Homogeneous permutations

Peter J. Cameron
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road, London E1 4NS, U.K.
p.j.cameron@qmul.ac.uk

Submitted: May 10, 2002; Accepted: 18 Jun, 2002; Published: Oct 31, 2002
MR Subject Classification: 05A99, 03C50

Abstract

There are just five Fraïssé classes of permutations (apart from the trivial class of permutations of a singleton set); these are the identity permutations, reversing permutations, composites (in either order) of these two classes, and all permutations. The paper also discusses infinite generalisations of permutations, and the connection with Fraïssé's theory of countable homogeneous structures, and states a few open problems. Links with enumeration results, and the analogous result for circular permutations, are also described.

1 What is an infinite permutation?

There are several ways of viewing a permutation of the finite set $\{1,\ldots,n\}$, giving rise to completely different infinite generalisations.

To an algebraist, a permutation is a bijective mapping from X to itself. This definition immediately extends to an arbitrary set. The set of all permutations of any set X is a group under composition, the *symmetric group* $\mathrm{Sym}(X)$.

A combinatorialist regards a permutation of $\{1,\ldots,n\}$ in passive form, as the elements of $\{1,\ldots,n\}$ arranged in a sequence $(a_1,a_2,\ldots,a_n)$. If we try to extend this definition to the infinite, we are immediately faced with a problem: what kind of sequence should we use? For example, should it be well-ordered?

A more satisfactory approach is to regard a permutation of $\{1,\ldots,n\}$ as a pair of total orders, where the first is the natural order and the second is the order $a_1 < a_2 < \cdots < a_n$ of the terms in the sequence. Thus a permutation is a relational structure over the language with two binary relational symbols (interpreted as total orders).

In this aspect, the infinite generalisation is clear, but the result is different from the other two. On an infinite set X, a pair of total orders do not correspond to a single permutation, but to a double coset $G_1\pi G_2$ in $\mathrm{Sym}(X)$, where G_1 and G_2 are the automorphism groups of the two total orders. (In the finite case, of course, a total order is rigid, so this double coset contains just the single permutation π.)

This representation also makes the notion of *subpermutation* clear; it is simply the induced substructure on a subset Y of X (the restriction of the two total orders to Y).

I will adopt this view of permutations here. Accordingly, a finite permutation will be regarded as a pair of total orders, each represented by a sequence. For example, the permutation usually written in passive form as $(2,4,1,3)$ might be represented as $(abcd, bdac)$. I will call 2413 the *pattern* of this structure. Thus, a finite permutation is the pattern of an isomorphism class of finite structures (each consisting of a set with two total orders). The two total orders are denoted $<_1$ and $<_2$.

2 Ages and amalgamation

A relational structure X is *homogeneous* if any isomorphism between finite substructures of X can be extended to an automorphism of X. The *age* of a relational structure X is the class of all finite structures embeddable in X.

The best-known homogeneous structure is the ordered set $\mathbb{Q}$. Fraïssé [8], taking this as a prototype, gave a necessary and sufficient condition for a class of finite structures to be the age of a countable homogeneous relational structure. The four conditions are listed below; a class $\mathfrak{C}$ of finite structures satisfying them is called a *Fraïssé class*.

(a) $\mathfrak{C}$ is closed under isomorphism.

(b) $\mathfrak{C}$ is closed under taking induced substructures.

(c) $\mathfrak{C}$ has only countably many members (up to isomorphism).

(d) $\mathfrak{C}$ has the *amalgamation property*: if $A, B_1, B_2 \in \mathfrak{C}$ and $f_i : A \to B_i$ are embeddings for $i = 1,2$, then there exist $C \in \mathfrak{C}$ and embeddings $g_i : B_i \to C$ for $i = 1,2$ such that $f_1 g_1 = f_2 g_2$ (where $f_1 g_1$ means the result of applying f_1 and then g_1).

