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THE EXTERNAL VERTICES CONJECTURE IN CASE N = 4*

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Abstract. The determinantal conjecture of M. Marcus and G. N. de Oliveira is known in many special cases. The case of 3×3 matrices was settled by N. Bebiano, J. K. Merikoski and J. da Providência. The case $n = 4$ remains open. In this article a technical conjecture is established implying a weakened form of the determinantal conjecture for $n = 4$.

Key words. Normal matrix, Unitary matrix, Determinant.

AMS subject classifications. 15A15, 15A57.

1. Introduction. Let $\mathbf{A}$ and $\mathbf{B}$ be normal $n \times n$ matrices with prescribed complex eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ respectively. We define the OM vertices $v(\sigma)$ by

$$v(\sigma) = \prod_{j=1}^{n} (\alpha_j - \beta_{\sigma(j)}),$$

where $\sigma \in S_n$ and S_n denotes the group of all permutations of $\{1, \ldots, n\}$.

The determinantal conjecture of M. Marcus [5] and G. N. de Oliveira [7] can be stated in the following way.

The de Oliveira – Marcus Conjecture (OMC)

We have

$$\det(\mathbf{A} - \mathbf{B}) \in \mathrm{co}\ \{v(\sigma); \sigma \in S_n\},$$

where co *denotes the convex hull taken in the complex plane.*

Actually the conjecture was originally stated slightly differently. We have replaced an addition by a subtraction. The conjecture is known in many special cases. It was established for $n = 3$ in N. Bebiano, J. K. Merikoski and J. da Providência [2]. It remains open however in the case $n \geq 4$. In an earlier paper [4], we proposed a related conjecture motivated by the work of J. K. Merikoski and A. Virtanen.

The external vertices conjecture (EVC)

Suppose that the OM vertices $v(\sigma)$ satisfy $\Re v(\sigma) \geq 1$ for all σ in S_n. Then there exist nonnegative numbers t_{IJ} such that

$$\Re v(\sigma) = 1 + \sum_{IJ} t_{IJ} P_{IJ}(\sigma) \qquad \forall \sigma \in S_n,$$

the sum being taken over all pairs of subsets I and J of $\{1, \ldots, n\}$ with the same number of elements.

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The notation $P_{IJ}(\sigma)$ is an extension of the usual permutation matrix notation

$$P_{IJ}(\sigma) = \begin{cases} 1 & \text{if } \sigma(I) = J, \\ 0 & \text{otherwise.} \end{cases}$$

The EVC is known for $n \leq 3$ and also in the case where $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are real [1]. The main result of this paper is the following

THEOREM 1.1. *The EVC is true for $n = 4$.*

One consequence of this result is the following.

COROLLARY 1.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be normal 4×4 matrices with prescribed complex eigenvalues $\alpha_1, \ldots, \alpha_4$ and $\beta_1, \ldots, \beta_4$ respectively. Suppose that $0 \notin \text{co } \{v(\sigma); \sigma \in S_4\}$. Then*

$$\det(\mathbf{A} - \mathbf{B}) \in \text{co } \{rv(\sigma); \sigma \in S_4, r \geq 1\}.$$

Proof. We assert that if H is a closed half-space not containing zero for which $v(\sigma) \in H$ for all $\sigma \in S_4$, then $\det(\mathbf{A} - \mathbf{B}) \in H$. By scaling and rotating the problem, we can always assume without loss of generality that $H = \{z; \Re z \geq 1\}$. Next, we infer the existence of a 4×4 unitary matrix $\mathbf{U}$ with determinant 1 such that

$$\det(\mathbf{A} - \mathbf{B}) = \sum_{\sigma \in S_4} s_{\mathbf{U}}(\sigma) v(\sigma)$$

where $s_{\mathbf{U}}(\sigma) = \text{sgn}(\sigma) \Re \left(\prod_{j=1}^{4} u_{j,\sigma(j)} \right)$, see J. K. Merikoski and A. Virtanen [6, Theorem 1] for details. So, since $s_{\mathbf{U}}(\sigma)$ is real, we have, using Theorem 1.1 that

$$\begin{aligned} \Re \det(\mathbf{A} - \mathbf{B}) &= \sum_{\sigma \in S_4} s_{\mathbf{U}}(\sigma) \Re v(\sigma) \\ &= \sum_{\sigma \in S_4} s_{\mathbf{U}}(\sigma) \left(1 + \sum_{IJ} t_{IJ} P_{IJ}(\sigma) \right), \\ &= 1 + \sum_{IJ} t_{IJ} |u_{IJ}|^2, \\ &\geq 1, \end{aligned}$$

where u_{IJ} denotes the corresponding cofactor of $\mathbf{U}$ and because

$$\sum_{\sigma \in S_4} s_{\mathbf{U}}(\sigma) P_{IJ}(\sigma) = |u_{IJ}|^2$$

by J. K. Merikoski and A. Virtanen [6, equation 16]. This proves the assertion. Let $\Delta = \text{co } \{v(\sigma); \sigma \in S_4\}$. To complete the proof, suppose that the conclusion fails. Then for $0 < s \leq 1$, we find that $s \det(\mathbf{A} - \mathbf{B}) \notin \Delta$ and this is also true for $s = 0$ by hypothesis. Therefore the two compact convex sets Δ and the line segment joining 0 to $\det(\mathbf{A} - \mathbf{B})$ do not meet. The separation theorem for convex sets now allows the construction of a closed half-space H containing Δ but neither 0 nor $\det(\mathbf{A} - \mathbf{B})$. $\square$

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2. Linear and Multiplicative Issues. Around the end of the nineteenth century, the representation theory of the symmetric group S_n was worked out by Alfred Young. He proved that the irreducible representations of S_n are in one to one correspondence with the partitions of n realized as *Ferrers diagrams*. We refer the reader to B. E. Sagan [9], the only book we know that allows the non-specialist easy access to this topic.

