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# GIRTH AND SUBDOMINANT EIGENVALUES FOR STOCHASTIC MATRICES*

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**Abstract.** The set $\mathcal{S}(g, n)$ of all stochastic matrices of order $n$ whose directed graph has girth $g$ is considered. For any $g$ and $n$, a lower bound is provided on the modulus of a subdominant eigenvalue of such a matrix in terms of $g$ and $n$, and for the cases $g = 1, 2, 3$ the minimum possible modulus of a subdominant eigenvalue for a matrix in $\mathcal{S}(g, n)$ is computed. A class of examples for the case $g = 4$ is investigated, and it is shown that if $g > 2n/3$ and $n \geq 27$, then for every matrix in $\mathcal{S}(g, n)$, the modulus of the subdominant eigenvalue is at least $(\frac{1}{5})^{1/(2\lfloor n/3 \rfloor)}$.

**Key words.** Stochastic matrix, Markov chain, Directed graph, Girth, Subdominant eigenvalue.

**AMS subject classifications.** 15A18, 15A42, 15A51.

**1. Introduction and preliminaries.** Suppose that $T$ is an irreducible stochastic matrix. It is well known that the spectral radius of $T$ is 1, and that in fact 1 is an eigenvalue of $T$ (with the all ones vector $\mathbf{1}$ as a corresponding eigenvector). Indeed, denoting the directed graph of $T$ by $D$ (see [2]), Perron-Frobenius theory (see [8]) gives more information on the spectrum of $T$, namely that the number of eigenvalues having modulus 1 coincides with the greatest common divisor of the cycle lengths in $D$. In particular, if that greatest common divisor is 1, it follows that the powers of $T$ converge. (This in turn leads to a convergence result for the iterates of a Markov chain with transition matrix $T$.) Denoting the eigenvalues of $T$ by $1 = \lambda_1(T) \geq |\lambda_2(T)| \geq \ldots \geq |\lambda_n(T)|$ (throughout we will use this convention in labeling the eigenvalues of a stochastic matrix), it is not difficult to see that the asymptotic rate of convergence of the powers of $T$ is governed by $|\lambda_2(T)|$. We refer to $\lambda_2(T)$ as a *subdominant eigenvalue* of $T$.

In light of these observations, it is natural to wonder whether stronger hypotheses on the directed graph $D$ will yield further information on the subdominant eigenvalue(s) of $T$. This sort of question was addressed in [6], where it was shown that if $T$ is a primitive stochastic matrix of order $n$ whose exponent (i.e. the smallest $k \in \mathbb{N}$ so that $T^k$ has all positive entries) is at least $\lfloor \frac{n^2-2n+2}{2} \rfloor + 2$, then $T$ has at least $2\lfloor (n-4)/4 \rfloor$ eigenvalues with moduli exceeding $\left(\frac{1}{2}\sin[\pi/(n-1)]\right)^{2/(n-1)}$. Thus a hypothesis on the directed graph $D$ can lead to information about the eigenvalues of $T$.

In this paper, we consider the influence of the *girth* of $D$ - that is, the length of the shortest cycle in $D$ - on the modulus of the subdominant eigenvalue(s) of $T$. (It is straightforward to see that the girth of $D$ is the smallest $k \in \mathbb{N}$ such that $trace(T^k) > 0$.) Specifically, let $\mathcal{S}(g, n)$ be the set of $n \times n$ stochastic matrices having

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digraphs with girth $g$. If $T \in \mathcal{S}(g,n)$, how large can $|\lambda_2(T)|$ be? How small can $|\lambda_2(T)|$ be?

We note that the former question is readily dealt with. If $g \geq 2$, consider the directed graph $G$ on $n$ vertices that consists of a single $g-$cycle, say on vertices $1,\ldots,g$, along with a directed path $n \to n-1 \to \ldots \to g+1 \to 1$. Letting $A$ be the $(0,1)$ adjacency matrix of $G$, it is straightforward to determine that $A \in \mathcal{S}(g,n)$, and that the eigenvalues of $A$ consist of the $g-$th roots of unity, along with the eigenvalue 0 of algebraic multiplicity $n-g$. In particular, $|\lambda_2(A)| = 1$, so we find that $\max\{|\lambda_2(T)||T \in \mathcal{S}(g,n)\} = 1$. Similarly, for the case $g = 1$, we note that the identity matrix of order $n$, $I_n$, is an element of $\mathcal{S}(1,n)$, and again we have $\max\{|\lambda_2(T)||T \in \mathcal{S}(1,n)\} = 1$.

The bulk of this paper is devoted to a discussion of how small $|\lambda_2(T)|$ can be if $T \in \mathcal{S}(g,n)$ (and hence, of how quickly the powers of $T$ can converge). To that end, we let $\lambda_2(g,n)$ be given by $\lambda_2(g,n) = inf\{|\lambda_2(T)||T \in \mathcal{S}(g,n)\}$.

Remark 1.1. We begin by discussing the case that $g = 1$. Let $J$ denote the $n \times n$ all ones matrix, and observe that for any $n \geq 2$, the $n \times n$ matrix $\frac{1}{n}J$ has the eigenvalues 1 and 0, the latter with algebraic and geometric multiplicity $n-1$. It follows immediately that that $\lambda_2(1,n) = 0$.

Indeed there are many stochastic matrices yielding this minimum value for $\lambda_2$, of all possible admissible Jordan forms. To see this fact, let $M$ be any nilpotent Jordan matrix of order $n-1$. Let $v_1,\ldots,v_{n-1}$ be an orthonormal basis of the orthogonal complement of $\mathbf{1}$ in $\mathbb{R}^n$, and let $V$ be the $n \times (n-1)$ matrix whose columns are $v_1,\ldots,v_{n-1}$. We find readily that for all sufficiently small $\epsilon > 0$, the matrix $T = \frac{1}{n}J + \epsilon VMV^T$ is stochastic; further, the Jordan form for $T$ is given by $[1] \oplus M$, so that the Jordan structure of $T$ corresponding to the eigenvalue 0 coincides with that of $M$. Evidently for such a matrix $T$, the powers of $T$ converge in a finite number of iterations; in fact that number of iterations coincides with the size of the largest Jordan block of $M$.

The following elementary result provides a lower bound on $\lambda_2(g,n)$ for $g \geq 2$.

Theorem 1.1. *Suppose that $g \geq 2$ and that $T \in \mathcal{S}(g,n)$. Then $|\lambda_2(T)| \geq 1/(n-1)^{\frac{1}{(g-1)}}$. Equality holds if and only if $g = 2$ and the eigenvalues of $T$ are 1 (with algebraic multiplicity 1) and $\frac{-1}{n-1}$ (with algebraic multiplicity $n-1$). In particular,*

$$\lambda_2(g,n) \geq 1/(n-1)^{\frac{1}{(g-1)}}. \tag{1.1}$$

*Proof.* Let the eigenvalues of $T$ be $1,\lambda_2,\ldots,\lambda_n$. Since $trace(T^{g-1}) = 0$, we find that $\sum_{i=2}^n \lambda_i^{g-1} = -1$. Hence, $(n-1)|\lambda_2|^{g-1} \geq \sum_{i=2}^n |\lambda_i|^{g-1} \geq |\sum_{i=2}^n \lambda_i^{g-1}| = 1$. The inequality on $|\lambda_2|$ now follows readily.

Now suppose that $|\lambda_2| = 1/(n-1)^{\frac{1}{(g-1)}}$. Inspecting the proof above, we find that $|\lambda_i| = |\lambda_2|, i = 3,\ldots,n$, and that since equality holds in the triangle inequality, it must be the case that each of $\lambda_2,\ldots,\lambda_n$ has the same complex argument. Thus $\lambda_2 = \lambda_i$ for each $i = 3,\ldots,n$. Since $trace(T) = 0$, we deduce that $\lambda_2 = -1/(n-1)$; but then $trace(T^2) = n/(n-1) > 0$, so that $g = 2$. The converse is straightforward. ∎

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REMARK 1.2. If $T \in \mathcal{S}(2,n)$ and $|\lambda_2(T)| = 1/(n-1)$, it is straightforward to see that the matrix $S = \frac{n-1}{n}T + \frac{1}{n}I_n$ has just two eigenvalues, 1 and 0, the latter with algebraic multiplicity $n-1$. In particular, $S$ is a matrix in $\mathcal{S}(1,n)$ such that $\lambda_2(S) = \lambda_2(1,n) = 0$.

REMARK 1.3. From Theorem 1.1, we see that if $\exists\, c > 0$ such that $g \geq cn$, then necessarily $\lambda_2(g,n) \geq 1/(n-1)^{\frac{1}{(cn-1)}}$. An application of l'Hospital's rule shows that $1/(n-1)^{\frac{1}{(cn-1)}} \to 1$ as $n \to \infty$. Consequently, we find that for each $c > 0$, and any $\epsilon > 0$, there is a number $N$ such that if $n > N$ and $g \geq cn$, then each matrix $T \in \mathcal{S}(g,n)$ has $|\lambda_2(T)| \geq 1-\epsilon$.

We close this section with a discussion of $\lambda_2(g,n)$ as a function of $g$ and $n$.

