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ON POLYNOMIALS IN TWO PROJECTIONS*

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Abstract. Conditions are established on a polynomial f in two variables under which the equality $f(P_1, P_2) = 0$ for two orthogonal projections P_1, P_2 is possible only if P_1 and P_2 commute.

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1. Introduction.

Denote by $\mathbb{C}^{n\times n}$ the set of all $n \times n$ matrices with complex entries. Let $P_1, P_2 \in \mathbb{C}^{n\times n}$ be two orthogonal projections, that is,

$$P_j = P_j^* = P_j^2, \quad j = 1, 2.$$

Then an arbitrary polynomial f in two variables P_1, P_2 has the form

$$f(P_1, P_2) = \sum c_{(m,i)} P_{(m,i)} \tag{1.1}$$

with m assuming natural values and $i \in \{1, 2\}$. Here $c_{(m,i)} \in \mathbb{C}$, and $P_{(m,i)}$ is the notation for an alternating product of m multiples P_1, P_2 starting with P_i.

We are interested in the question for which polynomials f it is true that

$$f(P_1, P_2) = 0 \text{ implies the commutativity of } P_1 \text{ and } P_2. \tag{1.2}$$

The particular case of a binomial f was considered earlier in [3], and the situation of three or four terms was dealt with in [1]. As a matter of fact, the problem of describing all polynomials f for which condition (1.2) does not imply that P_1 and P_2 commute was also stated in [1], though not explicitly and in slightly different terms.

Our approach is different from that proposed in [3, 1], and is based on a (known) canonical form in which a pair of orthogonal projections can be put by a unitary equivalence. In Section 2 we recall this canonical form and prove the general result. Section 3 is devoted to its particular cases. Finally, the infinite dimensional variations are discussed in Section 4.

2. Main result.

The following result is well known; relevant references will be given in Section 4.

LEMMA 2.1. *For any two orthogonal projections P_1, P_2 there exists a unitary transformation U such that U^*P_1U is the orthogonal sum of one dimensional blocks $0, 1$ and two dimensional blocks $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ while U^*P_2U is the sum of (the same*

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number of) one dimensional blocks $0, 1$ *and two dimensional blocks*

$$p_t = \begin{bmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{bmatrix}.$$

The number of two dimensional blocks coincides with the number of the eigenvalues of $P_1P_2P_1$ *different from 0 and 1 (and therefore lying strictly in between), and the parameter* t *runs through the set of all such eigenvalues (counting their multiplicities).*

To formulate our main result, we need to introduce certain notation. Namely, let us partition the set of indices in the right hand side of (1.1) into four disjoint classes, corresponding to m being odd or even and i equal 1 or 2. Accordingly, introduce four scalar polynomials:

$$\phi_1(t) = \sum c_{(2k+1,1)} t^k,\ \phi_2(t) = \sum c_{(2k,1)} t^{k-1},$$
$$\phi_3(t) = \sum c_{(2k+1,2)} t^k,\ \phi_4(t) = \sum c_{(2k,2)} t^{k-1} \tag{2.1}$$

(as usual, we agree that a sum with the void set of indices is equal to zero).

THEOREM 2.2. *Let* P_1, P_2 *be two orthogonal projections, and let* f *be a polynomial in* P_1, P_2 *given by* (1.1). *Then statement* (1.2) *holds if and only if all four polynomials* ϕ_j *defined by* (2.1) *do not vanish simultaneously at any point in* $(0, 1)$.

Proof. Let us start by computing $f(p, p_t)$, where p and p_t ($t \in (0, 1)$) are 2×2 Hermitian idempotent matrices introduced in Lemma 2.1. This is a technical part, needed both for the proof of necessity and sufficiency.

Direct computations show that

$$pp_t = \begin{bmatrix} t & \sqrt{t(1-t)} \\ 0 & 0 \end{bmatrix},\ pp_tp = tp,\ p_tp = \begin{bmatrix} t & 0 \\ \sqrt{t(1-t)} & 0 \end{bmatrix},\ \text{and } p_tpp_t = tp_t.$$

Consequently,

$$f(p, p_t) = \phi_1(t)p + \phi_2(t)pp_t + \phi_3(t)p_t + \phi_4(t)p_tp. \tag{2.2}$$

Necessity. Suppose that all four polynomials ϕ_j have a common zero (say, t_0) in $(0, 1)$. Then, according to (2.2), $f(p, p_{t_0}) = 0$. On the other hand, the orthogonal projections p, p_{t_0} do not commute, since

$$pp_t - p_tp = \begin{bmatrix} 0 & \sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & 0 \end{bmatrix}.$$

Sufficiency. Suppose that the polynomials ϕ_j, $j = 1, \ldots, 4$, do not have joint zeros on $(0, 1)$ and that the orthogonal projections P_1, P_2 do not commute. According to Lemma 2.1, $f(P_1, P_2)$ is then unitarily equivalent to the orthogonal sum in which at least one of the summands is a two dimensional block of the form $f(p, p_t)$ with $t \in (0, 1)$. Combining formula (2.2) with the fact that the matrices p, p_t, pp_t and p_tp are linearly independent, we conclude that $f(p, p_t) \neq 0$. Hence, $f(P_1, P_2) \neq 0$. $\square$

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3. Particular cases. Apparently, zero is the only root of a monomial. This simple observation, when combined with Theorem 2.2. leads to the following statement.

