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ALGEBRAIC CONNECTIVITY OF TREES WITH A PENDANT EDGE OF INFINITE WEIGHT*

A. BERMAN† AND K.-H. FÖRSTER‡

Abstract. Let G be a weighted graph. Let v be a vertex of G and let G_ω^v denote the graph obtained by adding a vertex u and an edge $\{v,u\}$ with weight ω to G. Then the algebraic connectivity $\mu(G_\omega^v)$ of G_ω^v is a nondecreasing function of ω and is bounded by the algebraic connectivity $\mu(G)$ of G. The question of when $\lim\limits_{\omega\to\infty}\mu(G_\omega^v)$ is equal to $\mu(G)$ is considered and answered in the case that G is a tree.

Key words. Weighted graphs, Trees, Laplacian matrix, Algebraic connectivity, Pendant edge.

AMS subject classifications. 5C50, 5C40, 15A18.

1. Introduction. A *weighted graph on n vertices* is an undirected simple graph G on n vertices such that with each edge e of G, there is an associated positive number $\omega(e)$ which is called the *weight* of e.

The *Laplacian matrix* of a weighted graph G on n vertices is the $n\times n$ matrix $L(G)=L=(l_{ij})$, where for each $i,j=1,\dots,n$,

$$l_{ij}=\begin{cases} -\omega(e) & \text{if} \quad i\neq j \quad \text{and} \quad e=\{i,j\},\\ 0 & \text{if} \quad i\neq j \quad \text{and} \quad i \text{ is not adjacent to } j,\\ \sum_{k\neq i} l_{ik} & \text{if} \quad i=j. \end{cases}$$

Clearly L is a singular M-matrix and positive semidefinite, so $\lambda_1(L)=0$, where for a symmetric matrix A we arrange the eigenvalues in nondecreasing order

$$\lambda_1(A)\leq\lambda_2(A)\leq\dots$$

Fiedler [3] showed that $\lambda_2(L)$ is positive iff G is connected and called it the *algebraic connectivity* of G. The algebraic connectivity of G will be denoted by $\mu(G)$.

In this paper G always denotes a connected weighted graph without loops.

Let G be a graph with n vertices. Let v be a vertex of G and let G_ω^v be the graph with $n+1$ vertices obtained by adding to G a vertex u and an edge $e=\{v,u\}$ with weight ω.

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THEOREM 1.1. *The algebraic connectivity $\mu(G^v_\omega)$ is a nondecreasing function of ω and for every ω and $n > 1$*

$$\mu(G^v_\omega) \leq \mu(G).$$

Proof. Let L_ω be the Laplacian matrix of G^v_ω and let $0 < \omega_1 \leq \omega_2$. Then $B = L_{\omega_2} - L_{\omega_1}$ is a singular rank one positive semidefinite matrix. By [7, Th. 4.3.1]

$$\lambda_k(L_{\omega_1}) \leq \lambda_k(L_{\omega_2}) \quad \text{for } k = 1 \ldots n,$$
$$\text{and for } k = 2, \quad \mu(G^v_{\omega_1}) \leq \mu(G^v_{\omega_2}).$$

To show that $\mu(G^v_\omega)$ is bounded, write L_ω as the sum of two block matrices

$$L_\omega = \begin{bmatrix} L(G) & 0 \\ & \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \\ & & \\ 0 & \begin{matrix} \omega & -\omega \\ -\omega & \omega \end{matrix} \end{bmatrix},$$

where $L(G)$ is $n \times n$ and the left upper zero block in the second matrix is $(n-1)\times(n-1)$. By [7, Th. 4.3.4 (a), the case $k = 2$],

$$\begin{aligned} \mu(G^v_\omega) &= \lambda_2(L_\omega) \leq \lambda_3(L(G) \oplus (0)) \\ &= \lambda_2(L(G)) = \mu(G). \qquad \square \end{aligned}$$

REMARK 1.2. The theorem is essentially a consequence of Cor. 4.2 of [6]. It is proved for trees in [8].

