehat{x};\mathbb{G}) = \infty$.

This generalizes Sun's result on backward errors for the unitary eigenvalue problem, for which b has the special form $b = e^{\imath\theta}x$.

For automorphism groups other than the unitary or real orthogonal groups, deriving an explicit expression for the structured backward error in (4.7) is an open problem.

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