The irreducible representations having a Ferrers diagram with at most two rows will be termed *Saxl representations*. The notation π_k stands for the Saxl representation with $n-k$ places in the first row and k places in the second row. This notation applies for $0 \le k \le \lfloor \frac{1}{2}n \rfloor$. Thus π_0 denotes the trivial representation.

On the other hand the *k-ply representation* $(0 \le k \le n)$ is given by

$$\sigma \longrightarrow (P_{IJ}(\sigma^{-1}))_{IJ}$$

where, on the right, the matrix is indexed by the subsets of $\{1, \ldots, n\}$ with k elements. It is not irreducible unless $k = 0$ or $k = n$ and will be denoted $\pi^{(k)}$. Again $\pi^{(0)}$ is just the trivial representation.

J. Saxl [10] worked out the the way in which $\pi^{(k)}$ breaks down into irreducible constituents

$$\pi^{(k)} = \bigoplus_{j=0}^{k} \pi_j \tag{2.1}$$

for $0 \le k \le \lfloor \frac{1}{2}n \rfloor$. For the other values of k it is clear that $\pi^{(k)}$ and $\pi^{(n-k)}$ are equivalent representations.

We define the space Saxl_n of Saxl functions on S_n to be the linear subspace of functions on S_n whose non abelian Fourier transform is carried on the set of Saxl representations. Alternatively, it is the linear span of the functions $P_{IJ}(\cdot)$. Since we have

$$v(\sigma) = \prod_{j=1}^{n} (\alpha_j - \beta_{\sigma(j)}) = \sum_{IJ} \alpha^I (-\beta)^J P_{IJ}(\sigma)$$

where $\alpha^I = \prod_{i \in I} \alpha_i$ and $(-\beta)^J = \prod_{j \in J}(-\beta_j)$, we see that an arbitrary OM vertex function v is a Saxl function and indeed Saxl_n is the linear span of such functions [4]. It turns out that $\dim(\mathrm{Saxl}_n) = C_n$, where C_n denotes the n^{th} Catalan number. In the case of interest in this article $\dim(\mathrm{Saxl}_4) = C_4 = 14$.

Aside from this linear structure, the OM vertex function has an obvious multiplicative structure, exploited for example in [3]. In this article we need to extend that structure. Let us denote $\alpha_{n+j} = \beta_j$ and then consider products

$$v(\mu) = \prod_{j=1}^{n} (\alpha_{\mu(2j-1)} - \alpha_{\mu(2j)}) \tag{2.2}$$

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as μ runs over S_{2n}. In general, the mapping $\mu \mapsto v(\mu)$ is not one to one and we identify μ_1 and μ_2 if they are equal as polynomials in $\alpha_1, \dots, \alpha_{2n}$. We call these equivalence classes OM roots. The term root comes from the passing resemblance to Lie algebra root systems. If the OM root (2.2) is written in term of $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ then each constituent factor has one of the four forms

$$(\alpha_p - \alpha_q), \quad (\beta_p - \beta_q), \quad (\alpha_p - \beta_q), \quad (\beta_p - \alpha_q),$$

and it is clear that the number k of factors of the first type is equal to the number of factors of the second type. Matching these together and applying an identity of the form

$$(\alpha_p - \alpha_q)(\beta_r - \beta_s) = (\alpha_p - \beta_r)(\alpha_q - \beta_s) - (\alpha_p - \beta_s)(\alpha_q - \beta_r)$$

to each matched pair, it can be seen that each OM root can be written in one of the forms $v(\sigma)$, $-v(\sigma)$ for $\sigma \in S_n$ if $k = 0$ and otherwise in the form

$$\sum_{\epsilon_1, \dots, \epsilon_k \in \{0,1\}} (-1)^{\sum_{j=1}^k \epsilon_j} v(\sigma \tau_1^{\epsilon_1} \cdots \tau_k^{\epsilon_k}),$$

where σ is some fixed permutation in S_n and $\tau_1, \dots, \tau_k$ are disjoint transpositions. Here we have $1 \le k \le \lfloor \frac{1}{2} n \rfloor$.

Thus, each root (thought of as a polynomial in $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$) can be written as a linear combination of the OM vertices $v(\sigma)$ with integer coefficients, say $v(\mu) = \sum_{\sigma \in S_n} c_\sigma v(\sigma)$. We extend the notation $P_{IJ}(\sigma)$ to $P_{IJ}(\mu)$ by defining $P_{IJ}(\mu) = \sum_{\sigma \in S_n} c_\sigma P_{IJ}(\sigma)$ where c_σ are as above. This definition is independent of the choice of (c_σ) since in fact for every I, J with $|I| = |J|$, there is a choice of complex numbers $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ that yields $v = P_{IJ}$ (see [4] for details).

For every root μ, there is a corresponding negative root $-\mu$ such that $v(-\mu) = -v(\mu)$. We further define the *level* of a root μ, denoted $\mathbb{1}(\mu)$ as $\sum_{\sigma \in S_n} c_\sigma$. The OM vertices have level 1, their negatives have level -1 and all other roots have level 0. In effect, $\mathbb{1}(\mu) = P_{\emptyset,\emptyset}(\mu)$.

In case $n = 4$, there are in fact, 210 distinct OM roots having one of the forms

- $v(\sigma)$, $\sigma \in S_4$ (24 cases).
- $-v(\sigma)$, $\sigma \in S_4$ (24 cases).
- $v(\sigma) - v(\sigma\tau)$ for $\sigma \in S_4$ and τ is a transposition (144 cases).
- $v(\sigma) - v(\sigma\tau_1) - v(\sigma\tau_2) + v(\sigma\tau_1\tau_2)$ for $\sigma \in S_4$ and where τ_1 and τ_2 are disjoint transpositions (18 distinct cases).