EXAMPLE 1.3. For the complete graphs $K_n, n > 1$, with all weights equal to 1

$$\lim_{\omega\to\infty} \mu((K_n)^v_\omega) = \frac{n+1}{2} < n = \mu(K_n).$$

EXAMPLE 1.4. For the cycles $C_n, n > 2$, with weights equal to 1

$$\lim_{\omega\to\infty} \mu((C_n)^v_\omega) = \mu(C_{n+1}) < \mu(C_n).$$

EXAMPLE 1.5. Let G be the graph obtained from K_4 by deleting an edge and let all the weights of G be equal to 1. If the degree of v is 3,

$$\lim_{\omega\to\infty} \mu(G^v_\omega) = 2 = \mu(G).$$

If the degree of v is 2

$$\lim_{\omega\to\infty} \mu(G^v_\omega) < \mu(G).$$

Since $\mu(G^v_\omega)$ is bounded by $\mu(G)$, it is natural to ask when does

$$\lim_{\omega\to\infty} \mu(G^v_\omega) = \mu(G).$$

We answer this question in Section 3, in the case that G is a tree. The needed background on the algebraic connectivity of trees is described in Section 2.

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2. Results on trees. Our paper relies heavily on the work of [12] so in this section we describe their main results and basic background on trees needed for these results and for the next section. In some cases we change the notation of [12].

THEOREM 2.1. [4, Th. 3.11] *Let T be a weighted tree with Laplacian matrix L and algebraic connectivity μ. Let y be an eigenvector of L associated with μ. Then exactly one of the following two cases occur:*

(a) *Some entry of y is 0.*

(b) *All entries of y are nonzero.*

In the first case there exists a unique vertex c such that $y_c = 0$ and c is adjacent to a vertex d with $y_d \neq 0$. In the second case there is a unique pair of vertices i and j adjacent in T such that $y_i y_j < 0$.

DEFINITION 2.2. A weighted tree T is said to be of type I with a characteristic vertex c if case (a) of Theorem 2.1 holds, and of type II with characteristic vertices i and j in case (b). We use also the notation I_c in the first case and $II_{i,j}$ in the second case.

The name characteristic vertices was coined in [11] by R. Merris who showed that if μ is not a simple eigenvalue, then all the corresponding eigenvectors yield the same type of tree and the same characteristic vertices.

DEFINITION 2.3. Let v be a vertex of a tree T. Let L_v be the matrix obtained by deleting the row and column of the Laplacian matrix of T that correspond to v. The matrix $M_{v,T} := L_v^{-1}$ is called the bottleneck matrix of T at v.

In [9] and [10], it is shown that the entry of $M_{v,T}$ that corresponds to the vertices k and l is

$$m_{kl} = \sum \quad \frac{1}{\omega(g)}$$

where the summation is on all edges g that lie on the intersection of the path between k and v and the path between l and v. The matrix $M_{v,T}$ is permutationally similar to a block diagonal matrix, where the number of blocks is the degree of v and each block is a positive matrix which corresponds to a unique branch at v.

For vertices u, v of a tree T let $v \to u$ denote the branch of T at v, that contains u. We denote by $M_{v \to u,T}$ the block of $M_{v,T}$ that corresponds to $v \to u$, and by $M_{v \not\to u,T}$ the matrix obtained from $M_{v,T}$ by deleting the rows and the columns corresponding to $M_{v\to u,T}$.

DEFINITION 2.4. A diagonal block of $M_{v,T}$ whose spectral radius is equal to $\rho(M_{v,T})$, where $\rho(A)$ denotes the spectral radius of the matrix A, is called a Perron block and the corresponding branch of T at v is called a Perron branch.

THEOREM 2.5. [9, Cor. 2.1] *Let T be a weighted tree. Then T is of type I with a characteristic vertex c, if and only if at c, T has more than one Perron branch.*

In this case, $\mu(T)$, the algebraic connectivity of T is equal to $\frac{1}{\rho(M_{c,T})}$.

Let e be an edge of a graph G. Replace the weight at e by ω and denote the resulting graph by G^e_ω. Observe that since $e = \{v, u\}$ is a pendant edge of G^v_ω, then $(G^v_\omega)^e_\omega = G^v_\omega$. Let G^e_∞ denote the family of weighted graphs $\{G^e_\omega, \omega > 0\}$, and let G^v_∞ denote the family of weighted graphs $\{G^v_\omega, \omega > 0\}$.

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THEOREM 2.6. [12, Corollary 1.1] *Let T be a weighted tree and let e be an edge of T. Then there exists a positive number ω_0 such that all the trees T^e_ω, $\omega_0 < \omega < \infty$, are of the same type and have the same characteristic vertices.*

The following definitions are used in [12].