The amalgamation property informally says that two structures with a common substructure can be glued together. Fraïssé further showed using a back-and-forth argument that, if $\mathfrak{C}$ is a Fraïssé class, then the countable homogeneous structure X whose age is $\mathfrak{C}$ is unique up to isomorphism. We call X the *Fraïssé limit* of $\mathfrak{C}$.

Some authors (for example, Hodges [9]) include also the *joint embedding property* here. This is the following apparent weakening of the amalgamation property: given $B_1, B_2 \in \mathfrak{C}$, there exists $C \in \mathfrak{C}$ such that both B_1 and B_2 can be embedded in C. These authors usually require a substructure to be non-empty; I will allow the empty structure (but assume that it is unique up to isomorphism). With this convention, the joint embedding property is a special case of the amalgamation property.

It is easy to see that conditions (a)–(c) above and the joint embedding property are necessary and sufficient for $\mathfrak{C}$ to be the age of some countable structure; but such a structure is by no means unique in general.

See Hodges [9], Chapter 6, for further discussion of this material.

Now we interpret (a)–(d) for the structures associated with permutations (sets with a pair of total orders). Since a pattern specifies an isomorphism class, (a) means that such a class is defined by a set $\mathcal{C}$ of patterns. Condition (b), called the *hereditary property*, of course means that $\mathcal{C}$ is defined by a set of excluded subpermutations. Condition (c) is vacuous. So the amalgamation property is the crucial condition. We will not always distinguish carefully between a class $\mathfrak{C}$ of relational structures and the corresponding class $\mathcal{C}$ of permutations!

The aim of this paper is to determine the Fraïssé classes of permutations (and so, implicitly, the countable homogeneous structures consisting of a set with a pair of total orders). The classes will be described in the next section, and the theorem proved in the section following. Note that Murphy [12] has considered the question of hereditary classes of permutations with the joint embedding property (that is, ages of infinite permutations).

Countable homogeneous graphs, digraphs and posets have been determined [10, 4, 13]. The result of this paper is analogous (though rather easier); but as far as I can see it does not follow from existing classifications.

Much effort has been devoted to enumerating the permutations in various classes. In particular, the Stanley–Wilf conjecture [1] asserts that a hereditary class not containing all permutations has at most c^n permutations on n points, for some constant c. On the other hand, Macpherson [11] showed that any *primitive* Fraïssé class of relational structures of arbitrary signature (one whose members do not carry a natural equivalence relation derived from the structure) has at least $c^n/p(n)$ members of given cardinality, provided that it has more than one member of some cardinality. (Here c is an absolute constant greater than 1, and p a polynomial.) Examples where the growth is no faster than exponential are comparatively rare. It would appear that permutations would be a good place to look for such examples: this was part of the motivation for the present paper. From this point of view, the main theorem of this paper is a disappointment: of the five Fraïssé classes of permutations defined below, $\mathcal{I}$ and $\mathcal{I}^*$ are trivial, $\mathcal{I}/\mathcal{I}^*$ and $\mathcal{I}^*/\mathcal{I}$ are imprimitive, and $\mathcal{U}$ consists of all permutations.

3 The examples

We begin by defining five classes of finite permutations.

$\mathcal{I}$: the class of identity permutations. This corresponds to two identical total orders, and is defined by the excluded pattern 21.

$\mathcal{I}^*$: the class of reversals, of the form $(n, n-1, \ldots, 1)$. This arises when the second order is the converse of the first, and is defined by the excluded pattern 12.

$\mathcal{I}/\mathcal{I}^*$: this is the class of increasing sequences of decreasing sequences of permutations, defined by the excluded patterns 231 and 312.

$\mathcal{I}^*/\mathcal{I}$: the class of decreasing sequences of increasing sequences, defined by the excluded patterns 213 and 132.

$\mathcal{U}$: the universal class of all finite permutations, where the two total orders are arbitrary.