PROPOSITION 1.2. *Fix $g$ and $n$ with $2 \leq g \leq n-1$. Then*
*a) $\lambda_2(g,n) \geq \lambda_2(g,n+1)$, and*
*b) $\lambda_2(g+1,n) \geq \lambda_2(g,n)$.*

*Proof.* a) Suppose that $T \in \mathcal{S}(g,n)$, and partition off the last row and column of $T$, say $T = \left[\begin{array}{c|c} T_1 & x \\ \hline y^T & 0 \end{array}\right]$. Now let $S$ be the stochastic matrix of order $n+1$ given by $S = \left[\begin{array}{c|c|c} T_1 & \frac{1}{2}x & \frac{1}{2}x \\ \hline y^T & 0 & 0 \\ \hline y^T & 0 & 0 \end{array}\right]$. Note that the digraph of $S$ is formed from that of $T$ by adding the vertex $n+1$, along with the arcs $i \to n+1$ for each $i$ such that $i \to n$ in the digraph of $T$, and the arcs $n+1 \to j$ for each $j$ such that $n \to j$ in the digraph of $T$. It now follows that the girth of the digraph of $S$ is also $g$, so that $S \in \mathcal{S}(g,n+1)$. Observe also that we can write $S$ as $S = ATB$, where the $(n+1) \times n$ matrix $A$ is given by $A = \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0^T & 1 \\ 0^T & 1 \end{array}\right]$, while the $n \times (n+1)$ matrix $B$ is given by $B = \left[\begin{array}{c|cc} I_{n-1} & 0 & 0 \\ \hline 0^T & \frac{1}{2} & \frac{1}{2} \end{array}\right]$.
It is straightforward to see that $BA = I_n$; from this we find that since the matrix $ATB$ and the matrix $TBA$ have the same nonzero eigenvalues, so do $S$ and $T$. In particular, $\lambda_2(S) = \lambda_2(T)$, and we readily find that $\lambda_2(g,n) \geq \lambda_2(g,n+1)$.
b) Let $\epsilon > 0$ be given, and suppose that $T \in \mathcal{S}(g+1,n)$ is such that $|\lambda_2(T)| < \lambda_2(g+1,n) + \epsilon/2$. Without loss of generality, we suppose that the digraph of $T$ contains the cycle $1 \to 2 \to 3 \to \ldots \to g+1 \to 1$. For each $x \in (0, T_{g,g+1})$, let $S(x) = T + xe_g(e_1 - e_{g+1})^T$, where $e_i$ denotes the $i$-th standard unit basis vector. Note that for each $x \in (0, T_{g,g+1})$,, $S(x) \in \mathcal{S}(g,n)$. By the continuity of the spectrum, there is a $\delta > 0$ such that for any $0 < x < \min\{\delta, T_{g,g+1}\}$, $|\lambda_2(S(x))| - |\lambda_2(T)| < \epsilon/2$. Hence we find that for $0 < x < \min\{\delta, T_{g,g+1}\}$ we have $\lambda_2(g,n) \leq |\lambda_2(S(x))| < |\lambda_2(T)| + \epsilon/2 < \lambda_2(g+1,n) + \epsilon$. In particular, we find that for each $\epsilon > 0$, $\lambda_2(g,n) \leq \lambda_2(g+1,n) + \epsilon$, from which we conclude that $\lambda_2(g,n) \leq \lambda_2(g+1,n)$. $\square$

## 2. Girths 2 and 3.

In this section, we use some elementary techniques to find $\lambda_2(2,n)$ and $\lambda_2(3,n)$. We begin with a discussion of the former.

THEOREM 2.1. *For any $n \geq 2, \lambda_2(2,n) = 1/(n-1)$.*

*Proof.* From Theorem 1.1, we have $\lambda_2(2,n) \geq 1/(n-1)$; the result now follows upon observing that the matrix $\frac{1}{n-1}(J-I) \in \mathcal{S}(2,n)$, and has eigenvalues 1 and $-1/(n-1)$, the latter with multiplicity $n-1$. $\square$

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Our next result shows that there is just one diagonable matrix that yields the minimum value $\lambda_2(2,n)$.

THEOREM 2.2. *Suppose that $T \in \mathcal{S}(2,n)$. Then $T$ is diagonable with $|\lambda_2(T)| = 1/(n-1)$ if and only if $T = \frac{1}{n-1}(J-I)$.*

*Proof.* Suppose that $T$ is diagonable, with $|\lambda_2(T)| = 1/(n-1)$; from Theorem 1.1 we find that the eigenvalue $\lambda_2 = -1/(n-1)$ has algebraic multiplicity $n-1$. Since $T$ is diagonable, the dimension of the $\lambda_2$-eigenspace is $n-1$. Let $x^T$ be the left Perron vector for $T$, normalized so that $x^T\mathbf{1} = 1$. It follows that there are right $\lambda_2$-eigenvectors $v_2, \ldots, v_n$ and left $\lambda_2$-eigenvectors $w_2, \ldots, w_n$ so that $T = \mathbf{1}x^T + \frac{-1}{n-1}\sum_{i=2}^n v_i w_i^T$ and $I = \mathbf{1}x^T + \sum_{i=2}^n v_i w_i^T$. Substituting, we see that $T = \frac{1}{n-1}(n\mathbf{1}x^T - I)$, and since $T$ has trace zero, necessarily, $x^T = \frac{1}{n}\mathbf{1}^T$, yielding the desired expression for $T$. The converse is straightforward. □

Our next example shows that other Jordan forms are possible for matrices yielding the minimum value $\lambda_2(2,n)$.

EXAMPLE 2.1. Consider the polynomial

$$\begin{aligned}(\lambda + \frac{1}{n-1})^{n-1} &= \sum_{j=0}^{n-1} \lambda^j \left(\frac{1}{n-1}\right)^{n-1-j} \binom{n-1}{j} \\ &= \lambda^{n-1} + \lambda^{n-2} + \sum_{j=0}^{n-3} \lambda^j \left(\frac{1}{n-1}\right)^{n-1-j} \binom{n-1}{j}.\end{aligned}$$

From the fact that $n-j > \frac{j}{n-1}$ for $j = 1, \ldots, n-2$, it follows readily that $(\frac{1}{n-1})^{n-1-j}\binom{n-1}{j} > (\frac{1}{n-1})^{n-j}\binom{n-1}{j-1}$ for each such $j$.

We thus find that $(\lambda-1)(\lambda + \frac{1}{n-1})^{n-1}$ can be written as $\lambda^n - \sum_{j=2}^n a_j \lambda^{n-j}$, where $a_j > 0$ for $j = 2, \ldots, n$, and $\sum_{j=2}^n a_j = 1$. Consequently, the companion matrix $C = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_2 & 0 \end{bmatrix}$ is in $\mathcal{S}(2,n)$, and $\lambda_2(C) = -1/(n-1)$. Note that since any eigenvalue of a companion matrix is geometrically simple, the eigenvalue $-1/(n-1)$ of $C$ has a single Jordan block of size $n-1$.

Next, we compute $\lambda_2(3,n)$ for odd $n$.

THEOREM 2.3. *Suppose that $n \geq 3$ is odd. If $T \in \mathcal{S}(3,n)$, then $|\lambda_2(T)| \geq \frac{\sqrt{n+1}}{n-1}$, with equality holding if and only if the eigenvalues of $T$ are* 1 *(with algebraic multiplicity one) and $\frac{-1 \pm i\sqrt{n}}{n-1}$ (with algebraic multiplicity $(n-1)/2$ each). Further, $\lambda_2(3,n) = \frac{\sqrt{n+1}}{n-1}$.*

*Proof.* Suppose that $T \in \mathcal{S}(3,n)$, and denote the eigenvalues of $T$ by 1, and $x_j + iy_j, j = 2, \ldots, n$ (where of course each complex eigenvalue appears with a corresponding complex conjugate). Since $trace(T) = 0$, we have $\sum_{j=2}^n x_j = -1$, while from the fact that $trace(T^2) = 0$, we have $1 + \sum_{j=2}^n (x_j^2 - y_j^2) = 0$. Consequently,

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$\sum_{j=2}^{n}(x_j^2 + y_j^2) = 1 + 2\sum_{j=2}^{n} x_j^2 \geq 1 + 2|\sum_{j=2}^{n} x_j|^2/(n-1) = \frac{n+1}{n-1}$, the inequality following from the Cauchy-Schwarz inequality, and the fact that $\sum_{j=2}^{n} x_j = -1$. Thus we find that $(n-1)|\lambda_2|^2 \geq \sum_{j=2}^{n}(x_j^2 + y_j^2) \geq \frac{n+1}{n-1}$, so that $|\lambda_2(T)| \geq \frac{\sqrt{n+1}}{n-1}$. Inspecting the proof above, we see that $|\lambda_2(T)| = \frac{\sqrt{n+1}}{n-1}$ if and only if each $x_j$ is equal to $-1/(n-1)$, and each $y_j^2$ is equal to $n/(n-1)^2$. The equality characterization now follows.

We claim that for each odd $n$, the companion matrix for the polynomial $(\lambda - 1)(\lambda - \frac{-1+i\sqrt{n}}{n-1})^{(n-1)/2}(\lambda - \frac{-1-i\sqrt{n}}{n-1})^{(n-1)/2} = (\lambda-1)(\lambda^2 + \frac{2}{n-1}\lambda + \frac{n+1}{(n-1)^2})^{(n-1)/2}$ is in fact a nonnegative matrix, from which it will follow that for each odd $n$, there is a matrix in $\mathcal{S}(3,n)$ having $\frac{-1+i\sqrt{n}}{n-1}$ as a subdominant eigenvalue. In order to prove that this companion matrix is nonnegative, it suffices to show that the coefficients of the polynomial $q(\lambda) = \left(\lambda^2 + \frac{2}{n-1}\lambda + \frac{n+1}{(n-1)^2}\right)^{(n-1)/2}$ are increasing with the powers of $\lambda$.