DEFINITION 2.7. The family of trees T^e_∞ is a type I tree at infinity with characteristic vertex c if there exists an $\omega_0 > 0$ such that for all $\omega \in [\omega_0, \infty)$, T^e_ω is of type I_c. Similarly, T^e_∞ is a type II tree at infinity with characteristic vertices i and j if there exists an $\omega_0 > 0$ such that for all $\omega \in [\omega_0, \infty)$, T^e_ω is of type $II_{i,j}$.

We now can state the main result of [12].

THEOREM 2.8. [12, Th.1.8] *Let $e = \{v, u\}$ be an edge that is not a pendant edge of a tree T. Let T_1 and T_2 be the resulting components arising from the deletion of e. Suppose $v \in T_1$, $u \in T_2$ and $\mu(T_1) \le \mu(T_2)$. Then $\lim_{\omega\to\infty} \mu(T^e_\omega) = \mu(T_1)$ iff T_1 is a tree of type I with a characteristic vertex, say, c, and one of the following conditions holds:*

(a) *T^e_∞ is of type I with a characteristic vertex c.*

(b) *c is incident to e and $\rho(M_{u,T_2}) \le \frac{1}{\mu(T_1)}$.*

We conclude the background results with the analogue of Theorem 2.5 for type II trees and two propositions that will be used in proving the main result.

THEOREM 2.9. [9, Th.1] *A weighted tree T is of type II iff at every vertex T has a unique Perron branch. If the characteristic vertices, i and j, of T are joined by an edge of weight θ, then there exists a number $0 < \gamma < 1$, such that*

$$\rho(M_{i\to j,T} - \frac{\gamma}{\theta} J) = \rho(M_{j\to i,T} - \frac{1-\gamma}{\theta} J), and$$

$$\mu(T) = \frac{1}{\rho(M_{i\to j,T} - \frac{\gamma}{\theta} J)} = \frac{1}{\rho(M_{j\to i,T} - \frac{1-\gamma}{\theta} J)},$$

where J denotes an all ones matrix.

PROPOSITION 2.10. [8, Cor. 1.1] *The characteristic vertices of T^v_ω lie on the path between the characteristic vertices of T and u.*

PROPOSITION 2.11. [12, Claim 3.2] *Let T be a tree. Let $\{i_k, j_k\}$ be edges in T with weights α_k, for $k = 1, 2$, such that the path from i_1 to j_2 contains j_1 and i_2, and let $0 < \gamma_1, \gamma_2 < 1$. Then*

$$\rho(M_{j_1\to i_1,T} - \frac{\gamma_1}{\alpha_1} J) < \rho(M_{j_1\to i_1,T}) < \rho(M_{j_2\to i_2,T} - \frac{\gamma_2}{\alpha_2} J).$$

3. Assigning an arbitrarily large weight to a pendant edge of a tree.

In this section we consider the case where T is a tree and u is a pendant vertex of $T \cup e$ where $e = \{v, u\}$ and $v \in T$.

In some sense the question in this case may be considered as a special case of the discussion in [12]. To do this, $\{u\}$ is to be considered as a "tree with algebraic connectivity ∞" and the spectral radius of an empty matrix has to be defined (for example as 0).

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Our discussion is based on the analysis of the limits of the bottleneck matrices of T_ω^v when ω increases to ∞; namely

$$M_{v,T_\omega^v} = M_{v,T} \oplus \left(\frac{1}{\omega}\right),$$

$$M_{u,T_\omega^v} = (M_{v,T} \oplus (0)) + \frac{1}{\omega} J$$

and if $s \neq v$ is a vertex of T,

$$M_{s,T_\omega^v} = \begin{pmatrix} M_{s,T} & M^{(v)} \\ M^{(v)t} & m_{vv} + \frac{1}{\omega} \end{pmatrix},$$

where $M^{(v)}$ is the column of $M_{s,T}$ corresponding to v and m_{vv} is the diagonal entry of $M_{s,T}$, corresponding to v. In particular, for all the branches of T at s that do not contain v, the diagonal blocks of M_{s,T_ω^v} and of $M_{s,T}$ are the same. Denoting $\lim\limits_{\omega\to\infty} M_{s,T_\omega^v}$ by M_{s,T_∞^v} we see that for $s \notin e$