These are all Fraïssé classes. Indeed, the countable homogeneous structures are clear in the first four cases: the first and second are $\mathbb{Q}$ (with the second order equal to or the reverse of the first); the third and fourth are the lexicographic product of $\mathbb{Q}$ with itself, with the second ordering reversed within blocks, resp. reversed between blocks. (Their automorphism groups are $\mathrm{Aut}(\mathbb{Q})$ in the first two cases, and the wreath product $\mathrm{Aut}(\mathbb{Q}) \wr \mathrm{Aut}(\mathbb{Q})$ in the third and fourth.) In the last case, since the orders are unrelated, we can amalgamate them independently.

The countable homogeneous structure corresponding to $\mathcal{U}$ has an explicit description as follows. The point set is $\mathbb{Q}^2$. Choose two real vectors (a,b) and (c,d), with b/a and d/c distinct irrationals satisfying $b/a+d/c \neq 0$. Now set $(x,y) <_1 (u,v)$ if $xa+yb < ua+vb$, and $(x,y) <_2 (u,v)$ if $xc+yd < uc+vd$. (To see this, note first that given two points $\mathbf{x} = (x,y)$ and $\mathbf{u} = (u,v)$, the remaining points (p,q) fall into three intervals divided by $\mathbf{x}$ and $\mathbf{u}$ with respect to the first order, and three intervals with respect to the second order; all nine combinations are non-empty. Using this, we find that all possible extensions of a given finite structure are realised.)

4 The main theorem

Theorem 1 *A class of finite permutations is a Fraïssé class if and only if it is one of the following: the identity permutation of $\{1\}$, $\mathcal{I}$, $\mathcal{I}^*$, $\mathcal{I}/\mathcal{I}^*$, $\mathcal{I}^*/\mathcal{I}$, or $\mathcal{U}$.*

Proof The trivial class is obviously a Fraïssé class, and we have observed that the same is true for the other five classes. We have to show that any Fraïssé class is one of these.

Let $\mathcal{C}$ be a Fraïssé class of permutations, and C its Fraïssé limit. We may assume that $\mathcal{C}$ contains permutations on more than one point.

First observe that, if $\mathcal{C}$ contains 2-element structure on which the orders agree, then it contains arbitrarily large such structures. For, by amalgamating a structure of length m with one of length n, where the last point of one is identified with the first point of the other, we obtain a structure of length $m+n-1$. So, in this case, $\mathcal{C}$ contains $\mathcal{I}$.

Dually, if $\mathcal{C}$ contains a two-point structure on which the orders disagree, then it contains $\mathcal{I}^*$.

We conclude that, if $\mathcal{C}$ is not equal to either $\mathcal{I}$ or $\mathcal{I}^*$, then it contains both of them. We may suppose that this is the case.

We further suppose that $\mathcal{C} \neq \mathcal{U}$. Then there is some structure X not contained in $\mathcal{C}$; we assume that X is minimal with this property. We show that X has three or four points. For suppose that $|X| = n > 4$. There are $n-1$ pairs of elements which are consecutive in each of the orders. Since $\binom{n}{2} > 2(n-1)$, there are points $x, y \in X$ consecutive in neither order. Then the only amalgam of $X \setminus \{x\}$ and $X \setminus \{y\}$ (identifying $X \setminus \{x,y\}$) is the given structure on X, since the relations between x and y are determined by the other points. Thus $X \in \mathcal{C}$, contrary to assumption.

Suppose first that $|X| = 3$. We know that the patterns 123 and 321 certainly occur. Now amalgamating (ab, ab) with (bc, cb) shows that we have either (abc, acb) (pattern 132) or (abc, cab) (pattern 312). The other three possible ways of amalgamating the two 2-element structures show that we have one of each of the following pairs:

- 312 or 213;
- 213 or 231;
- 231 or 132.

Thus one of the following holds:

(a) exactly two of these four patterns occur, necessarily either 132 and 213, or 312 and 231.

(b) exactly three of the four patterns occur; any one may be the missing one.

