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ON DETERMINING MINIMAL SPECTRALLY ARBITRARY PATTERNS*

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Abstract. A new family of minimal spectrally arbitrary patterns is presented which allow for arbitrary spectrum by using the Nilpotent-Jacobian method introduced in [J.H. Drew, C.R. Johnson, D.D. Olesky, and P. van den Driessche. Spectrally arbitrary patterns.*Lin. Alg. and Appl.* 308:121-137, 2000]. The novel approach here is the use of the Intermediate Value Theorem to avoid finding an explicit nilpotent realization of the new minimal spectrally arbitrary patterns.

Key words. Sign pattern, Spectrum, Nilpotent matrix.

AMS subject classifications. 15A18, 15A29, 05C50.

1. Introduction. A matrix $\mathcal{S}$ with entries in $\{+,-,0\}$ is a *sign pattern.* Let $\mathcal{S}=[s_{ij}]$ and $\mathcal{U}=[u_{ij}]$ be m by n sign patterns. If $u_{ij}=s_{ij}$ whenever $s_{ij}\neq 0$, then $\mathcal{U}$ is a *superpattern* of $\mathcal{S}$ and $\mathcal{S}$ is a *subpattern* of $\mathcal{U}$. A subpattern of $\mathcal{S}$ which is not $\mathcal{S}$ itself is a *proper subpattern* of $\mathcal{S}$. Similarly, a superpattern of $\mathcal{S}$ which is not $\mathcal{S}$ itself is a *proper superpattern* of $\mathcal{S}$.

For a given real number a, the sign of a is denoted by $\operatorname{sgn}(a)$, and is $+$, 0, or $-$ depending on whether a is positive, 0 or negative. The *sign pattern class* of an m by n sign pattern $\mathcal{S}=[s_{ij}]$ is defined by

$$Q(\mathcal{S})=\{A=[a_{ij}]\in\mathbb{R}^{m\times n} : \operatorname{sgn}(a_{ij})=s_{ij} \text{ for all } i,j\}.$$

A *nilpotent realization* of a sign pattern $\mathcal{S}$ is a real matrix $A\in Q(\mathcal{S})$ whose only eigenvalue is zero. We say a sign pattern $\mathcal{S}$ *requires* a property P if each matrix $A\in Q(\mathcal{S})$ has property P.

For a complex number λ, we take $\overline{\lambda}$ to denote the complex conjugate of λ. Let σ be a multi-list of complex numbers. Then σ is *self-conjugate* if and only if for each $\lambda\in\sigma$, $\overline{\lambda}$ occurs with the same multiplicity as λ's in σ. Note that if A is an n by n real matrix, then the spectrum of A is self-conjugate.

An n by n sign pattern $\mathcal{S}$ is a *spectrally arbitrary pattern* (*SAP*) if each self-conjugate multi-list of n complex numbers is the spectrum of a realization of $\mathcal{S}$, that is, if each monic real polynomial of degree n is the characteristic polynomial of a

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matrix in $Q(\mathcal{S})$. If $\mathcal{S}$ is a SAP and no proper subpattern of $\mathcal{S}$ is spectrally arbitrary, then $\mathcal{S}$ is a *minimal spectrally arbitrary pattern* (*MSAP*).

The question of the existence of a SAP arose in [3], where a general method (based on the Implicit Function Theorem) was given to prove that a sign pattern and all of its superpatterns are SAP. The first SAP of order n for each $n \geq 2$ was provided in [6]. Later, the method in [3], which we will call the *Nilpotent-Jacobian method* (*N-J method*), was reformulated in [1], and used to find MSAPs in [1], [2] and [5].

The N-J method is quite powerful, yet its Achilles' heel is the need to determine (not necessarily explicitly) an appropriate nilpotent realization in order to compute the Jacobian involved in the method. It is not an easy task to find an appropriate nilpotent realization, even for the well-structured antipodal tridiagonal pattern in [3], at which the Jacobian is nonzero (see [4]). In this paper, we show how to use the N-J method without explicitly constructing a nilpotent realization by using the Intermediate Value Theorem, and provide a new family of MSAPs.