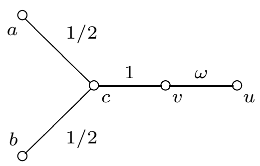
$$M_{s,T_\infty^v} = \begin{pmatrix} M_{s,T} & M^{(v)} \\ M^{(v)t} & m_{vv} \end{pmatrix} \tag{3.1}$$

and

$$M_{v,T_\infty^v} = M_{u,T_\infty^v} = M_{v,T} \oplus (0). \tag{3.2}$$

The reader should not be confused by the fact that T_∞^v denotes a family of trees while M_{s,T_∞^v} denotes a single matrix (up to permutation similarity).

EXAMPLE 3.1.



$$M_{v,T_\infty^v} = M_{u,T_\infty^v} = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{c,T_\infty^v} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

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$$M_{a,T^v_\infty} = M_{b,T^v_\infty} = \begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \end{pmatrix}.$$

REMARK 3.2. The matrices M_{s,T^v_∞} are of course singular, but they do contain information on $\lim_{\omega\to\infty} \mu(T^v_\omega)$.

As in the case of nonsingular bottleneck matrices we call the diagonal blocks of M_{s,T^v_∞} whose spectral radius is maximal, Perron blocks. The corresponding branches of T^v_ω do not depend on ω; they will be called the Perron branches of the family T^v_∞.

LEMMA 3.3. *If M_{s,T^v_∞} has more than one Perron block, then*

$$\lim_{\omega\to\infty} \mu(T^v_\omega) = \frac{1}{\rho(M_{s,T^v_\infty})}.$$

Proof. Consider the principal submatrix L_{s,T^v_ω} obtained from the Laplacian matrix of T^v_ω by deleting the row and column corresponding to s. Then

$$M_{s,T^v_\infty} = \lim_{\omega\to\infty} (L_{s,T^v_\omega})^{-1}.$$

By [7, Th. 4.3.15] for $r = n-1, k = 1$ and $k = 2$,

$$\lambda_1(L_{s,T^v_\omega}) \le \mu(T^v_\omega) \le \lambda_2(L_{s,T^v_\omega}),$$

so

$$\lim_{\omega\to\infty} \lambda_1(L_{s,T^v_\omega}) \le \lim_{\omega\to\infty} \mu(T^v_\omega) \le \lim_{\omega\to\infty} \lambda_2(L_{s,T^v_\omega}),$$

Since M_{s,T^v_∞} has at least two Perron blocks we obtain

$$\lim_{\omega\to\infty} \lambda_2(L_{s,T^v_\omega}) = \lim_{\omega\to\infty} \lambda_1(L_{s,T^v_\omega}) = \rho(M_{s,T^v_\infty}). \qquad \square$$

REMARK 3.4. If there exists an ω_0 such that two of the Perron blocks of M_{s,T^v_∞} are Perron blocks of M_{s,T^v_ω} for $\omega \ge \omega_0$ then

$$\lambda_2(L_{s,T^v_\omega}) = \lambda_1(L_{s,T^v_\omega}) \textit{ for } \omega \ge \omega_0.$$

LEMMA 3.5. *Let s be a vertex of T. Suppose M_{s,T^v_∞} has at least two Perron blocks and let t be another vertex of T. Then*

$$\rho(M_{t,T^v_\infty}) > \rho(M_{s,T^v_\infty})$$

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Proof. By assumption, the family T_∞^v has at least two Perron branches at s, so one of them, say $s \to x$, does not contain t. Let $t \to s$ be the branch at t that contains s. Then it contains the branch $s \to x$, and we obtain

$$\rho(M_{t,T_\infty^v}) \geq \rho(M_{t\to s,T_\infty^v}) > \rho(M_{s\to x,T_\infty^v}) = \rho(M_{s,T_\infty^v}),$$

where the strict inequality follows from [1, Cor. 2.1.5] and the fact that $M_{s\to x,T_\infty^v}$ is a submatrix of $M_{t\to s,T_\infty^v}$, which is positive. □

COROLLARY 3.6. *There is at most one vertex, say c, such that M_{c,T_∞^v} has more than one Perron block.*

DEFINITION 3.7. In the case that there is a vertex c such that M_{c,T_∞^v} has more than one Perron block, we will say that the family of trees T_∞^v is a trii (tree in infinity) of type Ic. If no such c exists we say that T_∞^v is a trii of type II.