We begin with case (a). Let A and B be structures (carrying two total orders). We use $A \nearrow B$ to denote the disjoint union of A and B, with $a <_1 b$ and $a <_2 b$ for all $a \in A$, $b \in B$.

Lemma 2 *Suppose that $\mathcal{C}$ is a Fraïssé class of permutations containing* 132 *and* 213*, Then, for any structures $A, B \in \mathcal{C}$, we have $(A \nearrow B) \in \mathcal{C}$.*

Proof First assume that $|A| = 1$, say $A = \{a\}$, and let x and y be the minimum elements of B in the two orders. If $x = y$, then amalgamate B with (ax, ax); otherwise, amalgamate it with (axy, ayx) (of pattern 132).

Dually, the result holds if $|B| = 1$ (using the pattern 213).

Now for the general case, we first construct $\{c\} \cup B$, with $c <_1 B$ and $c <_2 B$, and also $A \cup \{c\}$, with $A <_1 c$ and $A <_2 c$. Amalgamating these structures gives the result. ■

If both 312 and 231 are forbidden, then the binary relation defined by $x \sim y$ if the orders disagree on $\{x, y\}$ is an equivalence relation, and so the structure belongs to the class $\mathcal{I}/\mathcal{I}^*$. Lemma 2 shows that every permutation in this class belongs to $\mathcal{C}$. So $\mathcal{C} = \mathcal{I}/\mathcal{I}^*$.

Dually, if 132 and 213 are forbidden, then $\mathcal{C} = \mathcal{I}^*/\mathcal{I}$.

Now we turn to case (b) and show that this cannot occur. Suppose, without loss of generality, that only 132 is forbidden. (Interchanging either or both of the orders transforms this case into any of the others.) Now

- amalgamating (abc, bac) (with pattern 213) with (bcd, dbc) (with pattern 312) gives $(abcd, dbac)$;
- amalgamating (bde, dbe) (with pattern 213) with (abe, bea) (with pattern 231) gives $(abde, dbea)$;
- amalgamating $(abcd, dbac)$ with $(abde, dbea)$ gives $(abcde, dbeac)$.

But the last structure contains (bce, bec) with the excluded pattern 132, a contradiction.

Next suppose that $|X| = 4$. Our earlier argument shows that the forbidden patterns have the property that each of the six 2-subsets in an excluded 4-set must be adjacent in one of the two orders. The only permutations satisfying this condition are the two permutations 2413 and 3142.

But amalgamating $(abce, aceb)$ (with pattern 1342) with $(acde, dace)$ (with pattern 3124) gives $(abcde, daceb)$, containing $(abde, daeb)$ with pattern 3142. Similarly the other pattern can be formed by amalgamating $(abce, beca)$ with $(acde, ecad)$.

Finally, if $\mathcal{C}$ contains all four-element structures, then there is no minimal excluded pattern, and we have $\mathcal{C} = \mathcal{U}$. The proof is complete. ■

5 Circular permutations

A *circular order* on a finite set X is the ternary relation obtained by placing the points on a circle and taking all triples in anticlockwise order. In general, a circular order can be defined as a ternary relation such that the restriction to any finite set is a circular order (it suffices to consider restrictions to sets with at most four points [2]).

Now, by analogy, we can define a *circular permutation* to be a finite set carrying two distinct circular orders.

Since a circular order on n points is not rigid but admits the cyclic group C_n of order n as automorphism group, we see that a *pattern* (defining an isomorphism class of finite permutations) is not a single permutation but a double coset $C_n\pi C_n$, for some permutation π. The number of patterns is asymptotically $n!/n^2$; the exact values are given as sequence A002619 in the *Encyclopedia of Integer Sequences* [7].