Note that $q(\lambda) = \left((\lambda + \frac{1}{n-1})^2 + \frac{n}{(n-1)^2}\right)^{(n-1)/2}$. Applying the binomial expansion, and collecting powers of $\lambda$, we find that

$$(2.1)\, q(\lambda) = \sum_{l=0}^{n-1} \lambda^l \sum_{j=\lceil l/2 \rceil}^{(n-1)/2} \left(\frac{1}{n-1}\right)^{2j-l} \left(\frac{n}{(n-1)^2}\right)^{(n-1)/2-j} \binom{2j}{l}\binom{(n-1)/2}{j}.$$

Write $q(\lambda)$ as $\sum_{l=0}^{n-1} \lambda^l \alpha_l$. We claim that $\alpha_l \geq \alpha_{l-1}$ for each $l = 1, \ldots, n-1$, which will yield the desired result. Note that for each such $l$, the inequality $\alpha_l \geq \alpha_{l-1}$ is equivalent to $(n-1)\sum_{j=\lceil l/2 \rceil}^{(n-1)/2} \binom{2j}{l}\binom{(n-1)/2}{j}\frac{1}{n^j} \geq \sum_{j=\lceil (l-1)/2 \rceil}^{(n-1)/2} \binom{2j}{l-1}\binom{(n-1)/2}{j}\frac{1}{n^j}$. Observe that $(n-1)\binom{2j}{l} - \binom{2j}{l-1} = \frac{2j!}{(l-1)!(2j-l)!}(\frac{n-1}{l} - \frac{1}{2j-l+1}) \geq 0$, so in particular, if $l$ is even (so that $\lceil l/2 \rceil = \lceil (l-1)/2 \rceil$) it follows readily that $\alpha_l \geq \alpha_{l-1}$.

Finally, suppose that $l$ is odd with $1 \leq l \leq n-1$ and $l = 2r+1$. Then $\lceil l/2 \rceil = r+1$, $\lceil (l-1)/2 \rceil = r$, and since $2r+1 \leq n-1$, we find that $r \leq \frac{n-3}{2}$. In order to show that $\alpha_l \geq \alpha_{l-1}$, it suffices to show, in conjunction with the inequalities proven above, that $(n-1)\binom{2r+2}{2r+1}\binom{(n-1)/2}{r+1}\frac{1}{n^{r+1}} - \binom{2r+2}{2r}\binom{(n-1)/2}{r+1}\frac{1}{n^{r+1}} - \binom{2r}{2r}\binom{(n-1)/2}{r}\frac{1}{n^r} \geq 0$. That inequality can be seen to be equivalent to $2(\frac{n-1}{n}) - \frac{2r+1}{n} - \frac{1}{(n-1)/2-r} \geq 0$, and since we have $2(\frac{n-1}{n}) - \frac{2r+1}{n} - \frac{1}{(n-1)/2-r} \geq 2(\frac{n-1}{n}) - \frac{n-2}{n} - 1 = 0$, the desired inequality is thus established. Hence for odd $l$, we have $\alpha_l \geq \alpha_{l-1}$, and it now follows that there is a companion matrix $C \in \mathcal{S}(3,n)$ such that $|\lambda_2(C)| = \frac{\sqrt{n+1}}{n-1}$. $\square$

Example 2.2. Another class of matrices in $\mathcal{S}(3,n)$ yielding the minimum value for $|\lambda_2|$ arises in the following combinatorial context. A square $(0,1)$ matrix $A$ of order $n$ is called a *tournament matrix* if it satisfies the equation $A + A^T = J - I$. From that equation, one readily deduces that there are no cycles of length 2 in the digraph of a tournament matrix, and a standard result in the area asserts that the digraph associated with any tournament matrix either contains a cycle of length 3, or it has no cycles at all. Thus the digraph of any nonnilpotent tournament matrix necessarily has girth 3.

If, in addition, a tournament matrix $A$ satisfies the identity $A^TA = \frac{n+1}{4}I + \frac{n-3}{4}J = AA^T$, then $A$ is known as a *doubly regular* (or *Hadamard*) tournament ma-

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trix; note that necessarily $n \equiv 3 \ mod\ 4$ in that case. It turns out that doubly regular tournament matrices are co-existent with skew-Hadamard matrices, and so of course the question of whether there is a doubly regular tournament matrix in every admissible order is open, and apparently quite difficult.

In [3] it is shown that if $A$ is a doubly regular tournament matrix, then its eigenvalues consist of $\frac{n-1}{2}$ (of algebraic multiplicity one, and having $\mathbf{1}$ as a corresponding right eigenvector) and $\frac{-1}{2} \pm i\frac{\sqrt{n}}{2}$, each of algebraic multiplicity $(n-1)/2$. Consequently, we find that if $A$ is an $n \times n$ doubly regular tournament matrix, then $T = \frac{2}{n-1}A$ is in $\mathcal{S}(3, n)$ and has eigenvalues 1 and $\frac{-1 \pm i\sqrt{n}}{n-1}$, the latter with algebraic multiplicity $(n-1)/2$ each. From Theorem 2.3, we find that $|\lambda_2(T)| = \lambda_2(3, n)$.

We adapt the technique of the proof of Theorem 2.3 in order to compute $\lambda_2(3, n)$ for even $n$.

Theorem 2.4. *Suppose that $n \geq 4$ is even. If $T \in \mathcal{S}(3, n)$, then $|\lambda_2(T)| \geq \sqrt{\frac{n+2}{n^2-2n}}$, with equality holding if and only if the eigenvalues of $T$ are 1 (with algebraic multiplicity one), $-2/n$ (also with algebraic multiplicity one) and $\frac{-1}{n} \pm \frac{i}{n}\sqrt{\frac{n^2+n+2}{n-2}}$ (with algebraic multiplicity $(n-2)/2$ each). Further, $\lambda_2(3, n) = \sqrt{\frac{n+2}{n^2-2n}}$.*

*Proof.* Suppose that $T \in \mathcal{S}(3, n)$. Since $T$ is stochastic, it has 1 as an eigenvalue, and since $n$ is even, there is at least one more real eigenvalue for $T$, say $z$. Let $x_j + iy_j, j = 2, \ldots, n-1$, denote the remaining eigenvalues of $T$. From the fact that $trace(T) = 0$, we have $1 + z + \sum_{j=2}^{n-1} x_j = 0$, while $trace(T^2) = 0$ yields $1 + z^2 + \sum_{j=2}^{n-1}(x_j^2 - y_j^2) = 0$. Thus we have $\sum_{j=2}^{n-1}(x_j^2 + y_j^2) = 1 + z^2 + 2\sum_{j=2}^{n-1} x_j^2$. Consequently, we find that $(n-2)|\lambda_2|^2 \geq \sum_{j=2}^{n-1}(x_j^2 + y_j^2) = 1 + z^2 + 2\sum_{j=2}^{n-1} x_j^2 \geq 1 + z^2 + 2(1+z)^2/(n-2)$, the second inequality following from the Cauchy-Schwarz inequality. The expression $1 + z^2 + 2(1+z)^2/(n-2)$ is readily seen to be uniquely minimized when $z = -2/n$, with a minimum value of $\frac{n+2}{n}$. Hence we find that $(n-2)|\lambda_2|^2 \geq \frac{n+2}{n}$, and the lower bound on $|\lambda_2|$ follows.

Inspecting the argument above, we see that if $|\lambda_2(T)| = \sqrt{\frac{n+2}{n^2-2n}}$, then necessarily $z$ must be $-2/n$, each $x_j$ must be $-1/n$, while each $y_j^2$ is equal to $\frac{1}{n^2}\frac{n^2+n+2}{n-2}$. The characterization of equality now follows.

We claim that for each even $n$, there is a companion matrix in $\mathcal{S}(3, n)$ having $\frac{-1}{n} + \frac{i}{n}\sqrt{\frac{n^2+n+2}{n-2}}$ as a subdominant eigenvalue. To see the claim, first consider the polynomial $q(\lambda) = \left(\lambda - \left(\frac{-1}{n} - \frac{i}{n}\sqrt{\frac{n^2+n+2}{n-2}}\right)\right)^{(n-2)/2}\left(\lambda - \left(\frac{-1}{n} + \frac{i}{n}\sqrt{\frac{n^2+n+2}{n-2}}\right)\right)^{(n-2)/2} = \left((\lambda + \frac{1}{n})^2 + \frac{n^2+n+2}{n^2(n-2)}\right)^{(n-2)/2}$ and write it as $q(\lambda) = \sum_{l=0}^{n-2}\lambda^l a_l$, so that $(\lambda + 2/n)q(\lambda) = \lambda^{n-1} + \sum_{l=1}^{n-2}\lambda^l(a_{l-1} + 2a_l/n) + 2a_0/n$. As in the proof of Theorem 2.3, it suffices to show that in this last expression, the coefficients of $\lambda^l$ are nondecreasing in $l$. Also as in the proof of that theorem, we find that for each $l = 0, \ldots, n-2, a_l = \sum_{j=\lceil l/2 \rceil}^{(n-2)/2}(\frac{1}{n})^{2j-l}(\frac{n^2+n+2}{n^2(n-2)})^{(n-2)/2-j}\binom{2j}{l}\binom{(n-2)/2}{j}$; straightforward computations now reveal that the coefficients of $\lambda^{n-1}, \lambda^{n-2}, \lambda^{n-3}$ and $\lambda^{n-4}$ in the polynomial $(\lambda +$

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$2/n)q(\lambda)$ are $1, 1, 1$ and $\frac{2n^2-3n-2}{3n^2}$, respectively. We claim that for each $l = 1, \ldots, n-4, a_l \geq a_{l-1}$, which is sufficient to give the desired result.