REMARK 3.8.

(a) If the trees T_ω^v are of type I_c for all sufficiently large ω, then the family T_∞^v is a trii of type I_c (and also a type I tree at infinity with characteristic vertex c). In other words, if T_∞^v is a trii of type II, then for all ω large enough, T_ω^v are trees of type II.

(b) Suppose the trees T_ω^v are of type $\mathrm{II}_{p,q}$ for all sufficiently large ω, then by the representation of L_ω in the proof of Theorem 1.1 and by Theorem 2.1, $\{p, q\}$ cannot be the pendant edge $\{v, u\}$.

(c) It is possible that T_ω^v are of type II for all sufficiently large ω (so T_∞^v is a type II tree at infinity) but T_ω^v is a trii of type I; see Lemma 3.10 and Subcase 4 of Example 3.13 in the following discussion.

REMARK 3.9. The proof of Lemma 3.5 shows that if T is a tree of type I with a characteristic vertex c, then for any other vertex s of T

$$\rho(M_{s,T}) > \rho(M_{c,T}).$$

(This has already been established in Proposition 2 of [9].)

Consider Theorem 2.9 where T_ω^v is of type $\mathrm{II}_{i,j}$ and the weight of the edge $\{i, j\}$ is θ. Then for every ω (sufficiently large) there exist a number γ_ω, between 0 and 1, such that

$$\mu(T_\omega^v) = \frac{1}{\rho(M_{i\to j,T_\omega^v} - \frac{\gamma_\omega}{\theta}J)} = \frac{1}{\rho(M_{j\to i,T_\omega^v} - \frac{1-\gamma_\omega}{\theta}J)}.$$

What happens to the the number γ_ω when ω goes to ∞? We claim that $\lim_{\omega\to\infty}\gamma_\omega$ exists. Indeed, one of the branches corresponding to $M_{i\to j,T_\omega^v}$ and $M_{j\to i,T_\omega^v}$ does not contain u. Suppose it is the second, so $M_{j\to i,T_\omega^v} = M_{j\to i,T}$. The numbers $\mu(T_\omega^v)$ increase to a limit, see Theorem 1.1, so the numbers $\rho(M_{j\to i,T_\omega^v} - \frac{1-\gamma_\omega}{\theta}J)$ decrease to a limit , which means that the numbers 1-γ_ω increase to a limit. This limit is at most 1 since $0 < \gamma_\omega < 1$.

LEMMA 3.10. *If the trees T_ω^v are of type II, with characteristic vertices i, j for $\omega_0 < \omega < \infty$, and if $\gamma = \lim_{\omega\to\infty}\gamma_\omega = 0$, where $\rho(M_{i\to j,T_\omega^v} - \frac{\gamma_\omega}{\theta}J) = \rho(M_{j\to i,T_\omega^v} - \frac{1-\gamma_\omega}{\theta}J)$*

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and θ is the weight of the edge $\{i,j\}$, then T_∞^v is a trii of type I_i. Similarly, if $\gamma = 1$ then T_∞^v is a trii of type I_j.

Proof. $M_{j\to i,T_\infty^v} = (M_{i\nrightarrow j,T_\infty^v} \oplus (0)) + \frac{1}{\theta}J$ so if $\gamma = 0$, then $\rho(M_{i\to j,T_\infty^v}) = \rho(M_{j\to i,T_\infty^v} - \frac{1}{\theta}J) = \rho(M_{i\nrightarrow j,T_\infty^v})$ so T_∞^e is a trii of type I_i. ◻

COROLLARY 3.11. *If the trees T_ω^v are of type II and if T_∞^v is a trii of type II, then $0 < \gamma = \lim\limits_{\omega\to\infty} \gamma_\omega < 1$.*

REMARK 3.12. The tree T can be a tree of type I with a characteristic vertex, say c, or a tree of type II. In the first case there are 3 possibilities:

1 T_∞^v is a trii of type I_c,
2 T_∞^v is a trii of type I_s, where $s \neq c$,
3 T_∞^v is a trii of type II.

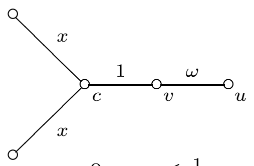
In the second case there are two possibilities:

4 T_∞^v is a trii of type I_s for some s,
5 T_∞^v is a trii of type II.

The following example demonstrates that all five subcases are possible.