From the main theorem, we can deduce the classification of Fraïssé classes of circular permutations:

Theorem 3 *There are just five Fraïssé classes of circular permutations containing structures with more than two points.*

Proof From any circular order C on a set A, and any point $a \in A$, we obtain a derived total order C_a on $A \setminus \{a\}$, where

$$C_a = \{(b,c) : (a,b,c) \in C\}.$$

Moreover, C can be recovered uniquely from C_a: for, if $b < c < d$ in the order C_a, then $(b,c,d) \in C$. Hence, from any circular permutation, on A and any $a \in A$, we obtain a derived permutation on $A \setminus \{a\}$. For any class $\mathcal{C}$ of finite circular permutations, let $\mathcal{C}'$ be the class of derived permutations; then $\mathcal{C}$ determines $\mathcal{C}'$, and $\mathcal{C}'$ determines at most one class $\mathcal{C}$.

It is easy to see that each of the five classes of permutations in the main theorem is the derived class of a class of circular permutations. For example, corresponding to $\mathcal{J}/\mathcal{J}^*$, take points on a circle partitioned into consecutive blocks; for the second circular order, reverse the order of the points within each block.

The proof is completed using Theorem 1 and the following lemma. ■

Lemma 4 *A class $\mathcal{C}$ of circular permutations is a Fraïssé class if and only if its derived class $\mathcal{C}'$ is a Fraïssé class of permutations.*

Proof As usual, the hereditary and amalgamation properties are the only ones which require attention. The argument here deals with the amalgamation property; the hereditary property is similar but easier.

Suppose that $\mathcal{C}$ has the amalgamation property. To amalgamate elements B_1, B_2 of the derived class $\mathcal{C}'$ over A, add a point a to A and construct the corresponding circular permutations, and then amalgamate these and derive the result with respect to a. Conversely, suppose that $\mathcal{C}'$ has the amalgamation property, and we wish to amalgamate $B_1, B_2 \in \mathcal{C}$ over the substructure A. Without loss of generality, $A \neq \emptyset$; choose $a \in A$ and amalgamate the derived structures with respect to a. ∎

6 Open problems

I conclude with some open problems arising from this paper.

Problem 1 Extend the main theorem of this paper to structures consisting of m total orders, where $m \geq 3$.

The last three problems depend on the concept of a *reduct* of a relational structure $(X, \mathcal{R})$. This is a relational structure $(X, \mathcal{S})$, where $\mathcal{S}$ is a family of relations, each of which has a first-order definition without parameters in the structure $(X, \mathcal{R})$. For example, if $<$ is a total order on X, and the betweenness relation B is defined by the rule that $B(x, y, z)$ holds if and only if either $x < y < z$ or $z < y < x$, then (X, B) is a reduct of $(X, <)$.

In the case of countable ω-categorical structures $(X, \mathcal{R})$ (which includes countable homogeneous structures over finite relational languages), a reduct is simply a relational structure $(X, \mathcal{S})$ such that $\mathrm{Aut}(X, \mathcal{S}) \geq \mathrm{Aut}(X, \mathcal{R})$. Moreover, in this case, a reduct is defined up to equivalence by its automorphism group, where two relational structures are equivalent if each is a reduct of the other. If X is countable, then a subgroup of $\mathrm{Sym}(X)$ is closed in the topology of pointwise convergence if and only if it is the automorphism group of a relational structure on X. So finding the reducts of $(X, \mathcal{R})$ is equivalent to finding the closed overgroups of $\mathrm{Aut}(X, \mathcal{R})$. I refer to Hodges [9] for further details.

The universal homogeneous countable total order is $(\mathbb{Q}, <)$; its reducts are itself, the derived betweenness relation, circular order and separation relation, and the empty relation (corresponding to the symmetric group) – see [2]. The reducts of the random graph were determined by Thomas [14].

Problem 2 Determine all reducts of the universal homogeneous permutation (up to equivalence).

There are 37 obvious reducts. Choosing independently a reduct of each order gives 25 possibilities; and reversals and interchange of the orders generate a dihedral group of order 8, with 10 subgroups, and similarly for reversing and interchanging the two derived circular orders; but we have now counted 8 reducts twice.

Among these reducts is a universal 2-dimensional poset (the intersection of $<_1$ and $<_2$) and a universal permutation graph (their agreement graph) – neither is homogeneous.