The claim is equivalent to proving that for each $l = 1, \ldots, n-4$, $n \sum_{j=\lceil l/2 \rceil}^{(n-2)/2} \left(\frac{n-2}{n^2+n+2}\right)^j \binom{2j}{l}\binom{(n-2)/2}{j} \geq \sum_{j=\lceil (l-1)/2 \rceil}^{(n-2)/2} \left(\frac{n-2}{n^2+n+2}\right)^j \binom{2j}{l-1}\binom{(n-2)/2}{j}$. Observe that $n\binom{2j}{l} - \binom{2j}{l-1} = \frac{2j!}{(l-1)!(2j-l)!}(\frac{n}{l} - \frac{1}{2j-l+1}) \geq 0$, so in particular, if $l$ is even (so that $\lceil l/2 \rceil = \lceil (l-1)/2 \rceil$) it follows readily that $a_l \geq a_{l-1}$. Now suppose that $l \geq 1$ is odd, say $l = 2r+1$, so that $\lceil l/2 \rceil = r+1$ and $\lceil (l-1)/2 \rceil = r$. Note also that since $l \leq n-4$, in fact $l \leq n-5$, so that $r \leq (n-6)/2$. In conjunction with the argument above, it suffices to show that $n(\frac{n-2}{n^2+n+2})^{r+1}\binom{2r+2}{2r+1}\binom{(n-2)/2}{r+1} - (\frac{n-2}{n^2+n+2})^{r+1}\binom{2r+2}{2r}\binom{(n-2)/2}{r+1} - (\frac{n-2}{n^2+n+2})^{r}\binom{2r}{2r}\binom{(n-2)/2}{r} \geq 0$. This last inequality can be seen to be equivalent to $\frac{2n(n-2)}{n^2+n+2} - (2r+1)\frac{n-2}{n^2+n+2} - \frac{1}{(n-2)/2-r} \geq 0$. Note that since $r \leq (n-6)/2$, we have $\frac{2n(n-2)}{n^2+n+2} - (2r+1)\frac{n-2}{n^2+n+2} - \frac{1}{(n-2)/2-r} \geq \frac{2n(n-2)}{n^2+n+2} - (n-5)\frac{n-2}{n^2+n+2} - \frac{1}{2} = \frac{n^2+5n-22}{2(n^2+n+2)} \geq 0$, the last since $n \geq 4$. Hence we have $a_l \geq a_{l-1}$ for each $l = 1, \ldots, n-4$, as desired. $\square$

The following result shows that the lower bound of (1.1) on $\lambda_2(g, n)$ is of the correct order of magnitude for $g = 3$. Its proof is immediate from Theorems 2.3 and 2.4.

Corollary 2.5. $lim_{n\to\infty}\lambda_2(3, n)\sqrt{n-1} = 1$.

**3. A class of examples for girth 4.** Our object in this section is to identify, for infinitely many $n$, a matrix $T \in \mathcal{S}(4, n)$ such that $|\lambda_2(T)|$ is of the same order of magnitude as $1/\sqrt[3]{n-1}$, the lower bound on $\lambda_2(4, n)$ arising from (1.1). Our approach is to identify a certain sequence of candidate spectra, and then show that each candidate spectrum is attained by an appropriate stochastic matrix.

Fix an integer $p \geq 3$, and let $r = \frac{1}{3p}$. Set $q = 9p^3 + 2p, l = 18p^3 + 9p^2 + p$ and $m = 9p^2 + 3p$. Letting $n = q + l + m + 1$, it follows that $(n-1)r^3 - 2r^2 - 2r - 1 = 0$. We would like to show that there is a matrix $T \in \mathcal{S}(4, n)$ whose eigenvalues are: 1 (with multiplicity 1), $-r$ (with multiplicity $q$), $re^{\pm\pi i/3}$ (each with multiplicity $l/2$) and $re^{\pm 2\pi i/3}$ (each with multiplicity $m/2$).

For each $j \in \mathbb{N}$, let

$$s_j = 1+q(-r)^j+(l/2)(re^{\pi i/3})^j+(l/2)(re^{-\pi i/3})^j+(m/2)(re^{2\pi i/3})^j+(m/2)(re^{-2\pi i/3})^j.$$

(Observe that if we could find the desired matrix $T$, then $s_j$ would just be the trace of $T^j$.) We find readily that $s_1 = s_2 = s_3 = 0$, while $s_4 = 1 - r^2, s_5 = 1 - r^4$, and $s_6 = 1 + r^3 + 2r^4 + 2r^5$. Finally, note that for any $j \in \mathbb{N}, s_{j+6} - 1 = r^6(s_j - 1)$.

Write the polynomial

$$(\lambda-1)(\lambda+r)^q(\lambda-re^{\pi i/3})^{\frac{l}{2}}(\lambda-re^{-\pi i/3})^{\frac{l}{2}}(\lambda-re^{2\pi i/3})^{\frac{m}{2}}(\lambda-re^{-2\pi i/3})^{\frac{m}{2}}$$

as $\lambda^n + \sum_{j=0}^{n-1} a_j\lambda^j$. Let $C_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & & \\ & & & & & \\ 0 & 0 & & \ldots & 0 & 1 \\ -a_0 & -a_1 & & \ldots & & -a_{n-1} \end{bmatrix}$ be the asso-

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ciated companion matrix, let $M_n = \begin{bmatrix} n & 0 & 0 & 0 & \dots & 0 \\ s_1 & n-1 & 0 & 0 & \dots & 0 \\ s_2 & s_1 & n-2 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & & & & \\ s_{n-1} & s_{n-2} & \dots & & s_1 & 1 \end{bmatrix}$, and let

$A_n = \begin{bmatrix} s_1 & n-1 & 0 & 0 & \dots & 0 \\ s_2 & s_1 & n-2 & 0 & \dots & 0 \\ s_3 & s_2 & s_1 & n-3 & 0 & \dots \\ & \ddots & \ddots & \ddots & & \\ & & & & & \\ s_n & s_{n-1} & & \dots & s_2 & s_1 \end{bmatrix}$. Following an idea from [7], we note that from the Newton identities, it follows that $C_nM_n = A_n$, so that $M_n^{-1}C_nM_n = M_n^{-1}A_n$. In particular, $C_n$ is similar to $M_n^{-1}A_n$. Much of our goal in this section is to show that $M_n^{-1}A_n$ is an irreducible nonnegative matrix. Since any irreducible nonnegative matrix with Perron value 1 is diagonally similar to a stochastic matrix, we will then conclude that there is a matrix $T \in \mathcal{S}(4, n)$ such that $|\lambda_2(T)| = r$.

Throughout the remainder of this section, we take the parameters $p, n, r$ and the sequence $\{s_j\}$ to be as defined above. In particular, we will rely on the facts that $p \geq 3, r \leq 1/9$ and $(n-1)r^3 - 2r^2 - 2r - 1 = 0$.

We begin with some technical results. In what follows, we use $0_k$ denote the $k$-vector of zeros.

Lemma 3.1. *Suppose that $k \in \mathbb{N}$ with $7 \leq k \leq n$. Then*

$$\begin{aligned} M_k\mathbf{1} = (k-3-r^2)\mathbf{1} + (3+r^2)e_1 + (2+r^2)e_2 + \\ (1+r^2)e_3 + r^2e_4 + r^3 \begin{bmatrix} 0_6 \\ \hline v \end{bmatrix}, \end{aligned} \tag{3.1}$$

*where $||v||_\infty = 1 + r + 2r^2$.*

*Proof.* Evidently the first four entries of $M_k\mathbf{1}$ are $k, k-1, k-2$ and $k-3$, respectively. For $j \geq 5$, the $j$-th entry of $M_k\mathbf{1}$ is $k-3+t_j$, where $t_j = \sum_{i=4}^{j}(s_i - 1)$. We have $t_4 = -r^2, t_5 = -r^2 - r^4, t_6 = -r^2 + r^3 + r^4 + 2r^5, t_7 = -r^2 + r^3 + r^4 + 2r^5 - r^6, t_8 = -r^2 + r^3 + r^4 + 2r^5 - 2r^6$, and $t_9 = -r^2 + r^3 + r^4 + 2r^5 - 3r^6$. In particular, for $4 \leq j \leq 9$, note that $-r \leq \frac{t_j + r^2}{r^3} \leq 1 + r + 2r^2$, with equality holding in the upper bound for $j = 6$. Also, for each $4 \leq j \leq 9$ and $i \in \mathbb{N}$, we have $t_{j+6i} = t_9\frac{1-r^{6i+6}}{1-r^6} + t_jr^{6i}$. We find that for such $i$ and $j$, $0 < \frac{t_{j+6i}+r^2}{r^3} \leq \frac{1}{r^3}(t_9/(1-r^6) + r^2 + r^{6i}t_6) \leq \frac{1}{r^3}(t_9/(1-r^6) + r^2 + r^6t_6)$. An uninteresting computation shows that the rightmost member is equal to $1 + r + 2r^2 + \frac{1}{1-r^6}(-3r^3 - 2r^5 + 2r^6 + 2r^7 + 4r^8 + r^{11} - r^{12} - r^{13} - 2r^{14})$. Since $r \leq 1/9$, it follows that this last quantity is strictly less than $1 + r + 2r^2$. Consequently, for any $j \geq 4$, we have $\frac{t_j + r^2}{r^3} \leq 1 + r + 2r^2$, with equality holding for $j = 6$. The result now follows. ◻

Proposition 3.2. *For each $1 \leq k \leq n$, we have*
*a) the offdiagonal entries of $M_k^{-1}$ are nonpositive, so that $M_k^{-1}$ is an M-matrix,*

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*b)* $M_k^{-1}\mathbf{1} \geq \frac{1}{k+1}\mathbf{1}$, *and*

*c)* $M_k^{-1}\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3} \end{bmatrix}$ *is a positive vector.*

*Proof.* We proceed by extended induction on $k$ using a single induction proof for all three statements. Note that each of a), b) and c) is easily established for $k = 1, \ldots, 6$. Suppose now that a), b) and c) hold for natural numbers up to and including $k - 1 \geq 6$.