EXAMPLE 3.13.

Subcase 1



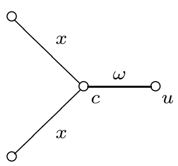
$$0 < x \le \tfrac{1}{2}.$$

Here

$$M_{c,T_\omega^v} = \begin{pmatrix} 1/x & 0 & 0 & 0 \\ 0 & 1/x & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1+\frac{1}{\omega} \end{pmatrix},$$

so for $x < \frac{1}{2}$, T_∞^v is a tree of type I with a characteristic vertex c and for $x = \frac{1}{2,}$ it is only a trii of type I_c.

Another example is when $c = v$



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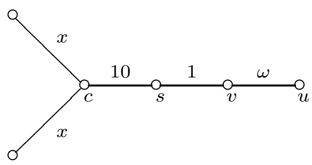
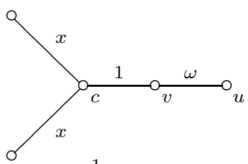
Subcase 2



where

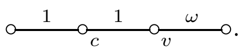
$$\rho\left[\begin{pmatrix} 1/x+0.1 & 0.1 & 0.1 \\ 0.1 & 1/x+0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 \end{pmatrix}\right]=2.$$

Subcase 3

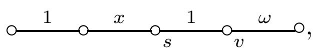


$$\tfrac{1}{2}<x\le 1,$$

or

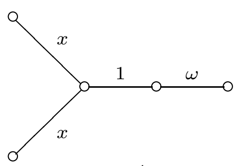


Subcase 4

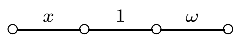


$$\rho\left[\begin{pmatrix} 1+1/x & 1/x \\ 1/x & 1/x \end{pmatrix}\right]=2.$$

Subcase 5
Here we suggest 3 examples:



$$x>1,$$



$$x\notin\{1/2,1\},$$

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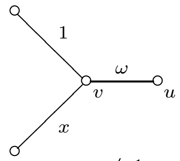

$x \neq 1.$

We are now ready to state and prove the main result.

THEOREM 3.14. *Let T be a tree. Then*

$$\lim_{\omega\to\infty} \mu(T^v_\omega) = \mu(T) \tag{3.3}$$

if and only if
(a) *T is a tree of type I with characteristic vertex say c,*
and
(b) $\rho(M_{c\to u,T^v_\infty}) \leq \rho(M_{c,T}) = \frac{1}{\mu(T)}$.

Proof. We prove the theorem by considering the five subcases of Remark 3.12, and showing that (3), (a) and (b) hold in Subcase 1 and only in this case, i.e. if and only if T and T^v_∞ are of type I_c for some vertex c.

Subcase 1: Obviously (a) holds. From (1) and (2) follows that $\rho(M_{c,T^v_\omega}) \geq \rho(M_{c,T})$. But if T and T^v_ω are of type I_c, then equality holds. Thus

$$\rho(M_{c\to u,T^v_\infty}) \leq \rho(M_{c,T}),$$

proving (b), and

$$\mu(T) = \frac{1}{\rho(M_{c,T})} = \frac{1}{\rho(M_{c,T^v_\omega})} = \lim_{\omega\to\infty} \mu(T^v_\omega),$$

proving (3). This completes the proof in Subcase 1.

If T is a tree of type I_c and (b) holds, then it follows easily that T^v_ω is a trii of type Ic. Therefore (b) does not hold in Subcases 2 and 3, while (a) obviously does not hold in Subcases 4 and 5. Now we will prove that (3) does not hold in the last four subcases.

Subcase 2: We have to show that (3) does not hold. Indeed

$$\begin{aligned}
\mu(T) = \frac{1}{\rho(M_{c,T})} &> \frac{1}{\rho(M_{s,T})} \\
&= \frac{1}{\rho(M_{s,T^v_\omega})}\text{, by Lemma 3.5} \\
&= \lim_{\omega\to\infty} \mu(T^v_\omega)\text{, by Lemma 3.3.}
\end{aligned}$$

Subcase 3: By Remark 3.8(a) the trees T^v_ω are for sufficiently large ω of type II, say of type $II_{p,q}$, see Theorem 2.6, and the edge $\{p, q\}$ of T has weight θ, by Remark

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3.8(b) it does not depend on ω. Without loss of generality, p lies on the path between q and c.