2. The N-J Method. Throughout, we take $p_A(x) = x^n - \alpha_1 x^{n-1} + \alpha_2 x^{n-2} - \cdots + (-1)^{n-1}\alpha_{n-1}x + (-1)^n \alpha_n$ to denote the characteristic polynomial of an n by n matrix A, and the Jacobian $\Delta = \det\left(\dfrac{\partial f_i}{\partial x_j}\right)$ is denoted by $\dfrac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)}$ where $f = (f_1,\ldots,f_n)$ is a function of $x_1,\ldots,x_n$ such that $\dfrac{\partial f_i}{\partial x_j}$ exists for all $i,j \in \{1,\ldots,n\}$. The matrix $\left(\dfrac{\partial f_i}{\partial x_j}\right)$ is called the *Jacobian matrix* of f.

The next theorem describes the N-J method for proving that a sign pattern and all of its superpatterns are spectrally arbitrary.

THEOREM 2.1. ([1], Lemma 2.1) *Let $\mathcal{S}$ be an n by n sign pattern, and suppose that there exists a nilpotent realization $M = [m_{ij}]$ of $\mathcal{S}$ with at least n nonzero entries, say, $m_{i_1 j_1},\ldots,m_{i_n j_n}$. Let X be the matrix obtained by replacing these entries in M by variables $x_1,\ldots,x_n$, and let $(-1)^k\alpha_k$ be the coefficients of $p_X(x)$ for $k = 1,2,\ldots,n$. If the Jacobian $\dfrac{\partial(\alpha_1,\ldots,\alpha_n)}{\partial(x_1,\ldots,x_n)}$ is nonzero at $(x_1,\ldots,x_n) = (m_{i_1 j_1},\ldots,m_{i_n j_n})$, then every superpattern of $\mathcal{S}$ is spectrally arbitrary.*

EXAMPLE 2.2. ([1], Example 2.2) Let $\mathcal{S} = \begin{bmatrix} + & - \\ + & - \end{bmatrix}$. Then $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ is a nilpotent realization of $\mathcal{S}$. Let $X = \begin{bmatrix} x_1 & -1 \\ 1 & x_2 \end{bmatrix}$. Then,

$$p_X(x) = x^2 - \alpha_1 x + \alpha_2,$$

where $\alpha_1 = x_1 + x_2$ and $\alpha_2 = x_1 x_2 + 1$. Thus

$$\Delta = \frac{\partial(\alpha_1,\alpha_2)}{\partial(x_1,x_2)} = \det\begin{bmatrix} 1 & 1 \\ x_2 & x_1 \end{bmatrix} = x_1 - x_2.$$

At $(x_1,x_2) = (1,-1)$, $\Delta = 2 \neq 0$. By Theorem 2.1, $\mathcal{S}$ is spectrally arbitrary.

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If some entries of $\mathcal{S}$ are replaced by 0, then α_2 of each realization of the resulting sign pattern has a fixed sign in $\{+,-,0\}$ and hence, the resulting sign pattern is not spectrally arbitrary. Therefore, $\mathcal{S}$ is a MSAP.

3. A new MSAP. Let n and r be positive integers with $2 \le r \le n$, and let

$$\mathcal{K}_{n,r} = \begin{bmatrix} + & - & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ + & 0 & - & 0 & & & & \vdots \\ + & 0 & 0 & - & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ & & & & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & & & & \ddots & - & 0 \\ + & 0 & \cdots & & & \cdots & 0 & - \\ 0 & \cdots & 0 & + & 0 & \cdots & 0 & - \end{bmatrix}_{n\times n},$$

where the positive entry in the last row is in column $n-r+1$. For $r > \frac{n}{2}$, the patterns $\mathcal{K}_{n,r}$ are spectrally arbitrary [2, Theorem 4.4]. The pattern $\mathcal{K}_{n,n}$ was shown to be a MSAP in [1] and $\mathcal{K}_{n,n-1}$ was shown to be a MSAP in [2]. In this paper, we show that $\mathcal{K}_{n,r}$ is a SAP (and in fact a MSAP) for all r with $2 \le r < n$.