3. Cross Ratios. The cross ratio of 4 complex numbers is given by

$$[\zeta_1, \zeta_2, \zeta_3, \zeta_4] = \frac{(\zeta_1 - \zeta_2)(\zeta_3 - \zeta_4)}{(\zeta_1 - \zeta_4)(\zeta_3 - \zeta_2)}.$$

It is fairly clear that $[\zeta_{\sigma(1)}, \zeta_{\sigma(2)}, \zeta_{\sigma(3)}, \zeta_{\sigma(4)}] = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]$, where σ is an element of the 4-group generated by the double transpositions of S_4. The quantity

$$\langle \zeta_1, \zeta_2, \zeta_3, \zeta_4 \rangle = \operatorname{sgn}(\Im[\zeta_1, \zeta_2, \zeta_3, \zeta_4]) \tag{3.1}$$

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has special significance. In the case that ζ_1, ζ_2, ζ_3 and ζ_4 are distinct points, we consider the circle[1] through the points $\zeta_2, \zeta_3, \zeta_4$ and traversed in that order. The quantity in (3.1) is $+1$ if ζ_1 is on the left of this circular path (i.e. inside the circle if the circle is being traversed anticlockwise and outside it if it is being traversed clockwise) and the quantity is -1 if ζ_1 is on the right of the circular path. The quantity is zero if ζ_1 actually lies on the circular path. More generally $\langle\zeta_1, \zeta_2, \zeta_3, \zeta_4\rangle = 0$ if the four points ζ_1, ζ_2, ζ_3 and ζ_4 lie on a circle. It makes sense to overrule the definition (3.1) by setting $\langle\zeta_1, \zeta_2, \zeta_3, \zeta_4\rangle = 0$ in case that ζ_1, ζ_2, ζ_3 and ζ_4 ! are not distinct. We have

$$\langle\zeta_{\sigma(1)}, \zeta_{\sigma(2)}, \zeta_{\sigma(3)}, \zeta_{\sigma(4)}\rangle = \operatorname{sgn}(\sigma)\langle\zeta_1, \zeta_2, \zeta_3, \zeta_4\rangle \tag{3.2}$$

for all σ in S_4.

There is a way of understanding $\langle\alpha_1, \alpha_2, \beta_1, \beta_2\rangle$ in terms of the root geometry in case $n = 2$ (and which is inherited in root geometry for $n \geq 2$). We have three roots summing to zero

$$(\beta_1 - \alpha_1)(\alpha_2 - \beta_2) + (\alpha_2 - \beta_1)(\alpha_1 - \beta_2) + (\alpha_1 - \alpha_2)(\beta_1 - \beta_2) = 0.$$

Whenever we have $w_1, w_2, w_3 \in \mathbb{C}$ with $w_1 + w_2 + w_3 = 0$ we can distinguish three cases.

1. The points w_1, w_2, w_3 line on a straight line passing through the origin. In the example above, this is the case where $\alpha_1, \alpha_2, \beta_1, \beta_2$ lie on a circle and $\langle\alpha_1, \alpha_2, \beta_1, \beta_2\rangle = 0$.
2. The points w_1, w_2, w_3 are arranged around the origin in anticlockwise order, the angle between consecutive points in the cycle being strictly between zero and π. In the case above $\langle\alpha_1, \alpha_2, \beta_1, \beta_2\rangle = -1$.
3. The points w_1, w_2, w_3 are arranged around the origin in clockwise order, the angle between consecutive points in the cycle being strictly between $-\pi$ and zero. In the case above $\langle\alpha_1, \alpha_2, \beta_1, \beta_2\rangle = 1$.

4. The Simplex Algorithm. It will be impossible to give in this paper a complete proof of Theorem 1.1 because it is based on calculations that could only be carried out with a digital computer. Most of the calculations involve the Simplex Algorithm. The Simplex Algorithm involves a system of linear equations $\mathbf{Ax} = \mathbf{b}$ and one seeks to maximize or minimize a linear function $f(\mathbf{x})$ subject to the constraints $x_j \geq 0$, $j = 1, \ldots, J$. We will need three types of computation. We use fractional arithmetic so that roundoff error is never a problem.

1. The first step of the algorithm, which determines if the feasible set is empty or if it is non-empty determines an extreme point.
2. The determination of whether the strict problem $\mathbf{Ax} = \mathbf{b}$ with the constraints $x_j > 0$, $j = 1, \ldots, J$ has a feasible solution. If the regular problem does not have a solution, neither does the strict problem. If the regular problem does have a solution, we can maximize each of the variables x_j in turn for the regular problem. If there is a variable x_j for which $\max x_j = 0$, then the strict problem does not have a solution. If $\max x_j > 0$ for all j, then by taking a

[1] In this paper, we will always consider a straight line to be a special case of a circle.

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ELA
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convex combination we see that the strict problem does have a solution. (In practice, whenever we visit an extreme point, we take note of which variables are strictly positive, thus eliminating much of the work).

3. Facet finding. We use the gift wrapping method described in [8]. Note that finding the facets of a bounded polytope is equivalent to finding all the extreme points of the dual (polar) polytope. This can be achieved by combining the Simplex Method with a depth first search. Each "wrap" in the gift wrapping method on the original polytope corresponds to a single step in the Simplex Algorithm on the dual polytope. In our problem we have non-simplicial facets which usually create havoc. We deal with this by spontaneous symmetry breaking. For example, if one perturbs the extreme points of a cube, each originally square facet breaks up into two triangles.

Proposition 4.1. *Let*

$$\Gamma = \left\{ f; f \in \mathrm{Saxl}_4,\ f = \sum_{I,J \textit{ with } |I|=|J|} t_{IJ} P_{IJ},\ t_{IJ} \geq 0 \right\}.$$

The facet inequalities of the cone Γ are of two types

1. $f(\sigma) \geq 0$ *for some fixed* $\sigma \in S_4$.
2. $u(f_{\rho_1,\rho_2}) \geq 0$ *for some fixed linear functional* u *on* Saxl_4 *and some* $\rho_1, \rho_2 \in S_4$. *We have denoted* $f_{\rho_1,\rho_2}(\sigma) = f(\rho_1 \sigma \rho_2^{-1})$.