By Proposition 2.10 the vertices p and q lie on the path between c and u. Let i be a neighbor of c such that c, p and q lie on the path between i and u and $c \to i$ is a Perron branch of T. Then we obtain

$$\begin{aligned}
\lim_{\omega\to\infty} \mu(T^v_\omega) &= \lim_{\omega\to\infty} \frac{1}{\rho(M_{q\to p,T^v_\omega} - \frac{\gamma_\omega}{\theta}J)}, \text{ by Theorem 2.9,}\\
&= \lim_{\omega\to\infty} \frac{1}{\rho(M_{q\to p,T} - \frac{\gamma_\omega}{\theta}J)}, \text{ since } q\to p \text{ is in } T,\\
&= \frac{1}{\rho(M_{q\to p,T} - \frac{\gamma}{\theta}J)}, \text{ where } 0<\gamma<1, \text{ by Corollary 3.11,}\\
&< \frac{1}{\rho(M_{c\to i,T})}, \text{ by Proposition 2.11,}\\
&= \frac{1}{\rho(M_{c,T})}, \text{ since } c\to i \text{ is a Perron branch of } T,\\
&= \mu(T), \text{ since } T \text{ is of type } I_c,
\end{aligned}$$

so (3) does not hold.

Subcase 4: Here again we have to show that (3) does not hold. Suppose T is of type II_{ij}, where j lies on the path from i to u. Let θ and γ be as in Theorem 2.9. Since T^v_ω is a trii of type I_s, the by Proposition 2.10, s lies on the path from i to u. Therefore

$$\begin{aligned}
\lim_{\omega\to\infty} \mu(T^v_\omega) &= \lim_{\omega\to\infty} \frac{1}{\rho(M_{s\to i,T^v_\omega})} = \lim_{\omega\to\infty} \frac{1}{\rho(M_{s\to i,T})}\\
&\le \frac{1}{\rho(M_{j\to i,T})} < \frac{1}{\rho(M_{j\to i,T} - \frac{\gamma}{\theta}J)} = \mu(T).
\end{aligned}$$

Subcase 5: Here T is of, say, type II_{ij} and for ω large enough, T^v_ω are of, say, type II_{pq}, where by Proposition 2.10, we may take, without loss of generality, p and q to lie between i and q. Let θ and γ be as in Theorem 2.9 for the edge $\{i,j\}$ in T and let $\hat\theta$ and γ_ω be the corresponding pair for the edge $\{p,q\}$ in T^v_ω. Observe that $\hat\theta$ does not depend on ω by Remark 3.8(b). Let $\hat\gamma = \lim_{\omega\to\infty}\gamma_\omega$. By Corollary 3.13 we have $0<\gamma<1$. Now

$$\begin{aligned}
\lim_{\omega\to\infty} \mu(T^v_\omega) &= \lim_{\omega\to\infty} \frac{1}{\rho(M_{q\to p,T^v_\omega} - \frac{\gamma_\omega}{\hat\theta}J)} = \lim_{\omega\to\infty} \frac{1}{\rho(M_{q\to p,T} - \frac{\gamma_\omega}{\hat\theta}J)}\\
&= \frac{1}{\rho(M_{q\to p,T} - \frac{\hat\gamma}{\hat\theta}J)} < \frac{1}{\rho(M_{j\to i,T} - \frac{\gamma}{\theta}J)} = \mu(T),
\end{aligned}$$

where the inequality follows from Proposition 2.11. $\square$

Acknowledgments. We are indebted to Mr. Felix Goldberg [5] for suggesting Example 1.5 which shows that G does not have to be a tree for $\lim\limits_{\omega\to\infty}\mu(G^v_\omega) = \mu(G)$

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to hold. The question of when does $\lim_{\omega\to\infty} \mu(G_\omega^v) = \mu(G)$, when G is a general graph, seems to be much more difficult than the one in the case that G is a tree. We are grateful to the referee for his or her important remarks and for suggesting that Propositions 1.3 and 1.4, as well as Lemma 2.2 of [2], may be useful in dealing with the general case.

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