It is convenient to consider matrices $A \in Q(\mathcal{K}_{n,r})$ of the form

$$A = \begin{bmatrix} a_1 & -1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ a_2 & 0 & -1 & 0 & & & & \vdots \\ a_3 & 0 & 0 & -1 & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\ & & & & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & & & & \ddots & -1 & 0 \\ a_{n-1} & 0 & \cdots & 0 & \cdots & \cdots & 0 & -1 \\ 0 & \cdots & 0 & b_r & 0 & \cdots & 0 & -1 \end{bmatrix}_{n\times n}, \tag{3.1}$$

where $a_j > 0$ for $j = 1, \ldots, n-1$ and $b_r > 0$.

We first give a definition and a little result on the positive zeros of real polynomials (of finite degree). For a real polynomial $f(t)$, we set

$$Z_f = \{a > 0 \mid f(a) = 0\}.$$

If Z_f is nonempty, the minimum of Z_f is denoted by $\min(Z_f)$.

Proposition 3.1. *Let $f(t)$ and $g(t)$ be real polynomials, and $h(t) = g(t) - tf(t)$. Suppose that $f(0)$, $g(0) > 0$, Z_f and Z_g are nonempty, and $\min(Z_g) < \min(Z_f)$. Then Z_h is nonempty and $\min(Z_h) < \min(Z_g)$.*

Proof. Let $t_g = \min(Z_g)$. Since $t_g < \min(Z_f)$ and $f(0) > 0$, we have $f(t_g) > 0$. Thus,

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$$h(t_g) = g(t_g) - t_g f(t_g) = -t_g f(t_g) < 0.$$

Since $h(0) = g(0) > 0$, the Intermediate Value Theorem implies that there exists a real number a such that $0 < a < t_g$ and $h(a) = 0$. Thus, Z_h is nonempty and $\min(Z_h) < \min(Z_g)$. □

Using Proposition 3.1, we show the existence of a nilpotent realization of $\mathcal{K}_{n,r}$.

LEMMA 3.2. *For $n \geq 3$, $\mathcal{K}_{n,r}$ has a nilpotent realization.*

Proof. Let $A \in Q(\mathcal{K}_{n,r})$ be of the form (3.1). For convenience, we set $a_0 = 1$. From [1, p. 262], we deduce that the coefficients of the characteristic polynomial of A satisfy

$$\begin{array}{lcl} \alpha_1 &=& a_1 - 1, \\ \alpha_j &=& a_j - a_{j-1} \text{ for } j = 2, \ldots, r-1 \ (r \geq 3), \\ \alpha_j &=& a_j - a_{j-1} + b_r a_{j-r} \text{ for } j = r, \ldots, n-1 \ (r \geq 2), \text{ and} \\ \alpha_n &=& b_r a_{n-r} - a_{n-1}. \end{array} \tag{3.2}$$

Let $a_1 = \cdots = a_{r-1} = 1$, $b_r = t$. Then $\alpha_1 = \alpha_2 = \cdots = \alpha_{r-1} = 0$. Note that if $r = n$, then setting $t = 1$ gives a solution to (3.2), and hence $\mathcal{K}_{n,n}$ has a nilpotent realization.