We will detail the functional u at the beginning of the next section. It will be normalized so that $u(\mathbb{1}) = 1$.

Proof. The first step is to define a base of the cone by intersecting it with the hyperplane $\sum_{\sigma \in S_4} f(\sigma) = 1$. We then use computer calculation using the gift-wrapping method to determine the facets of the resulting polytope and hence also of Γ. □

Proposition 4.2. *Let $\alpha_1, \ldots, \alpha_4$ and $\beta_1, \ldots, \beta_4$ be such that $\Re v(\sigma) \geq 1$ for all $\sigma \in S_4$. Then $u(\Re v) \geq 1$ where u is as in Proposition 4.1.*

Proof of Theorem 1.1 given Proposition 4.2. We check that $\Re v - \mathbb{1}$ satisfies the facet inequalities of the cone Γ. The first set of inequalities is obvious by hypothesis. For the second set, we need to verify that $u((\Re v - \mathbb{1})_{\rho_1,\rho_2}) \geq 0$ for all $\rho_1, \rho_2 \in S_4$. But

$$(\Re v - \mathbb{1})_{\rho_1,\rho_2} = \Re(v_{\rho_1,\rho_2}) - \mathbb{1} \tag{4.1}$$

and

$$v_{\rho_1,\rho_2}(\sigma) = \prod_{j=1}^{4} \left(\alpha_{\rho_2(j)} - \beta_{\rho_1(\sigma(j))} \right)$$

are a perfectly good set of OM vertices. But v_{ρ_1,ρ_2} satisfies the hypotheses of Proposition 4.2 for all $\rho_1, \rho_2 \in S_4$ and therefore $u(\Re(v_{\rho_1,\rho_2}) - \mathbb{1}) \geq 0$. Combining this with (4.1) gives the desired result. □

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5. The Linear Domain. We give below 10 ways of writing the functional u. Recall that $f \in \mathrm{Saxl}_4$, so that there are many linear relations between the $f(\sigma)$. This accounts for the multitude of equivalent expressions.

$$\begin{aligned}
u(f) &= \frac{1}{3}\Big(f(3214) + f(4132) + f(3421) + f(2341) - f(3241)\Big)\\
&= \frac{1}{3}\Big(f(3214) + f(4132) + f(2431) + f(4321) - f(4231)\Big)\\
&= \frac{1}{3}\Big(f(3124) + f(4132) + f(3412) + f(2341) - f(3142)\Big)\\
&= \frac{1}{3}\Big(f(3124) + f(1432) + f(4312) + f(2341) - f(1342)\Big)\\
&= \frac{1}{3}\Big(f(2314) + f(4123) + f(3412) + f(2431) - f(2413)\Big)\\
&= \frac{1}{3}\Big(f(3214) + f(4123) + f(4312) + f(2431) - f(4213)\Big)\\
&= \frac{1}{3}\Big(f(2314) + f(4123) + f(1432) + f(3421) - f(1423)\Big)\\
&= \frac{1}{3}\Big(f(2134) + f(4123) + f(3412) + f(2341) - f(2143)\Big)\\
&= \frac{1}{3}\Big(f(3124) + f(2314) + f(1432) + f(4321) - f(1324)\Big)\\
&= \frac{1}{3}\Big(f(2134) + f(3214) + f(1432) + f(4321) - f(1234)\Big).
\end{aligned}$$

Consider the last two terms in the first expression where $f = \Re v$.

$$\begin{aligned}
&\Re(v(2341) - v(3241))\\
&= \Re\Big(\Big((\alpha_1 - \beta_2)(\alpha_2 - \beta_3) - (\alpha_1 - \beta_3)(\alpha_2 - \beta_2)\Big)\Big((\alpha_3 - \beta_4)(\alpha_4 - \beta_1)\Big)\Big)\\
&= \Re\Big((\alpha_1 - \alpha_2)(\beta_2 - \beta_3)(\alpha_3 - \beta_4)(\alpha_4 - \beta_1)\Big) = \Re v(\nu)
\end{aligned}$$

for a certain root ν. If the hypotheses of Proposition 4.2 hold and $\Re v(\nu) \geq 0$, then we get

$$u(\Re v) = \frac{1}{3}\Big(\Re v(3214) + \Re v(4132) + \Re v(3421) + \Re v(\nu)\Big) \geq \frac{1}{3}(1 + 1 + 1 + 0) = 1$$

giving the conclusion of Proposition 4.2. If Proposition 4.2 fails, it follows that $\Re v(\mu) > 0$ where $\mu = -\nu$. There are many such inequalities apparent from the definition of u. We want however to discover all such root inequalities and to state a result asserting that they cannot all be true.

We define 28 expressions ω_j $(j = 1, 2, \ldots, 28)$ by $\omega_1 = (\alpha_2 - \alpha_1)$, $\omega_2 = (\alpha_3 - \alpha_1)$, $\omega_3 = (\alpha_3 - \alpha_2)$, $\omega_4 = (\alpha_4 - \alpha_1)$, $\omega_5 = (\alpha_4 - \alpha_2)$, $\omega_6 = (\alpha_4 - \alpha_3)$, $\omega_7 = (\beta_1 - \alpha_1)$, $\omega_8 = (\beta_1 - \alpha_2)$, $\omega_9 = (\beta_1 - \alpha_3)$, $\omega_{10} = (\beta_1 - \alpha_4)$, $\omega_{11} = (\beta_2 - \alpha_1)$, $\omega_{12} = (\beta_2 - \alpha_2)$, $\omega_{13} = (\beta_2 - \alpha_3)$, $\omega_{14} = (\beta_2 - \alpha_4)$, $\omega_{15} = (\beta_2 - \beta_1)$, $\omega_{16} = (\beta_3 - \alpha_1)$, $\omega_{17} = (\beta_3 - \alpha_2)$,