First, we consider statement a). We have $M_k^{-1} = \left[\begin{array}{c|c} 1/k & 0^T \\ \hline -y & M_{k-1}^{-1} \end{array}\right]$, where $y$ can be written as $y = \frac{1}{k}\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline M_{k-4}^{-1}\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k-1} \end{bmatrix} \end{array}\right]$. From part c) of the induction hypothesis, it follows that $y$ is a nonnegative vector, while from part a) of the induction hypothesis, the offdiagonal entries of $M_{k-1}^{-1}$ are also nonpositive. Hence all offdiagonal entries of $M_k^{-1}$ are nonpositive, which completes the proof of the induction step for statement a).

Next, we consider statement b). From Lemma 3.1, it follows that

$$M_k^{-1}\mathbf{1} = \frac{1}{k-3-r^2}\left(\mathbf{1} - (3+r^2)M_k^{-1}e_1 - (2+r^2)M_k^{-1}e_2 - \right.$$
$$\left. (1+r^2)M_k^{-1}e_3 - r^2M_k^{-1}e_4 + r^3M_k^{-1}\left[\begin{array}{c} 0_6 \\ \hline v \end{array}\right]\right),$$

for some vector $v$ with $||v||_\infty = 1 + r + 2r^2$. The first four entries of $M_k^{-1}\mathbf{1}$ are $1/k, 1/(k-1), 1/(k-2)$ and $1/(k-3)$, respectively, so it remains only to show that $M_k^{-1}\mathbf{1} \geq \frac{1}{k+1}\mathbf{1}$ in positions after the fourth.

Let $trunc_4(M_k^{-1}\mathbf{1})$ denote the vector formed from $M_k^{-1}\mathbf{1}$ by deleting its first four entries. Noting that the entries of $M_k^{-1}e_1, M_k^{-1}e_2, M_k^{-1}e_3$, and $M_k^{-1}e_4$ are nonpositive after the fourth position, it follows that $trunc_4(M_k^{-1}\mathbf{1}) \geq \frac{1}{k-3-r^2}\mathbf{1} + \frac{r^3}{k-3-r^2}\left[\begin{array}{c} 0_2 \\ \hline M_{k-6}^{-1}v \end{array}\right]$.

From part b) of the induction hypothesis, $M_{k-6}^{-1}\mathbf{1}$ is a positive vector, and from part a) of the induction hypothesis, $M_{k-6}^{-1}$ is an M-matrix. Note that $M_{k-6}^{-1}$ has diagonal entries $1/(k-6), 1/(k-7), \ldots, 1/2, 1$. Letting $u_i$ be the $i$-th row sum of $M_{k-6}^{-1}$, it follows that $||e_i^T M_{k-6}^{-1}||_1 = 1/(k-5+i) + (1/(k-5+i) - u_i) \leq 2/(k-5+i) \leq 2$. Letting $|||\bullet|||_\infty$ denote the absolute row sum norm (induced by the infinity norm for vectors), we conclude that $|||M_{k-6}^{-1}|||_\infty \leq 2$. Hence $M_{k-6}^{-1}v \geq -2||v||_\infty\mathbf{1} = -2(1+r+2r^2)\mathbf{1}$. As

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a result, we have $\frac{1}{k-3-r^2}\mathbf{1}+\frac{r^3}{k-3-r^2}\left[\frac{0_2}{M_{k-6}^{-1}v}\right] \geq \frac{1}{k-3-r^2}\mathbf{1}-2(1+r+2r^2)\frac{r^3}{k-3-r^2}\mathbf{1} = \frac{1-2r^3(1+r+2r^2)}{k-3-r^2}\mathbf{1}$.

Since $(k-1)r^3 \leq 2r^2+2r+1$, we have

$$\frac{1-2r^3(1+r+2r^2)}{k-3-r^2} \geq \frac{1-2(1+r+2r^2)(1+2r+2r^2)/(k-1)}{k-3-r^2} \geq \frac{k-3.8325}{(k-1)(k-3)},$$

the last inequality following from the fact that $r \leq 1/9$. Since $k \geq 7$, we find readily that $\frac{k-3.8325}{(k-1)(k-3)} \geq \frac{1}{k+1}$. Putting the inequalities together, we have $M_k^{-1}\mathbf{1} \geq \frac{1}{k+1}\mathbf{1}$, which completes the proof of the induction step for statement b).

Finally, we consider statement c). We have $\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3} \end{bmatrix} = \mathbf{1} + \begin{bmatrix} s_4-1 \\ s_5-1 \\ \vdots \\ s_{k+3}-1 \end{bmatrix} =$ $\mathbf{1} + \begin{bmatrix} -r^2 \\ -r^4 \\ r^3(1+2r+2r^2) \\ -r^6 \\ -r^6 \\ -r^6 \\ \hline 0_{k-6} \end{bmatrix} + \begin{bmatrix} 0_6 \\ \hline s_{10}-1 \\ \vdots \\ s_{k+3}-1 \end{bmatrix}$. Recall that for $4 \leq j \leq 9$ and $i \in \mathbb{N}$, $s_{j+6i}-1 = r^{6i}(s_j-1)$, so that $\frac{|s_{j+6i}-1|}{r^8} \leq \frac{|s_j-1|}{r^2} \leq 1$. Hence $\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3} \end{bmatrix} = \mathbf{1} - r^2e_1 - r^4e_2 + r^3(1+2r+2r^2)e_3 - r^6(e_4+e_5+e_6) + r^8\left[\frac{0_6}{v}\right]$, where $||v||_\infty \leq 1$. Thus we have

$$M_k^{-1}\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3} \end{bmatrix} = M_k^{-1}\mathbf{1} - M_k^{-1}(r^2e_1+r^4e_2+r^6(e_4+e_5+e_6)) + r^3(1+2r+2r^2)M_k^{-1}e_3 + r^8\left[\frac{0_6}{M_{k-6}^{-1}v}\right].$$

Certainly the first six entries of $M_k^{-1}\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3} \end{bmatrix}$ are positive, so it remains only to show that the remaining entries are positive. Note also that the entries of $M_k^{-1}(r^2e_1+$

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$r^4e_2 + r^6(e_4 + e_5 + e_6))$ below the sixth position are all nonpositive, that $M_k^{-1}e_3 = \left[\begin{array}{c} 0_6 \\ \hline \frac{-1}{k-2}M_{k-6}^{-1}\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k-3}\end{bmatrix}\end{array}\right]$, and that the infinity norm of $\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k-3}\end{bmatrix}$ is bounded above by $s_6 = 1 + r^3(1 + 2r + 2r^2)$.

Let $trunc_6\left(M_k^{-1}\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3}\end{bmatrix}\right)$ denote the vector formed from $M_k^{-1}\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3}\end{bmatrix}$ by deleting its first six entries, and define $trunc_6(M_k^{-1}\mathbf{1})$ similarly. From the considerations above, we find that

$$trunc_6\left(M_k^{-1}\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k+3}\end{bmatrix}\right) \geq$$

$$trunc_6(M_k^{-1}\mathbf{1}) - \frac{r^3(1 + 2r + 2r^2)}{k-2}M_{k-6}^{-1}\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k-3}\end{bmatrix} + r^8M_{k-6}^{-1}v.$$

As above, since $M_{k-6}^{-1}$ is an M-matrix, we find that $|||M_{k-6}^{-1}|||_\infty \leq 2$. Applying b), and using the bound on the norm of $M_{k-6}^{-1}$, we have

$$trunc_6(M_k^{-1}\mathbf{1}) - \frac{r^3(1 + 2r + 2r^2)}{k-2}M_{k-6}^{-1}\begin{bmatrix} s_4 \\ s_5 \\ \vdots \\ s_{k-3}\end{bmatrix} + r^8M_{k-6}^{-1}v \geq$$

$$\frac{1}{k+1}\mathbf{1} - \frac{r^3(1 + 2r + 2r^2)(2 + 2r^3 + 4r^4 + 4r^5)}{k-2}\mathbf{1} - 2r^8\mathbf{1}.$$

Thus, it is sufficient to show that $\frac{1}{k+1} - \frac{r^3(1+2r+2r^2)(2+2r^3+4r^4+4r^5)}{k-2} - 2r^8 > 0$.