Suppose $r < \frac{n}{2}$. In order to show that there exists a nilpotent realization of $\mathcal{K}_{n,r}$, it is sufficient to determine the existence of positive numbers $a_r, a_{r+1}, \ldots, a_{n-1}, t$ satisfying the following equations (obtained by setting α_j's to be zero for all $j = r, \ldots, n$):

$$\begin{array}{rcl} a_r(t) &=& a_{r-1}(t) - ta_0(t) = 1 - t \\ a_{r+1}(t) &=& a_r(t) - ta_1(t) = 1 - 2t \\ a_{r+2}(t) &=& a_{r+1}(t) - ta_2(t) = 1 - 3t \\ && \vdots \\ a_{2r-1}(t) &=& a_{2r-2}(t) - ta_{r-1}(t) = 1 - rt \\ a_{2r}(t) &=& a_{2r-1}(t) - ta_r(t) \\ a_{2r+1}(t) &=& a_{2r}(t) - ta_{r+1}(t) \\ && \vdots \\ a_{n-1}(t) &=& a_{n-2}(t) - ta_{n-1-r}(t) \\ 0 &=& a_{n-1}(t) - ta_{n-r}(t). \end{array} \tag{3.3}$$

For $j = r, r+1, \ldots, n-1$, the polynomials in (3.3) satisfy

$$a_j(t) = a_{j-1}(t) - ta_{j-r}(t) \tag{3.4}$$

Let $h(t) = a_{n-1}(t) - ta_{n-r}(t)$. It can be easily checked that $a_j(0) = 1$ for all $j = r, r+1, \ldots, n-1$, and $h(0) = 1$. Since $a_{r+i}(t) = 1 - (i+1)t$ for $i = 0, 1, \ldots, r-1$, the zero of $a_{r+i}(t)$ is $\dfrac{1}{i+1}$ for $i = 0, 1, \ldots, r-1$. Hence,

$$\min(Z_{a_{2r-1}}) < \min(Z_{a_{2r-2}}) < \cdots < \min(Z_{a_{r+1}}) < \min(Z_{a_r}). \tag{3.5}$$

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Since, by (3.5) $\min(Z_{a_{2r-1}}) < \min(Z_{a_r})$, and $a_{2r}(t) = a_{2r-1}(t) - ta_r(t)$, Proposition 3.1 implies that $\min(Z_{a_{2r}}) < \min(Z_{a_{2r-1}})$. Likewise, by repeatedly using Proposition 3.1, we have

$$\min(Z_h) < \min(Z_{a_{n-1}}) < \cdots < \min(Z_{a_{r+1}}) < \min(Z_{a_r}).$$

When $\frac{n}{2} \le r < n$,

$$a_{r+i}(t) = 1 - (i+1)t \text{ for } i = 0, 1, \ldots, n-r-1. \tag{3.6}$$

Hence, by Proposition 3.1, $\min(Z_h) < \min(Z_{a_{n-1}}) < \cdots < \min(Z_{a_{r+1}}) < \min(Z_{a_r})$.

Thus, if $\min(Z_h)$ is denoted by t_h, then $a_j(t_h) > 0$ for all $j = r, r+1, \ldots, n-1$. Therefore, there exists a nilpotent realization of $\mathcal{K}_{n,r}$ for $n \ge 3$ when $a_1 = \cdots = a_{r-1} = 1$, $b_r = t_h$, and $a_j = a_j(t_h)$ for $j = r, r+1, \ldots, n-1$. □

Throughout the remainder of this section $t_h = \min(Z_h)$, and a_j^0 denotes the positive numbers $a_j(t_h)$ for $j = r, r+1, \ldots, n-1$, and $a_j^0 = 1$ for $j = 0, 1, \ldots, r-1$, where $a_j(t)$'s are polynomials in either (3.3) or (3.6), and $h(t) = a_{n-1}(t) - ta_{n-r}(t)$.