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$\omega_{18} = (\beta_3 - \alpha_3)$, $\omega_{19} = (\beta_3 - \alpha_4)$, $\omega_{20} = (\beta_3 - \beta_1)$, $\omega_{21} = (\beta_3 - \beta_2)$, $\omega_{22} = (\beta_4 - \alpha_1)$, $\omega_{23} = (\beta_4 - \alpha_2)$, $\omega_{24} = (\beta_4 - \alpha_3)$, $\omega_{25} = (\beta_4! - \alpha_4)$, $\omega_{26} = (\beta_4 - \beta_1)$, $\omega_{27} = (\beta_4 - \beta_2)$ and $\omega_{28} = (\beta_4 - \beta_3)$.

We also define a set of 57 roots using the shorthand $\Omega_{a,b,c,d} = \omega_a\omega_b\omega_c\omega_d$.
$v(\mu_1) = -\Omega_{26,16,12,6}$, $v(\mu_2) = -\Omega_{27,16,8,6}$, $v(\mu_3) = -\Omega_{28,11,8,6}$, $v(\mu_4) = -\Omega_{22,18,15,5}$, $v(\mu_5) = -\Omega_{28,11,9,5}$, $v(\mu_6) = -\Omega_{22,19,15,3}$, $v(\mu_7) = -\Omega_{28,12,9,4}$, $v(\mu_8) = -\Omega_{24,21,10,1}$, $v(\mu_9) = -\Omega_{27,18,10,1}$, $v(\mu_{10}) = \Omega_{23,20,11,6}$, $v(\mu_{11}) = \Omega_{23,21,7,6}$, $v(\mu_{12}) = \Omega_{22,21,9,5}$, $v(\mu_{13}) = \Omega_{24,20,11,5}$, $v(\mu_{14}) = \Omega_{27,18,7,5}$, $v(\mu_{15}) = \Omega_{26,16,14,3}$, $v(\mu_{16}) = \Omega_{22,21,10,3}$, $v(\mu_{17}) = \Omega_{27,16,10,3}$, $v(\mu_{18}) = \Omega_{28,14,7,3}$, $v(\mu_{19}) = \Omega_{28,15,4,3}$, $v(\mu_{20}) = \Omega_{24,17,15,4}$, $v(\mu_{21}) = \Omega_{23,20,13,4}$, $v(\mu_{22}) = \Omega_{26,17,13,4}$, $v(\mu_{23}) = \Omega_{23,21,9,4}$, $v(\mu_{24}) = \Omega_{25,17,15,2}$, $v(\mu_{25}) = \Omega_{26,17,14,2}$, $v(\mu_{26}) = \Omega_{25,20,12,2}$, $v(\mu_{27}) = \Omega_{27,19,8,2}$, $!v(\mu_{28}) = \Omega_{28,14,8,2}$, $v(\mu_{29}) = \Omega_{27,20,5,2}$, $v(\mu_{30}) = \Omega_{25,18,15,1}$, $v(\mu_{31}) = \Omega_{25,20,13,1}$, $v(\mu_{32}) = \Omega_{26,19,13,1}$, $v(\mu_{33}) = \Omega_{26,21,6,1}$, $v(\mu_{34}) = \Omega_{22,17,13,10}$, $v(\mu_{35}) = \Omega_{23,16,13,10}$, $v(\mu_{36}) = \Omega_{22,18,12,10}$, $v(\mu_{37}) = \Omega_{24,16,12,10}$, $v(\mu_{38}) = \Omega_{23,18,11,10}$, $v(\mu_{39}) = \Omega_{24,17,11,10}$, $v(\mu_{40}) = \Omega_{22,17,14,9}$, $v(\mu_{41}) = \Omega_{23,16,14,9}$, $v(\mu_{42}) = \Omega_{22,19,12,9}$, $v(\mu_{43}) = \Omega_{25,16,12,9}$, $v(\mu_{44}) = \Omega_{23,19,11,9}$, $v(\mu_{45}) = \Omega_{25,17,11,9}$, $v(\mu_{46}) = \Omega_{22,18,14,8}$, $v(\mu_{47}) = \Omega_{24,16,14,8}$, $v(\mu_{48}) = \Omega_{22,19,13,8}$, $v(\mu_{49}) = \Omega_{25,16,13,8}$, $v(\mu_{50}) = \Omega_{24,19,11,8}$, $v(\mu_{51}) = \Omega_{25,18,11,8}$, $v(\mu_{52}) = \Omega_{23,18,14,7}$, $v(\mu_{53}) = \Omega_{24,17,14,7}$, $v(\mu_{54}) = \Omega_{2!3,19,13,7}$, $v(\mu_{55}) = \Omega_{25,17,13,7}$, $v(\mu_{56}) = \Omega_{24,19,12,7}$, $v(\mu_{57}) = \Omega_{25,18,12,7}$.