COROLLARY 3.1. *Suppose that expression* (1.1) *contains exactly one term with odd (or even) number of multiples and starting with* P_1 *(or* P_2*). Then* (1.2) *holds.*

The situation is only slightly more complicated when one of the polynomials ϕ_j is a binomial, since the latter has exactly one non-zero root.

COROLLARY 3.2. *Suppose that one of the polynomials* (2.1) *is a binomial, say* $d_1t^{k_1} + d_2t^{k_2}$*. Then either of conditions*

(i) $d_1d_2 > 0$,

(ii) $(k_1 - k_2)(|d_1| - |d_2|) \le 0$,

(iii) *at least one of the remaining polynomials* ϕ_j *assumes a non-zero value at* $x_0 = (-d_2/d_1)^{1/(k_1-k_2)}$,

is sufficient for (1.2) *to hold.*

Proof. Under condition (i) or (ii) the roots of $d_1t^{k_1}+d_2t^{k_2}$ lie outside of $(0, 1)$, and Theorem 2.2 applies in a trivial way. If conditions (i) and (ii) do not hold, then x_0 is the only root of $d_1t^{k_1} + d_2t^{k_2}$ lying in $(0, 1)$, so that the applicability of Theorem 2.2 is guaranteed by condition (iii). ☐

We can now give an alternative proof for the following result from [3].

THEOREM 3.3. *Let*

$$P_{(m,i)} = P_{(l,k)} \text{ with } (m, i) \neq (l, k). \tag{3.1}$$

Then P_1 *and* P_2 *commute.*

Proof. Consider $f(P_1, P_2) = P_{(m,i)} - P_{(l,k)}$. If $m - l$ is even (but non-zero) and $i = k$, then exactly one of the polynomials ϕ_j associated with f is different from zero, and this polynomial is in fact a binomial $t^{s_1} - t^{s_2}$ with $s_1 \neq s_2$. Commutativity of P_1 and P_2 follows then from Corollary 3.2, since condition (ii) of this statement is met. In all other cases there are two non-trivial polynomials ϕ_j, both being monomials, so that Corollary 3.1 applies. ☐

It is straightforward (and was also observed in [3]) that the converse of Theorem 3.3 holds provided that $m, l \ge 2$. The necessary and sufficient conditions for (3.1) to hold can be easily established when $\min\{m, l\} = 1$ but they are not worth the further discussion.

THEOREM 3.4. *Let the polynomial* (1.1) *contain at most five summands. Suppose that the set of the indices* (m, i) *is such that the following possibilities are excluded:*

(i) *all indices are of the same type, that is, all* m *are of the same evenness and all* i *are the same;*

(ii) *there are two different types of the indices, one of them corresponding to exactly two coefficients of the opposite sign.*

Then statement (1.2) *holds.*

Proof. Suppose first that the number of types of the indices present (that is, the number of non-zero polynomials (2.1) associated with f) is three or higher. Then at least one of ϕ_j is a monomial, and Corollary 3.1 applies. This corollary applies also when there are two non-zero polynomials ϕ_j but one of them is still a monomial. It

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remains to consider the case when there are exactly two non-zero polynomials ϕ_j, neither of which being a monomial. At least one of them must then be a binomial, and its coefficients are of the same sign due to condition (ii). This is the situation (i) of Corollary 3.2.□

A particular case of Theorem 3.4, dealing with four summands and not utilizing the properties of the coefficients, was proved in [1][1].

4. Infinite dimensional setting. Lemma 2.1 is a finite dimensional adaptation of the following result.

THEOREM 4.1. *Let P_1 and P_2 be two orthogonal projections acting on a Hilbert space $\mathfrak{H}$. Then there exist an orthogonal decomposition*

$$\mathfrak{H} = \mathfrak{M}_{00} \oplus \mathfrak{M}_{01} \oplus \mathfrak{M}_{10} \oplus \mathfrak{M}_{11} \oplus \mathfrak{M}_0 \oplus \mathfrak{M}_1, \tag{4.1}$$

a Hermitian operator H on $\mathfrak{M}_0$ with the spectrum in $[0,1]$ not having $0,1$ as its eigenvalues, and a unitary operator $W\colon \mathfrak{M}_1 \to \mathfrak{M}_0$ such that with respect to decomposition (4.1):

$$\begin{aligned} P_1 &= I \oplus I \oplus 0 \oplus 0 \oplus \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \\ P_2 &= I \oplus 0 \oplus 0 \oplus I \oplus \begin{bmatrix} I & 0 \\ 0 & W^* \end{bmatrix} \begin{bmatrix} H & \sqrt{H(I-H)} \\ \sqrt{H(I-H)} & I-H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}. \end{aligned} \tag{4.2}$$