Since $r^3 \leq \frac{2r^2+2r+1}{k-1}$, it follows that $\frac{1}{k+1} - \frac{r^3(1+2r+2r^2)(2+2r^3+4r^4+4r^5)}{k-2} - 2r^8 \geq \frac{1}{k+1} - \frac{2(1+2r+2r^2)^2(k-1+(1+2r+2r^2)^2)}{(k-1)^2(k-2)} - \frac{2r^2(1+2r+2r^2)^2}{(k-1)^2}$. Now using the fact that $r \leq 1/9$, it eventually follows that $\frac{1}{k+1} - \frac{2(1+2r+2r^2)^2(k-1+(1+2r+2r^2)^2)}{(k-1)^2(k-2)} - \frac{2r^2(1+2r+2r^2)^2}{(k-1)^2} \geq \frac{k^3-6.54k^2+1.84k-2.62}{(k+1)(k-2)(k-1)^2}$. This last is positive, since $k \geq 7$. This completes the proof of the induction step for statement c). $\square$

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The preceding results lead to the following.

THEOREM 3.3. *$M_n^{-1}A_n$ is an irreducible nonnegative matrix.*

*Proof.* We claim that for each $4 \le k \le n$, $M_k^{-1}A_k$ is irreducible and nonnegative. The statement clearly holds if $k = 4$, and we proceed by induction. Suppose that the claim holds for some $4 \le k \le n-1$. Note that $M_{k+1} = \left[\begin{array}{c|c} k+1 & 0^T \\ \hline s & M_k \end{array}\right]$, where $s = \begin{bmatrix} s_1 \\ \vdots \\ s_k \end{bmatrix}$. We also have $A_{k+1} = \left[\begin{array}{c|c} 0 & ke_1^T \\ \hline \sigma & A_k \end{array}\right]$, where $\sigma = \begin{bmatrix} s_2 \\ \vdots \\ s_{k+1} \end{bmatrix}$. It then follows that $M_{k+1}^{-1}A_{k+1} = \left[\begin{array}{c|c} 0 & \frac{k}{k+1}e_1^T \\ \hline \frac{1}{k+1}M_k^{-1}\sigma & M_k^{-1}A_k - \frac{k}{k+1}M_k^{-1}se_1^T \end{array}\right]$.

From the induction hypothesis, $M_k^{-1}A_ke_j \ge 0$ for each $1 \le j \le k$. Note also that $M_k^{-1}A_ke_1 = M_k^{-1}s \ge 0$, so that the first column of $M_k^{-1}A_k - \frac{k}{k+1}M_k^{-1}se_1^T$ is just $\frac{1}{k}M_k^{-1}s$, which is nonnegative, and has the same zero-nonzero pattern as the first column of $M_k^{-1}A_k$. Thus the $(2,2)$ block of $M_{k+1}^{-1}A_{k+1}$ is nonnegative and irreducible by the induction hypothesis, while the $(1,2)$ block is a nonnegative nonzero vector. Further, from Proposition 3.2 it follows that $M_k^{-1}\sigma$ is also nonnegative and nonzero. Hence $M_{k+1}^{-1}A_{k+1}$ is both nonnegative and irreducible, completing the induction step.□

Here is the main result of this section; it follows from Theorem 3.3.

THEOREM 3.4. *For infinitely many $n$, $\lambda_2(4,n) \le r$, where $r$ is the positive root of the equation $(n-1)r^3 - 2r^2 - 2r - 1 = 0$.*

REMARK 3.1. Let $f(x) = (n-1)x^3 - 2x^2 - 2x - 1$. A straightforward computation shows that for all sufficiently large $n$, $f((n-1)^{-\frac{1}{3}} + (n-1)^{-\frac{2}{3}}) > 0$. It now follows that for all sufficiently large $n$, the positive root $r$ for the function $f$ satisfies $r < (n-1)^{-\frac{1}{3}} + (n-1)^{-\frac{2}{3}}$.

The following is immediate from Theorem 1.1, Theorem 3.4 and Remark 3.1.

COROLLARY 3.5. *$\liminf_{n\to\infty} \lambda_2(4,n)\sqrt[3]{n-1} = 1$.*

**4. Bounds for large girth.** At least part of the motivation for the study of $\lambda_2(g,n)$ is to develop some insight when $g$ is large relative to $n$. As noted in Remark 1.3, if both $n$ and $g$ are large, then we expect $\lambda_2(g,n)$ to be close to 1, so that any primitive matrix in $\mathcal{S}(g,n)$ will give rise to a sequence of powers that converges only very slowly. The purpose of this section is to quantify these notions more precisely. To that end, we focus on the case that $g > 2n/3$.

The following result is useful. Its proof appears in [4] and (essentially) in [6] as well.

LEMMA 4.1. *Suppose that $g > n/2$ and that $T \in \mathcal{S}(g,n)$. Then the characteristic polynomial for $T$ has the form $\lambda^n - \sum_{j=g}^n a_j\lambda^{n-j}$, where $a_j \ge 0, j = g, \ldots, n$ and $\sum_{j=g}^n a_j = 1$.*

Our next result appears in [5].

LEMMA 4.2. *Suppose that $g > 2n/3$ and that $T \in \mathcal{S}(g,n)$. Then $T$ has an eigenvalue of the form $\rho e^{i\theta}$, where $\theta \in [2\pi/n, 2\pi/g]$, and where $\rho \ge r(\theta)$, where $r(\theta)$ is the (unique) positive solution to the equation $r^g\sin(n\theta) - r^n\sin(g\theta) = \sin((n-g)\theta)$.*

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REMARK 4.1. It is shown in [5] that there is a one-to-one correspondence between the family of complex numbers $r(\theta)e^{i\theta}, \theta \in [2\pi/n, 2\pi/g]$, and a family of roots of the polynomial $\lambda^n - \alpha\lambda^{n-g} - (1-\alpha), \alpha \in [0,1]$. Specifically, [5] shows that for each $\alpha \in [0,1]$, there is a $\theta \in [2\pi/n, 2\pi/g]$ such that $r(\theta)e^{i\theta}$ is a root of $\lambda^n - \alpha\lambda^{n-g} - (1-\alpha)$, and conversely that for each $\theta \in [2\pi/n, 2\pi/g]$, there is an $\alpha \in [0,1]$ such that $\lambda^n - \alpha\lambda^{n-g} - (1-\alpha)$ has $r(\theta)e^{i\theta}$ as a root. As $\alpha$ runs from 0 to 1, $\theta$ runs from $2\pi/n$ to $2\pi/g$, while $r(\theta)e^{i\theta}$ interpolates between $e^{2\pi i/n}$ and $e^{2\pi i/g}$.

The following result produces lower bounds on $\lambda_2(g,n)$ for $g > 2n/3$ and for $g \geq 3(n+3)/4$.

THEOREM 4.3. *a) Suppose that* $n \geq 27$ *and that* $g > 2n/3$. *Then* $\lambda_2(g,n) \geq (\frac{1}{5})^{1/l(n)}$, *where* $l(n) = 2\lfloor \frac{n}{3} \rfloor + 1$ *if* $n \equiv 0,1$ *mod* 3, *and* $l(n) = 2\lceil \frac{n}{3} \rceil$ *if* $n \equiv 2$ *mod* 3.
*b) If* $n \geq 3(n+3)/4$, *then* $\lambda_2(g,n) \geq (\frac{2\sqrt{7}-1}{7})^{1/(3\lceil \frac{n}{4} \rceil)}$.

*Proof.* a) Let $k = \lfloor \frac{n}{3} \rfloor$, so that $n = 3k+i$, for some $0 \leq i \leq 2$. Since $g > 2n/3$, it follows that $g \geq 2k+1$ if $i = 0,1$, and $g \geq 2k+2$ if $i = 2$. Let $j_0 = 1, j_1 = 1$ and $j_2 = 2$. From Proposition 1.2 b), we find that $\lambda_2(g,n) \geq \lambda_2(2k+j_i, 3k+i)$. From Lemma 4.2 it follows that for each $T \in \mathcal{S}(2k+j_i, 3k+i)$, there is a $\theta \in [2\pi/(3k+i), 2\pi/(2k+j_i)]$ such that $|\lambda_2(T)| \geq r$, where $r$ is the positive solution to the equation $r^{2k+j_i}\sin((3k+i)\theta) - r^{3k+i}\sin((2k+j_i)\theta) = \sin((k+i-j_i)\theta)$. Evidently for such an $r$ we have $r^{2k+j_i}(\sin((3k+i)\theta) - \sin((2k+j_i)\theta)) \geq \sin((k+i-j_i)\theta)$, and it now follows that $\lambda_2(g,n)^{2k+j_i} \geq \min\{\frac{\sin((k+i-j_i)\theta)}{\sin((3k+i)\theta)-\sin((2k+j_i)\theta)} | \theta \in [2\pi/(3k+i), 2\pi/(2k+j_i)]\}$. In order to establish the desired inequality, it suffices to show that for each $\theta \in [2\pi/(3k+i), 2\pi/(2k+j_i)], 5\sin((k+i-j_i)\theta) \geq \sin((3k+i)\theta) - \sin((2k+j_i)\theta)$.