PROPOSITION 5.1. *There do not exist complex numbers $\alpha_1, \ldots, \alpha_4$ and $\beta_1, \ldots, \beta_4$ such that $\Re v(\mu_j) > 0$ for $j = 1, \ldots, 57$.*

Proof of Proposition 4.2 given Proposition 5.1. For each fixed zero-level root ν, we use the Simplex Algorithm to examine the polytope defined by the equations

$$\sum_{\sigma \in S_4} t(\sigma)P_{IJ}(\sigma) + tP_{IJ}(\nu) = u(P_{IJ}), \tag{5.1}$$

as I and J run over all subsets of $\{1, 2, 3, 4\}$ with $|I| = |J|$. We are seeking $t(\sigma)$ and t nonnegative. If there is such a solution, then by linearity, we get

$$\sum_{\sigma \in S_4} t(\sigma)f(\sigma) + tf(\nu) = u(f)$$

for all $f \in \mathrm{Saxl}_4$. In particular, putting $f = \mathbb{1}$ and using the fact that ν is zero level, we get

$$\sum_{\sigma \in S_4} t(\sigma) = \sum_{\sigma \in S_4} t(\sigma)\mathbb{1}(\sigma) + t\mathbb{1}(\nu) = u(\mathbb{1}) = 1.$$

Now suppose that $\Re v(\sigma) \geq 1$ for all $\sigma \in S_4$. Then if also $\Re v(\nu) \geq 0$,

$$u(\Re v) = \sum_{\sigma \in S_4} t(\sigma)\Re v(\sigma) + t\Re v(\nu) \geq \sum_{\sigma \in S_4} t(\sigma) = 1$$

and the Proposition is proved. So, assuming that Proposition 4.2 is false, we obtain $\Re v(\mu) > 0$ where $\mu = -\nu$. In this way we obtain a list of 33 roots μ_j $(j = 1, 2, \ldots, 33)$ such that $\Re v(\mu_j) > 0$. The remaining 24 roots (listed as μ_j for $j = 34, 35, \ldots, 57$) with this property are the OM vertices themselves and come directly from the hypothesis that $\Re v(\sigma) \geq 1$ for all $\sigma \in S_4$. Proposition 5.1 now yields the desired contradiction. $\square$

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6. The Multiplicative Domain. *Proof of Proposition 5.1.* Our task is to show that

$$\Re v(\mu_k) > 0 \qquad (k = 1, 2, \ldots, 57) \tag{6.1}$$

is impossible. We assume that (6.1) is true for some set of complex numbers $\alpha_1, \ldots, \alpha_4, \beta_1, \ldots, \beta_4$. In this case we can (and do) perturb these points so that no four of them lie on a circle while maintaining the veracity of (6.1). We then define θ_k $(k = 1, 2, \ldots, 57)$ in the range $-\pi/2 < \theta_k < \pi/2$ by $\theta_k = \arg(v(\mu_k))$.

Next, we look for multiplicative relations between the $v(\mu_k)$. One of these is $v(\mu_1)v(\mu_8) = v(\mu_{33})v(\mu_{37})$. Consequently

$$\theta_1 + \theta_8 - \theta_{33} - \theta_{37} = 2n\pi,$$

where n is an integer. But $-2\pi < \theta_1 + \theta_8 - \theta_{33} - \theta_{37} < 2\pi$ and we are forced to conclude that $n = 0$. Thus, we obtain a linear relation

$$\theta_1 + \theta_8 - \theta_{33} - \theta_{37} = 0$$

between the θ_k which is just one of a possible 24. However, the dimension of the space spanned by such linear relations is 20. We will call these *type one* relations.

A representative of another type of multiplicative relation is $v(\mu_1)v(\mu_9)v(\mu_{48}) = -v(\mu_2)v(\mu_{32})v(\mu_{36})$ which leads to

$$\theta_1 + \theta_9 + \theta_{48} - \theta_2 - \theta_{32} - \theta_{36} = n\pi, \tag{6.2}$$

where n is now an odd integer. But $-3\pi < \theta_1 + \theta_9 + \theta_{48} - \theta_2 - \theta_{32} - \theta_{36} < 3\pi$, so that $n = \pm 1$, giving us 2 alternatives. There are 64 such relations and ostensibly 2^{64} possibilities. However, the situation is not as bad as this. We first look for equivalence between such relations modulo the type one relations[2]. We find that there are 28 equivalence classes. So, writing down these relations we have

$$\sum_{\ell=1}^{3} \theta_{m_{k,\ell}} - \sum_{\ell=4}^{6} \theta_{m_{k,\ell}} = \epsilon_k \pi, \quad k = 1, 2, \ldots, 28$$

where $\epsilon_k = \pm 1$ for $k = 1, 2, \ldots, 28$. We will call these *type two* relations. Furthermore, these 28 relations are not linearly independent modulo (the linear span of) the type one relations. The dimension modulo the type one relations is 16. We write the relations down in such a way that the first 16 form a basis modulo the type one relations. This means that there are only $2^{16} = 65536$ choices for $(\epsilon_k)_{k=1}^{16}$. Furthermore, for $i = 17, 18, \ldots, 28$, ϵ_i is a linear combination of ϵ_k for $k = 1, 2, \ldots, 16$. When we calculate these ϵ_i as $(\epsilon_k)_{k=1}^{16}$ runs over the 65536 possibilities, not all of them satisfy $\epsilon_i = \pm 1$. This allows the number of choices of signs to be reduced from 65536 to 1872.

[2] Two such relations are said to be equivalent if either the sum or the difference of the expressions corresponding to the left-hand side of (6.2) are in the linear span of the type one relations. This forces the corresponding signs to be either opposite or equal respectively.

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ELA
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We now use the simplex algorithm in the multiplicative domain. Let

$$x_j = \frac{1}{\pi}\theta_j + \frac{1}{2} = \frac{1}{\pi}\left(\theta_j + \frac{\pi}{2}\right), \qquad x_{j+57} = \frac{1}{2} - \frac{1}{\pi}\theta_j = \frac{1}{\pi}\left(\frac{\pi}{2} - \theta_j\right),$$

for $j = 1, 2, \ldots, 57$. Then $x_j > 0$ for $j = 1, 2, \ldots, 114$, and the type one and type two relations hold along with the equations $x_j + x_{57+j} = 1$ for $j = 1, 2, \ldots, 57$. All in all, 93 equations for 114 variables. We check to see if the feasible set for the strict simplex problem is empty. This occurs for 1464 sign patterns of the current 1872 cases, now leaving 408 sign patterns to be eliminated.