As a matter of fact, the summands in (4.1) and the operator H are defined uniquely. Namely,

$$\mathfrak{M}_{00} = \operatorname{Im} P_1 \cap \operatorname{Im} P_2,\ \mathfrak{M}_{01} = \operatorname{Im} P_1 \cap \operatorname{Ker} P_2,$$
$$\mathfrak{M}_{10} = \operatorname{Ker} P_1 \cap \operatorname{Ker} P_2,\ \mathfrak{M}_{11} = \operatorname{Ker} P_1 \cap \operatorname{Im} P_2,$$

$\mathfrak{M}_0$ is the orthogonal complement of $\mathfrak{M}_{00} \oplus \mathfrak{M}_{01}$ in $\operatorname{Im} P_1$, and H is the restriction of $P_1P_2P_1$ onto $\mathfrak{M}_0$. The non-trivial part of Theorem 4.1 is the statement that H is unitarily equivalent to the restriction of $(I-P_1)(I-P_2)(I-P_1)$ onto the orthogonal complement $\mathfrak{M}_1$ of $\mathfrak{M}_{10} \oplus \mathfrak{M}_{11}$ in $\operatorname{Ker} P_1$; the operator W arises from this unitary equivalence. The operators P_1 and P_2 commute if and only if the subspace $\mathfrak{M}_0$ (and therefore $\mathfrak{M}_1$) has zero dimension, that is, the orthogonal sums in (4.1), (4.2) degenerate to the first four summands.

Theorem 4.1 in various disguises can be found, for example, in [7, 4, 5], and some of its applications are in [9, 8, 6].

The question on the validity of (1.2) can be asked in the setting of Hilbert spaces. Using the notation (2.1) and formula (4.2), it is easy to see that $f(P_1,P_2)$ is the sum of a diagonal operator acting on $\mathfrak{M}_{00} \oplus \mathfrak{M}_{01} \oplus \mathfrak{M}_{10} \oplus \mathfrak{M}_{11}$ with the operator unitarily equivalent to

$$\begin{bmatrix} \phi_1(H) + H(\phi_2+\phi_3+\phi_4)(H) & (\phi_2+\phi_3)(H)\sqrt{H(I-H)} \\ (\phi_3+\phi_4)(H)\sqrt{H(I-H)} & \phi_3(H)(I-H) \end{bmatrix}.$$

[1]Added in proof: The case of four summands was also considered in [2, Theorem 2.3].

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Due to the injectivity of H and $I - H$, the latter operator is zero if and only if $\phi_j(H) = 0$ for $j = 1, \ldots, 4$. The spectral mapping theorem implies then that the polynomials ϕ_j have a common root on $(0, 1)$, unless the summands $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are missing in the decomposition (4.1). The latter happens if and only if the operators P_1 and P_2 commute, which proves the operator version of Theorem 2.2. From here it follows that the results of Section 3 also remain valid verbatim in the infinite dimensional setting.

In fact, our results hold in a (formally) more general situation when P_1 and P_2 are selfadjoint idempotent elements of any C^*-algebra $\mathcal{A}$, because any such $\mathcal{A}$ is isomorphic to a subalgebra of the algebra of bounded linear operators on a Hilbert space $\mathfrak{H}$. On the other hand, the selfadjointness condition is essential: even Theorem 3.3 is not valid if P_1, P_2 are idempotent non-selfadjoint matrices.

REFERENCES

[1] J. K. Baksalary, O. M. Baksalary, and P. Kik. Generalization of a property of orthogonal projectors. *Linear Algebra Appl.*, 2005. In press.

[2] O. M. Baksalary and P. Kik. On commutativity of projectors. *Linear Algebra Appl.*, 2006. In press.

[3] J. K. Baksalary, O. M. Baksalary, and T. Szulc. A property of orthogonal projectors. *Linear Algebra Appl.*, 354:35–39, 2002. Ninth special issue on linear algebra and statistics.

[4] C. Davis. Separation of two linear subspaces. *Acta Sci. Math. Szeged*, 19:172–187, 1958.

[5] P. L. Halmos. Two subspaces. *Trans. Amer. Math. Soc.*, 144:381–389, 1969.

[6] C. Hillar, C. R. Johnson, and I. M. Spitkovsky. Positive eigenvalues and two-letter generalized words. *Electron. J. Linear Algebra*, 9:21–26 (electronic), 2002.

[7] M. G. Krein, M. A. Krasnoselski, and D. P. Milman. On defect numbers of operators on Banach spaces and related geometric problems. *Trudy Inst. Mat. Akad Nauk Ukrain. SSR*, 11:97–112, 1948.

[8] I. M. Spitkovsky. Once more on algebras generated by two projections. *Linear Algebra Appl.*, 208/209:377–395, 1994.

[9] N. Vasilevsky and I. Spitkovsky. On the algebra generated by two projections. *Doklady Akad. Nauk Ukrain. SSR, Ser. A*, 8:10–13, 1981.