While we made use of the characteristic polynomial associated with the patterns $\mathcal{V}_n^*(I)$ in [1], unlike for $\mathcal{V}_n^*(I)$, for $\mathcal{K}_{n,r}$ we do not always end up with a Jacobian matrix whose pattern requires a signed determinant. Thus we now develop some propositions to show that the Jacobian $\dfrac{\partial(\alpha_1, \ldots, \alpha_n)}{\partial(a_1, \ldots, a_{n-1}, b_r)}$ is nonzero at the nilpotent realization of $\mathcal{K}_{n,r}$, $(1, a_2^0, \ldots, a_{n-1}^0, t_h)$.

First, we consider the matrix

$$A_{k,r} = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 1 & 0 & & & & \vdots \\ \vdots & 0 & -1 & 1 & \ddots & & & \vdots \\ 0 & \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ t_h & 0 & & & \ddots & \ddots & 0 & 0 \\ 0 & t_h & \ddots & & & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & -1 & 1 \\ 0 & \cdots & 0 & t_h & 0 & \cdots & 0 & -1 \end{bmatrix}_{k\times k},$$

where t_h in the last row is in column $k-r+1$, and $k \ge 1$, $r \ge 2$. If $r > k$, $A_{k,r}$ does not have any t_h as an entry. In addition, $\det(A_{0,r})$ is defined to be 1. Now, we find an explicit form of the determinant of $A_{k,r}$.

Proposition 3.3. *If $2 \le r < n$ and $0 \le k < n$, then*

$$\det(A_{k,r}) = \begin{cases} (-1)^k a_k^0 & \text{when} \quad 2 \le r \le k, \\ (-1)^k & \text{when} \quad r > k. \end{cases}$$

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Proof. The proof is by induction on k. The cases for $k = 0, 1$ are clear. Suppose $k = 2$. If $r = k$, then

$$\det(A_{2,2}) = \det \begin{bmatrix} -1 & 1 \\ t_h & -1 \end{bmatrix} = 1 - t_h.$$

Since $r < n$, by (3.3) we have $1 - t_h = a_r^0$. Since $r = k = 2$, $1 - t_h = (-1)^k a_k^0$. If $r > 2 = k$, then

$$\det(A_{2,r}) = \det \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = (-1)^2.$$

Assume $k > 2$. For $2 < k \le n-1$ and $r > k$, $A_{k,r}$ is an upper triangular matrix each of whose diagonal entries is -1. Hence, $\det(A_{k,r}) = (-1)^k$.

Suppose $r \le k$. By cofactor expansion along the first column,

$$\det(A_{k,r}) = (-1)\det(A_{k-1,r}) + (-1)^{r+1} t_h \det(A_{k-r,r}).$$

If $k \ge 2r$, then $k - r \ge r$ and $k - 1 \ge r$. By induction and (3.4),

$$\begin{aligned} \det(A_{k,r}) &= (-1)(-1)^{k-1} a_{k-1}^0 + (-1)^{r+1} t_h (-1)^{k-r} a_{k-r}^0 \\ &= (-1)^k a_{k-1}^0 + (-1)^{k+1} t_h a_{k-r}^0 \\ &= (-1)^k (a_{k-1}^0 - t_h a_{k-r}^0) \\ &= (-1)^k a_k^0 \quad \text{since } k \ge 2r. \end{aligned}$$

Next, suppose $r + 1 \le k < 2r$. Since $k - 1 \ge r$ and $k - r < r$, by induction, and (3.4),

$$\begin{aligned} \det(A_{k,r}) &= (-1)(-1)^{k-1} a_{k-1}^0 + (-1)^{r+1} t_h (-1)^{k-r} \\ &= (-1)^k a_{k-1}^0 + (-1)^{k+1} t_h \\ &= (-1)^k (a_{k-1}^0 - t_h) \\ &= (-1)^k (a_{k-1}^0 - t_h a_{k-r}^0) \\ &= (-1)^k a_k^0. \end{aligned}$$