To that end, set $t = (k+i-j_i)\theta$, so that $t \in [\frac{2\pi}{3} - \frac{2\pi(3j_i-2i)}{3(3k+i)}, \pi - \frac{\pi(3j_i-2i)}{2k+j_i}] \subset [\frac{2\pi}{3} - \frac{2\pi}{3k}, \pi - \frac{\pi}{2k+2}]$. Set $b_i = \frac{3j_i-2i}{k+i-j_i}$; we find that $(3k+i)\theta = 3t + b_i t$ and that $(2k+j_i)\theta = 2t + b_i t$. We claim that for each $t \in [2\pi/3 - 2\pi/(3k), \pi - \pi/(2k+2)], 5\sin(t) \geq \sin(3t+b_i t) - \sin(2t+b_i t)$. Let $\cos(t) = x$, so that $-1 < x < 0$. Our claim is equivalent to proving that

$$(4.1) \qquad (5 - (4x^2-2x-1)\cos(b_i t))\sqrt{1-x^2} \geq (x-1)(4x^2+2x-1)\sin(b_i t).$$

From the hypothesis, it follows that $k \geq 9$, so we find that $\sin(b_i t), \cos(b_i t) \geq 0$. First, we note that if $-1 < x \leq -\frac{1+\sqrt{5}}{4}$, then we have $4x^2 - 2x - 1 > 4x^2 + 2x - 1 \geq 0$, so that the left side of (4.1) is positive while the right side is nonpositive.

Next, note that if $-\frac{1+\sqrt{5}}{4} < x \leq \frac{1-\sqrt{5}}{4}$, then $4x^2 - 2x - 1 \geq 0 > 4x^2 + 2x - 1$. It then follows that $(5 - (4x^2-2x-1)\cos(b_i t))\sqrt{1-x^2} \geq \sqrt{1-x^2}(6+2x-4x^2) \equiv f(x)$, while $(x-1)(4x^2+2x-1)\sin(b_i t) \leq (x-1)(4x^2+2x-1) \equiv g(x)$. For $-\frac{1+\sqrt{5}}{4} < x \leq \frac{1-\sqrt{5}}{4}$, we find readily that $f(x)$ is an increasing function of $x$, so that in particular, $f(x) \geq \sqrt{\frac{5-\sqrt{5}}{2}}\left(\frac{3-\sqrt{5}}{4}\right)\left(\frac{7+\sqrt{5}}{2}\right) \approx 1.0368312...$ on that interval. A straightforward computation also reveals that $g(x)$ is increasing on the interval $[-\frac{1+\sqrt{5}}{4}, \frac{1-\sqrt{10}}{6}]$, and is maximized on $[-1,0]$ at $x = \frac{1-\sqrt{10}}{6}$, with $g(\frac{1-\sqrt{10}}{6}) = \left(\frac{-5-\sqrt{10}}{6}\right)\left(4\left(\frac{1-\sqrt{10}}{6}\right)^2 + \frac{1-\sqrt{10}}{3} - 1\right) \approx 1.63$. Since $\frac{1-\sqrt{10}}{6} > -.7$, we find from these considerations that for $-\frac{1+\sqrt{5}}{4} < x \leq -.7$ we have $g(x) \leq g(-.7) \approx .748 <$

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1.036. On the other hand, if $-.7 < x \le \frac{1-\sqrt{5}}{4}$, , then $f(x) \ge f(-.7) \approx 1.88 > 1.63$. It now follows that for each $-\frac{1+\sqrt{5}}{4} < x \le \frac{1-\sqrt{5}}{4}$, $f(x) \ge g(x)$.

Finally, if $\frac{1-\sqrt{5}}{4} < x < 0$, the left side of (4.1) is easily seen to exceed $5\sqrt{1-(\frac{1-\sqrt{5}}{4})^2}$, which in turn exceeds the maximum value for $g(x)$ on $[-1, 0]$. We conclude that (4.1) holds, as desired.

b) Let $k = \lfloor \frac{n}{4} \rfloor$, so that $n = 4k + i$ for some $i = 0, 1, 2, 3$. Since $g \ge 3(n+3)/4$, then we have $g \ge 3k + (9 + 3i)/4$. If $i = 0$, then $g \ge 3k$, while if $i = 1, 2, 3$, then $g \ge 3k + 3$. Consequently, we have $\lambda_2(g, n) \ge \lambda_2(3k, 4k)$ if $i = 0$, and $\lambda_2(g, n) \ge \lambda_2(3(k+1), 4(k+1))$ if $i = 1, 2, 3$, or equivalently, $\lambda_2(g, n) \ge \lambda_2(3\lceil \frac{n}{4} \rceil, 4\lceil \frac{n}{4} \rceil)$.

Set $j = \lceil \frac{n}{4} \rceil$. From Lemma 4.2, we find that $\lambda_2(3j, 4j)^{3j} \ge \min\{\frac{\sin(j\theta)}{\sin(4j\theta)-\sin(3j\theta)} | \theta \in [2\pi/(4j), 2\pi/(3j)]\}$. We claim that $\min\{\frac{\sin(j\theta)}{\sin(4j\theta)-sin(3j\theta)} | \theta \in [2\pi/(4j), 2\pi/(3j)]\} = (\frac{2\sqrt{7}-1}{7})$, from which the result will follow.

To see the claim, let $x = \cos(j\theta)$ and note that $x \in [-1/2, 0]$. Further, we have $\sin(4j\theta) - \sin(3j\theta) = \sin(j\theta)(8x^3 - 4x^2 - 4x + 1)$. Consequently, $\min\{\frac{\sin(j\theta)}{\sin(4j\theta)-\sin(3j\theta)} | \theta \in [2\pi/(4j), 2\pi/(3j)]\} = \min\{\frac{1}{8x^3-4x^2-4x+1} | x \in [-1/2, 0]\}$. The claim now follows from a standard calculus computation. $\square$

Remark 4.2. Note that $\frac{2\sqrt{7}-1}{7} \approx 0.6130718...$ .

Remark 4.3. We note that Theorem 4.3 provides an estimate on $r(\theta)$ for the case that $g > 2n/3$; that estimate is a clear improvement on that of [6], which proves a lower bound of $\left(\frac{1}{2}\sin[\pi/(n-1)]\right)^{2/(n-1)}$ on that quantity.

Our final result considers the case that $n \to \infty$, while $n - g$ is fixed. In the proof, we use the notation $O(\frac{1}{n^k})$ to denote a sequence $s_n$ with the property that $n^k s_n$ is a bounded sequence.

Theorem 4.4. *Suppose that $i \ge 1$ is fixed. Then* $\lambda_2(n-i, n) \ge 1 - \frac{\pi^2 i^2}{2n^3} + O(\frac{1}{n^4})$.

*Proof.* From Lemma 4.2, we find that for $n > 3i$ we have

$$\lambda_2(n-i, n) \ge \left(\min\left\{\frac{\sin(i\theta)}{\sin(n\theta) - \sin((n-i)\theta)} | \theta \in [2\pi/n, 2\pi/(n-i)]\right\}\right)^{\frac{1}{n-i}}.$$

Let $\theta_0$ be a critical point of the function $\frac{\sin(i\theta)}{\sin(n\theta)-\sin((n-i)\theta)}$ on the interval $[2\pi/n, 2\pi/(n-i)]$. Then we have

$$\sin(i\theta_0)(n\cos(n\theta_0) - (n-i)\cos((n-i)\theta_0)) = i\cos(i\theta_0)(\sin(n\theta_0) - \sin((n-i)\theta_0)).$$

Let $\theta_0 = \frac{2\pi}{n} + \frac{a\pi}{n^2}$ where $a = O(1)$. We then have $n\theta_0 = 2\pi + \frac{a\pi}{n}, (n-i)\theta_0 = 2\pi - \left(\frac{(2i-a)\pi}{n} + \frac{ia\pi}{n^2}\right)$ and $i\theta_0 = \frac{2\pi i}{n} + \frac{\pi ai}{n^2}$. Expanding the equation above for $\theta_0$ to terms in $O(\frac{1}{n^3})$, we have $\left(\frac{2\pi i}{n} + \frac{\pi ai}{n^2}\right)\left[n\left(1 - \frac{a^2\pi^2}{2n^2}\right) - (n-i)\left(1 - \frac{(2i-a)^2\pi^2}{2n^2}\right)\right] = i\left(1 - \frac{4\pi^2 i^2}{2n^2}\right)\left[\frac{a\pi}{n} + \frac{(2i-a)\pi}{n} + \frac{ia\pi}{n^2}\right] + O(\frac{1}{n^3})$. Collecting terms and simplifying eventually yields $\frac{(2i-a)^2-a^2}{n^2}\pi^2 = O(\frac{1}{n^3})$,from which we conclude that $a = i + O(\frac{1}{n})$.

Next, we write $\theta_0 = \frac{2\pi}{n} + \frac{i\pi}{n^2} + \frac{b\pi}{n^3}$, where $b = O(1)$. As above, we find that $n\theta_0 = 2\pi + \frac{i\pi}{n} + \frac{b\pi}{n^2}, (n-i)\theta_0 = 2\pi - \left(\frac{i\pi}{n} + \frac{(i^2-b)\pi}{n^2} + \frac{ib\pi}{n^3}\right)$ and $i\theta_0 = \frac{2\pi i}{n} + \frac{\pi i^2}{n^2} + \frac{\pi bi}{n^3}$.