To make further progress, we use the interplay between the cross ratio and the roots.

We observe that

$$-\frac{v(\mu_{30})}{v(\mu_{51})} = -\frac{\Omega_{25,18,15,1}}{\Omega_{25,18,11,8}} = -\frac{(\beta_2 - \beta_1)(\alpha_2 - \alpha_1)}{(\beta_2 - \alpha_1)(\beta_1 - \alpha_2)} = [\beta_2, \beta_1, \alpha_2, \alpha_1]$$

and similarly

$$\frac{v(\mu_{57})}{v(\mu_{30})} = [\beta_2, \alpha_2, \alpha_1, \beta_1] \text{ and } \frac{v(\mu_{50})}{v(\mu_{56})} = [\beta_2, \alpha_1, \beta_1, \alpha_2].$$

Note that since the four points $\beta_2, \beta_1, \alpha_2, \alpha_1$ do not lie on a circle, none of the above cross ratios is real. It follows from (3.2) that

$$\text{(6.3)} \qquad -\operatorname{sgn}\left(\Im\frac{v(\mu_{30})}{v(\mu_{51})}\right) = \operatorname{sgn}\left(\Im\frac{v(\mu_{57})}{v(\mu_{30})}\right) = \operatorname{sgn}\left(\Im\frac{v(\mu_{50})}{v(\mu_{56})}\right).$$

Now $\arg\left(\frac{v(\mu_j)}{v(\mu_k)}\right) = \theta_j - \theta_k$ and $-\pi < \theta_j - \theta_k < \pi$, so $\operatorname{sgn}\left(\Im\frac{v(\mu_j)}{v(\mu_k)}\right) = 1$ implies $0 < \theta_j - \theta_k < \pi$ and similarly, $\operatorname{sgn}\left(\Im\frac{v(\mu_j)}{v(\mu_k)}\right) = -1$ implies $0 < \theta_k - \theta_j$. Therefore, either

$$\text{(6.4)} \qquad x_{51} - x_{30} > 0,\ x_{57} - x_{30} > 0 \text{ and } x_{50} - x_{56} > 0$$

or

$$\text{(6.5)} \qquad x_{30} - x_{51} > 0,\ x_{30} - x_{57} > 0 \text{ and } x_{56} - x_{50} > 0.$$

It turns out that there are 38 such possible conclusions arising from the following

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ELA
http://math.technion.ac.il/iic/ela

triples.