Lastly, suppose $r \le k < r + 1$, i.e. $k = r$. By induction and (3.3),

$$\begin{aligned} \det(A_{k,r}) &= (-1)(-1)^{k-1} + (-1)^{r+1} t_h (-1)^{k-r} \\ &= (-1)^k (1 - t_h) \\ &= (-1)^k a_k^0. \quad \square \end{aligned}$$

For $l \ge 1$, $r \ge 2$, and $c_j > 0$ $(j = 1, \dots, l)$, let

$$B_{l,r} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & c_1 \\ -1 & 1 & 0 & 0 & & & & \vdots & c_2 \\ 0 & -1 & 1 & 0 & & & & & \vdots \\ \vdots & 0 & -1 & 1 & \ddots & & & & \vdots \\ 0 & \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ t_h & 0 & & & & & \ddots & \vdots & \vdots \\ 0 & t_h & \ddots & & & & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & 1 & c_{l-1} \\ 0 & \cdots & 0 & t_h & 0 & \cdots & 0 & -1 & c_l \end{bmatrix}_{l\times l},$$

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where t_h in the last row is in column $l-r$. If $r \geq l$, $B_{l,r}$ does not have any t_h entry.

PROPOSITION 3.4. *If* $1 \leq l \leq n$ *and* $2 \leq r < n$, *then* $\det(B_{l,r}) > 0$.

Proof. The proof is by induction on l. The case for $l = 1$ is clear. For $l = 2$,

$$\det(B_{2,r}) = \det \begin{bmatrix} 1 & c_1 \\ -1 & c_2 \end{bmatrix} = c_2 + c_1 > 0.$$

Assume $l \geq 3$ and proceed by induction. By the cofactor expansion along the first row, the determinant of $B_{l,r}$ is

(3.7)

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & c_2 \\ -1 & 1 & 0 & 0 & & & & \vdots & c_3 \\ 0 & -1 & 1 & 0 & & & & & \vdots \\ \vdots & 0 & -1 & 1 & \ddots & & & & \vdots \\ 0 & \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ t_h & 0 & & & & & \ddots & \vdots & \vdots \\ 0 & t_h & \ddots & & & & \ddots & 0 & c_{l-2} \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & 1 & c_{l-1} \\ 0 & \cdots & 0 & t_h & 0 & \cdots & 0 & -1 & c_l \end{bmatrix} + (-1)^{l+1} c_1 \det(A_{l-1,r}).$$

By the induction hypothesis, the first term in (3.7) is positive. If $r > l-1$, Proposition 3.3 implies that the second term in (3.7) is $(-1)^{l+1}c_1(-1)^{l-1} = (-1)^{2l}c_1 > 0$. If $2 \leq r \leq l-1$, Proposition 3.3 implies that the second term in (3.7) is $(-1)^{l+1}c_1(-1)^{l-1}a^0_{l-1} = c_1 a^0_{l-1} > 0$. Hence, $\det(B_{l,r}) > 0$. □

LEMMA 3.5. *Let* $A \in Q(\mathcal{K}_{n,r})$ *be of the form* (3.1). *When* $2 \leq r < n$, *the Jacobian* $\dfrac{\partial(\alpha_1, \ldots, \alpha_n)}{\partial(a_1, \ldots, a_{n-1}, b_r)}$ *is positive at* $(1, a^0_2, \ldots, a^0_{n-1}, t_h)$.

Proof. By (3.2), the Jacobian matrix at $(1, a^0_2, \ldots, a^0_{n-1}, t_h)$ is the block lower