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From this it follows that

$$\frac{\sin(i\theta_0)}{\sin(n\theta_0)-\sin((n-i)\theta_0)} = \frac{\frac{2\pi i}{n}+\frac{\pi i^2}{n^2}+\frac{\pi b i}{n^3}-\frac{4\pi^3 i^3}{3n^3}+O(\frac{1}{n^4})}{\frac{2\pi i}{n}+\frac{\pi i^2}{n^2}+\frac{\pi b i}{n^3}-\frac{\pi^3 i^3}{3n^3}+O(\frac{1}{n^4})} = 1-\frac{\pi^2 i^2}{2n^2}+O\left(\frac{1}{n^3}\right). \tag{4.2}$$

Thus we have $\lambda_2(n-i,n) \geq \left(1-\frac{\pi^2 i^2}{2n^2}+O(\frac{1}{n^3})\right)^{\frac{1}{n-i}} = 1-\frac{\pi^2 i^2}{2n^3}+O(\frac{1}{n^4})$, as desired. □

Remark 4.4. Suppose that we have a matrix $T \in \mathcal{S}(n-1,n)$. Then the characteristic polynomial of $T$ is given by $p_\alpha(\lambda) \equiv \lambda^n - \alpha\lambda-(1-\alpha)$, for some $\alpha \in [0,1]$. Conversely, for each $\alpha \in [0,1]$, there is a matrix $T \in \mathcal{S}(n-1,n)$ whose characteristic polynomial is $p_\alpha$, namely the companion matrix of that polynomial. Thus we see that the eigenvalues of matrices in $\mathcal{S}(n-1,n)$ are in one-to-one correspondence with the roots of polynomials of the form $p_\alpha, \alpha \in [0,1]$. For such a polynomial, we say that a root $\lambda$ is a *subdominant root* if $\lambda \neq 1$ and $\lambda$ has maximum modulus among the roots of the polynomial that are distinct from 1. In particular, we find that discussing the subdominant roots of the polynomials $p_\alpha, \alpha \in [0,1]$ is equivalent to discussing the subdominant eigenvalues of the matrices in $\mathcal{S}(n-1,n)$.

Fix a value of $n \geq 4$. It follows from Corollary 2.1 of [5] that for each $\alpha \in [0,1]$, there is precisely one root of $p_\alpha$ whose argument lies in $[2\pi/n, 2\pi/(n-1)]$ (including multiplicities). Denote that root by $\sigma(\alpha)$. Evidently an analogous statement holds for the interval $[2\pi-2\pi/(n-1), 2\pi-2\pi/n]$, and we claim that in fact $\sigma(\alpha)$ and $\overline{\sigma(\alpha)}$ are subdominant roots for $p_\alpha$.

To see the claim, first suppose that $\alpha \in (0,1)$, and that $z_1$ and $z_2$ are two roots of $p_\alpha$ of equal moduli. Writing $z_1 = \rho e^{i\theta_1}, z_2 = \rho e^{i\theta_2}$, and substituting each into the equation $p_\alpha(\lambda)=0$, we find that $\rho^{2n} = |\alpha\rho e^{i\theta_1}+1-\alpha|^2 = |\alpha\rho e^{i\theta_2}+1-\alpha|^2$. It follows that $\alpha^2\rho^2+(1-\alpha)^2+2\alpha(1-\alpha)\rho\cos(\theta_1) = \alpha^2\rho^2+(1-\alpha)^2+2\alpha(1-\alpha)\rho\cos(\theta_2)$, from which we conclude that $\cos(\theta_1)=\cos(\theta_2)$. Consequently, we find that for each $\alpha \in (0,1)$, if $z_1$ and $z_2$ are roots of $p_\alpha$ that have equal moduli, then either $z_1 = z_2$ or $z_1 = \overline{z_2}$.

For each $\alpha \in [0,1]$, denote the roots of $p_\alpha$ that are distinct from 1 and whose argument fall outside of $[2\pi/n, 2\pi/(n-1)] \cup [2\pi - 2\pi/(n-1), 2\pi-2\pi/n]$ by $\gamma_1(\alpha),\ldots,\gamma_{n-3}(\alpha)$, labeled in nondecreasing order according to their arguments. Suppose that $\exists\, \alpha_1,\alpha_2 \in (0,1)$ such that $|\sigma(\alpha_1)| > \max\{|\gamma_i(\alpha_1)||i=1,\ldots,n-3\}$ and $|\sigma(\alpha_2)| < \max\{|\gamma_i(\alpha_2)||i=1,\ldots,n-3\}$. From the continuity of the roots of $p_\alpha$ in the parameter $\alpha$, and the intermediate value theorem, we find that $\exists\, \alpha_3 \in (0,1)$ such that $|\sigma(\alpha_3)| = \max\{|\gamma_i(\alpha_3)||i=1,\ldots,n-3\}$. Hence for some $i$ we have either $\gamma_i(\alpha_3)=\sigma(\alpha_3)$ or $\gamma_i(\alpha_3) = \overline{\sigma(\alpha_3)}$, a contradiction since the argument of $\gamma_i$ falls outside of $[2\pi/n, 2\pi/(n-1)] \cup [2\pi-2\pi/(n-1), 2\pi-2\pi/n]$. Consequently, we find that one of the following alternatives must hold: either $|\sigma(\alpha)| > \max\{|\gamma_i(\alpha)||i=1,\ldots,n-3\}$ for all $\alpha \in (0,1)$, or $|\sigma(\alpha)| < \max\{|\gamma_i(\alpha)||i=1,\ldots,n-3\}$ for all $\alpha \in (0,1)$.

Next, we claim that for all sufficiently small $\alpha > 0, \sigma(\alpha)$ is a subdominant eigenvalue of $p_\alpha$. To see this, observe that at $\alpha = 0$, the roots of $p_\alpha$ that are distinct from 1 are given by $e^{2\pi i j/n}, 1 \leq j \leq n-1$. Note that since these roots are distinct, there is a neighbourhood of $\alpha=0$ on which each root of $p_\alpha$ is a differentiable function of $\alpha$.

Fix an index $l$ such that either $1 \leq l < (n-2)/2$ or $(n-2)/2 < l \leq n-3$ and

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consider $\gamma_l(\alpha)$. We write $\gamma_l(\alpha) = \rho e^{i\theta}$, where on the right hand side, the explicit dependence on $\alpha$ is suppressed. Considering the real and imaginary parts of the equation $p_\alpha(\rho e^{i\theta}) = 0$, we find that for each $0 < \alpha \leq 1$ we have

$$\rho^n \cos(n\theta) - 1 = \alpha(\rho\cos(\theta) - 1) \tag{4.3}$$

and

$$\rho^{n-1} \sin(n\theta) = \alpha \sin(\theta). \tag{4.4}$$

In particular, crossmultiplying (4.3) and (4.4), canceling the common factor of $\alpha$, and simplifying, we find that for each $0 < \alpha \leq 1$, we have

$$\rho^{n-1} \sin(n\theta) - \rho^n \sin((n-1)\theta) = \sin(\theta). \tag{4.5}$$

(Observe that in fact (4.5) also holds when $\alpha = 0$, since then $\rho = 1$ and $\theta = \frac{2\pi(l+1)}{n}$.) Differentiating (4.4) with respect to $\alpha$ and evaluating at $\alpha = 0$, it follows that $\frac{d\theta}{d\alpha}|_{\alpha=0} = \frac{\sin(2\pi(l+1)/n)}{n}$. Differentiating (4.5) with respect to $\alpha$ (via the chain rule) and evaluating at $\alpha = 0$ then yields $\frac{d\rho}{d\alpha}|_{\alpha=0} = -\frac{1-\cos(2\pi(l+1)/n)}{n}$. Similar arguments show that if $l = (n-2)/2$, then $\frac{d\rho}{d\alpha}|_{\alpha=0} = \frac{-2}{n}$, and that $\frac{d|\sigma|}{d\alpha}|_{\alpha=0} = -\frac{1-\cos(2\pi/n)}{n}$.

We conclude that for all sufficiently small $\alpha > 0, |\sigma(\alpha)| = 1 - \alpha\left(\frac{1-\cos(2\pi/n)}{n}\right) + O(\alpha^2) > 1-\alpha\left(\frac{1-\cos(2\pi(l+1)/n)}{n}\right) + O(\alpha^2) = |\gamma_l(\alpha)|, l = 1\ldots, n-3$. Hence, for such $\alpha$, $\sigma$ (and $\overline{\sigma}$) are subdominant roots of $p_\alpha$. From the considerations above, we conclude that for each $\alpha \in [0,1], \sigma(\alpha)$ is a subdominant root of $p_\alpha$, as claimed.

From the claim, it now follows that $\lambda_2(n-1,n) = \min\{|\sigma(\alpha)||\alpha \in [0,1]\} = \min\{r(\theta)|r(\theta)^{n-1}\sin(n\theta) - r(\theta)^n \sin((n-1)\theta) = \sin(\theta), r(\theta) > 0, \theta \in [2\pi/n, 2\pi/(n-1)]\}$. Arguing as in Theorem 4.4, there is a $\theta_0 \in [2\pi/n, 2\pi/(n-1)]$ such that $\frac{\sin(\theta_0)}{\sin(n\theta_0)-\sin((n-1)\theta_0)} = 1-\frac{\pi^2}{2n^2}+O\left(\frac{1}{n^3}\right)$, which yields $\left(1-\frac{\pi^2}{2n^2}+O\left(\frac{1}{n^3}\right)\right)^{1/n} \geq r(\theta_0) \geq \lambda_2(n-1,n)$. Applying Theorem 4.4, we find that $\left(1-\frac{\pi^2}{2n^2}+O\left(\frac{1}{n^3}\right)\right)^{1/n} \geq \lambda_2(n-1,n) \geq \left(1-\frac{\pi^2}{2n^2}+O\left(\frac{1}{n^3}\right)\right)^{1/(n-1)}$. But since both the upper and lower bounds on $\lambda_2(n-1,n)$ can be written as $1-\frac{\pi^2}{2n^3}+O(\frac{1}{n^4})$, we conclude that $\lambda_2(n-1,n) = 1-\frac{\pi^2}{2n^3}+O(\frac{1}{n^4})$.

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