$+\frac{v(\mu_{17})}{v(\mu_{2})}$	$+\frac{v(\mu_{12})}{v(\mu_{16})}$	$+\frac{v(\mu_{3})}{v(\mu_{5})}$	$+\frac{v(\mu_{28})}{v(\mu_{3})}$	$+\frac{v(\mu_{22})}{v(\mu_{25})}$	$+\frac{v(\mu_{10})}{v(\mu_{21})}$
$-\frac{v(\mu_{30})}{v(\mu_{51})}$	$+\frac{v(\mu_{57})}{v(\mu_{30})}$	$+\frac{v(\mu_{50})}{v(\mu_{56})}$	$-\frac{v(\mu_{24})}{v(\mu_{45})}$	$+\frac{v(\mu_{55})}{v(\mu_{24})}$	$+\frac{v(\mu_{44})}{v(\mu_{54})}$
$+\frac{v(\mu_{6})}{v(\mu_{42})}$	$-\frac{v(\mu_{48})}{v(\mu_{6})}$	$+\frac{v(\mu_{42})}{v(\mu_{48})}$	$-\frac{v(\mu_{20})}{v(\mu_{39})}$	$+\frac{v(\mu_{18})}{v(\mu_{19})}$	$+\frac{v(\mu_{38})}{v(\mu_{52})}$
$+\frac{v(\mu_{4})}{v(\mu_{36})}$	$-\frac{v(\mu_{46})}{v(\mu_{4})}$	$+\frac{v(\mu_{36})}{v(\mu_{46})}$	$+\frac{v(\mu_{9})}{v(\mu_{17})}$	$+\frac{v(\mu_{24})}{v(\mu_{30})}$	$+\frac{v(\mu_{15})}{v(\mu_{25})}$
$-\frac{v(\mu_{31})}{v(\mu_{49})}$	$+\frac{v(\mu_{55})}{v(\mu_{31})}$	$+\frac{v(\mu_{47})}{v(\mu_{53})}$	$-\frac{v(\mu_{26})}{v(\mu_{43})}$	$+\frac{v(\mu_{14})}{v(\mu_{29})}$	$+\frac{v(\mu_{41})}{v(\mu_{52})}$
$-\frac{v(\mu_{21})}{v(\mu_{35})}$	$+\frac{v(\mu_{54})}{v(\mu_{21})}$	$+\frac{v(\mu_{35})}{v(\mu_{54})}$	$-\frac{v(\mu_{13})}{v(\mu_{39})}$	$+\frac{v(\mu_{27})}{v(\mu_{29})}$	$+\frac{v(\mu_{34})}{v(\mu_{48})}$
$-\frac{v(\mu_{10})}{v(\mu_{38})}$	$+\frac{v(\mu_{44})}{v(\mu_{10})}$	$+\frac{v(\mu_{36})}{v(\mu_{42})}$	$+\frac{v(\mu_{8})}{v(\mu_{37})}$	$-\frac{v(\mu_{39})}{v(\mu_{8})}$	$+\frac{v(\mu_{37})}{v(\mu_{39})}$
$-\frac{v(\mu_{16})}{v(\mu_{34})}$	$+\frac{v(\mu_{36})}{v(\mu_{16})}$	$+\frac{v(\mu_{34})}{v(\mu_{36})}$	$-\frac{v(\mu_{23})}{v(\mu_{41})}$	$+\frac{v(\mu_{44})}{v(\mu_{23})}$	$+\frac{v(\mu_{41})}{v(\mu_{44})}$
$-\frac{v(\mu_{12})}{v(\mu_{40})}$	$+\frac{v(\mu_{42})}{v(\mu_{12})}$	$+\frac{v(\mu_{40})}{v(\mu_{42})}$	$-\frac{v(\mu_{11})}{v(\mu_{52})}$	$+\frac{v(\mu_{32})}{v(\mu_{33})}$	$+\frac{v(\mu_{46})}{v(\mu_{48})}$
$+\frac{v(\mu_{12})}{v(\mu_{4})}$	$+\frac{v(\mu_{21})}{v(\mu_{23})}$	$+\frac{v(\mu_{30})}{v(\mu_{31})}$	$+\frac{v(\mu_{30})}{v(\mu_{4})}$	$+\frac{v(\mu_{21})}{v(\mu_{31})}$	$+\frac{v(\mu_{12})}{v(\mu_{23})}$
$-\frac{v(\mu_{32})}{v(\mu_{48})}$	$+\frac{v(\mu_{11})}{v(\mu_{33})}$	$+\frac{v(\mu_{46})}{v(\mu_{52})}$	$-\frac{v(\mu_{25})}{v(\mu_{40})}$	$+\frac{v(\mu_{53})}{v(\mu_{25})}$	$+\frac{v(\mu_{40})}{v(\mu_{53})}$
$-\frac{v(\mu_{15})}{v(\mu_{41})}$	$+\frac{v(\mu_{47})}{v(\mu_{15})}$	$+\frac{v(\mu_{41})}{v(\mu_{47})}$	$-\frac{v(\mu_{22})}{v(\mu_{34})}$	$+\frac{v(\mu_{55})}{v(\mu_{22})}$	$+\frac{v(\mu_{34})}{v(\mu_{55})}$
$+\frac{v(\mu_{1})}{v(\mu_{37})}$	$-\frac{v(\mu_{43})}{v(\mu_{1})}$	$+\frac{v(\mu_{37})}{v(\mu_{43})}$	$+\frac{v(\mu_{9})}{v(\mu_{36})}$	$-\frac{v(\mu_{38})}{v(\mu_{9})}$	$+\frac{v(\mu_{36})}{v(\mu_{38})}$
$-\frac{v(\mu_{27})}{v(\mu_{48})}$	$+\frac{v(\mu_{13})}{v(\mu_{29})}$	$+\frac{v(\mu_{34})}{v(\mu_{39})}$	$-\frac{v(\mu_{17})}{v(\mu_{35})}$	$+\frac{v(\mu_{37})}{v(\mu_{17})}$	$+\frac{v(\mu_{35})}{v(\mu_{37})}$
$-\frac{v(\mu_{14})}{v(\mu_{52})}$	$+\frac{v(\mu_{26})}{v(\mu_{29})}$	$+\frac{v(\mu_{41})}{v(\mu_{43})}$	$+\frac{v(\mu_{2})}{v(\mu_{47})}$	$-\frac{v(\mu_{49})}{v(\mu_{2})}$	$+\frac{v(\mu_{47})}{v(\mu_{49})}$
$+\frac{v(\mu_{9})}{v(\mu_{30})}$	$+\frac{v(\mu_{15})}{v(\mu_{17})}$	$+\frac{v(\mu_{24})}{v(\mu_{25})}$	$-\frac{v(\mu_{28})}{v(\mu_{46})}$	$+\frac{v(\mu_{47})}{v(\mu_{28})}$	$+\frac{v(\mu_{36})}{v(\mu_{37})}$
$-\frac{v(\mu_{18})}{v(\mu_{52})}$	$+\frac{v(\mu_{20})}{v(\mu_{19})}$	$+\frac{v(\mu_{38})}{v(\mu_{39})}$	$+\frac{v(\mu_{7})}{v(\mu_{42})}$	$-\frac{v(\mu_{43})}{v(\mu_{7})}$	$+\frac{v(\mu_{42})}{v(\mu_{43})}$
$+\frac{v(\mu_{5})}{v(\mu_{44})}$	$-\frac{v(\mu_{45})}{v(\mu_{5})}$	$+\frac{v(\mu_{44})}{v(\mu_{45})}$	$+\frac{v(\mu_{3})}{v(\mu_{50})}$	$-\frac{v(\mu_{51})}{v(\mu_{3})}$	$+\frac{v(\mu_{50})}{v(\mu_{51})}$
$+\frac{v(\mu_{3})}{v(\mu_{10})}$	$+\frac{v(\mu_{25})}{v(\mu_{28})}$	$+\frac{v(\mu_{21})}{v(\mu_{22})}$	$+\frac{v(\mu_{5})}{v(\mu_{12})}$	$+\frac{v(\mu_{2})}{v(\mu_{3})}$	$+\frac{v(\mu_{16})}{v(\mu_{17})}$

For each of the remaining 408 sign patterns and each of the 38 such conditions, we check to see if the strict simplex problem has an empty feasible set, for each choice of sign in the analogue of (6.3). Adding in the three additional conditions and three additional variables yields a simplex tableau with 96 equations and 117 variables. For 384 sign patterns, there is at least one of the 38 conditions where the feasible set is empty for both choices of sign in the analogue of (6.3). These sign patterns can be eliminated leaving only 24 sign patterns to be handled. For each of these 24 sign patterns, we remember when we can deduce the sign of each analogue of (6.3). For example, if when the three conditions in (6.4) are appended to the simplex tableau we have a strict simplex problem with empty feasible set, then we can assert that (6.5) holds and that the value of (6.3) is -1.

For one of the 24 remaining sign patterns, let us suppose that there are ℓ such conditions remembered. Then we can build a strict simplex problem with $93 + 3\ell$ equations and $114 + 3\ell$ variables which incorporates *all* of the remembered conditions. In all cases the feasible set is empty. All sign patterns have now been eliminated and Proposition 5.1 is proved. $\square$

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