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triangular matrix

$$
J = \left[\begin{array}{cccc|cccccccc}
1 & 0 & \cdots & 0 & & & & & & & & \\
-1 & 1 & \ddots & \vdots & & & & & O & & & \\
 & \ddots & \ddots & 0 & & & & & & & & \\
0 & & -1 & 1 & & & & & & & & \\
\hline
0 & & & -1 & 1 & 0 & \cdots & \cdots & & \cdots & 0 & 1 \\
t_h & \ddots & & & -1 & 1 & 0 & & & & 0 & 1 \\
 & t_h & \ddots & & 0 & -1 & 1 & 0 & & & 0 & a_2^0 \\
 & & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots \\
 & & & t_h & 0 & & & \ddots & \ddots & \ddots & \vdots & \vdots \\
 & & & & t_h & \ddots & & & \ddots & \ddots & 0 & \vdots \\
 & & & & \vdots & \ddots & \ddots & & \ddots & -1 & 1 & a_{n-r-1}^0 \\
 & & & & 0 & \cdots & t_h & 0 & \cdots & 0 & -1 & a_{n-r}^0
\end{array}\right].
$$

Note that the $(1,1)$-block of J has order $r-1$ and its determinant is 1. Since the t_h in the last row of J is in column $n-r$, the $(2,2)$-block of J is of the form $B_{n-r+1,r}$. By Proposition 3.4, $\det(B_{n-r+1,r}) > 0$. Thus, the Jacobian at $(1, a_2^0, \ldots, a_{n-1}^0, t_h)$ is positive. ▯

THEOREM 3.6. *For $n > r \geq 2$, every superpattern of $\mathcal{K}_{n,r}$ is a SAP.*

Proof. Let $A \in Q(\mathcal{K}_{n,r})$ be of the form (3.1). By Lemma 3.2, when $a_1 = \cdots = a_{r-1} = 1$, $a_j = a_j^0$ for $j = r, r+1, \ldots, n-1$, and $b_r = t_h$, the resulting matrix is nilpotent. Moreover, by Lemma 3.5, the Jacobian $\dfrac{\partial(\alpha_1, \ldots, \alpha_n)}{\partial(a_1, \ldots, a_{n-1}, b_r)}$ is nonzero at $(1, a_2^0, \ldots, a_{n-1}^0, t_h)$. Thus, Theorem 2.1 implies that every superpattern of $\mathcal{K}_{n,r}$ is a SAP for $n \geq 3$. ▯

THEOREM 3.7. *If $n \geq r \geq 2$, then $\mathcal{K}_{n,r}$ is a MSAP.*

Proof. As already noted, $\mathcal{K}_{n,n}$ is a MSAP. So assume $n > r$.

We first note that any irreducible subpattern of $\mathcal{K}_{n,r}$ with less than $2n-1$ nonzero entries is not a SAP [1, Theorem 6.2].

For $i \in \{1, 2, \ldots, n-1\}$, let $\mathcal{S}_i$ be the sign pattern obtained by replacing the $(i, i+1)$-entry in $\mathcal{K}_{n,r}$ by 0. Then $\mathcal{S}_i$ is a 2 by 2 block lower triangular matrix. In particular, the sign of the trace of each of the diagonal blocks of $\mathcal{S}_i$ is fixed and so each block requires a nonzero eigenvalue. Consequently, no subpattern of $\mathcal{S}_i$ is spectrally arbitrary.

Likewise, if the $(n, n-r+1)$-entry, $(1,1)$-entry, or (n,n)-entry of $\mathcal{K}_{n,r}$ is replaced by 0, the resultant pattern will require a signed eigenvalue.

Next, let $A \in Q(\mathcal{K}_{n,r})$ be of the form (3.1). Then, by (3.2), $\alpha_1 = a_1 - 1$, $\alpha_j = a_j - a_{j-1}$ for $j = 2, \ldots, r-1$, $\alpha_r = a_r + b_r - a_{r-1}$, $\alpha_j = a_j + a_{j-r}b_r - a_{j-1}$ for $j = r+1, \ldots, n-1$, and $\alpha_n = a_{n-r}b_r - a_{n-1}$. Suppose that for some $j \in \{2, \ldots, n-1\}$,

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a_j is replaced by 0. Then α_{j+1} of the resulting matrix is positive. Thus no proper subpattern of $\mathcal{K}_{n,r}$ is spectrally arbitrary. $\square$

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