Are there any others?

Problem 3 Which infinite permutations are reducts of homogeneous structures over finite relational languages?

As an example to illustrate this problem, I note that the class of *N-free permutations* (those containing neither of the patterns 2413 and 3142) is the age of an infinite permutation which is a reduct of a homogeneous structure, even though it is not itself a Fraïssé class, as we have seen.

Let (T,r) be a finite rooted binary tree, in which the two children of each non-leaf are ordered. Let c be an arbitrary colouring of the internal vertices of T with two colours (black and white). Let X be the set of leaves of T (excluding r if necessary). For $x,y \in X$, $x \neq y$, let $x \wedge y$ denote the last non-leaf common to the paths rx and ry. Now consider the following relations on X:

- A graph, in which $x \sim y$ if $x \wedge y$ is black. This graph is a *cograph* [5] or *N-free graph* [6]; that is, it contains no induced path of length 3. Every N-free graph can be so represented, though the representation is not unique.

- A ternary relation defined by the rule that $x|yz$ if $x \wedge y = x \wedge z \neq y \wedge z$.

Covington [6] showed that the structures consisting of the graph and ternary relation obtained from all triples (T,r,c) in this way is a Fraïssé class. Our class will be a slight variant of Covington's.

From the data (T,r,c), we obtain a permutation as follows. Let $<_1$ be the order on X defined in the usual way by depth-first search in T, and $<_2$ the order defined by the modified depth-first search in which the children of a white non-leaf are visited in reverse order. The agreement graph of this pair of orders is precisely the N-free graph defined above; so the permutation excludes 2413 and 3142. Any permutation excluding these patterns can be so represented.

Let $\mathfrak{C}$ be the class of structures with two total orders and a ternary relation, derived in this way from triples (T,r,c), where (T,r) is a rooted binary tree and c a 2-colouring of its non-leaves. Then $\mathfrak{C}$ is a Fraïssé class. The proof is not given here, as it is almost identical to that in [6]. If we take the Fraïssé limit and ignore the ternary relation, we obtain a universal N-free permutation.

Problem 4 Which infinite circular permutations are reducts of homogeneous structures over finite relational languages?

Note that, analogous to the N-free permutations, there is a class of pentagon-free circular permutations (similar to the pentagon-free two-graphs defined in [3]).

References

[1] M. Bóna, *Exact and asymptotic enumeration of permutations with subsequence conditions*, Ph.D. thesis, M.I.T, 1997.

[2] P. J. Cameron, Transitivity of permutation groups on unordered sets, *Math. Z.* **48** (1976), 127–139.

[3] P. J. Cameron, Counting two-graphs related to trees, *Electronic J. Combinatorics* **2** (1995), #R4 (8pp). Available from `http://www.combinatorics.org`

[4] G. Cherlin, The classification of countable homogeneous directed graphs and countable homogeneous *n*-tournaments, *Memoirs Amer. Math. Soc.* **621**, American Mathematical Society, Providence, RI, 1998.

[5] D. G. Corneil, Y. Perl and L. Stewart, Cographs: recognition, applications and algorithms, *Congr. Numerantium* **43** (1984), 249–258.

[6] J. Covington, A universal structure for N-free graphs, *Proc. London Math. Soc.* (3) **58** (1989), 1–16.

[7] *Encyclopedia of Integer Sequences*, available from
`http://www.research.att.com:80/~njas/sequences/`

[8] R. Fraïssé, Sur certains relations qui généralisent l'ordre des nombres rationnels, *C. R. Acad. Sci. Paris* **237** (1953), 540–542.

[9] W. Hodges, *A Shorter Model Theory*, Cambridge University Press, Cambridge, 1997.

[10] A. H. Lachlan and R. E. Woodrow, Countable ultrahomogeneous undirected graphs, *Trans. Amer. Math. Soc.* **262** (1980), 51–94.

[11] H. D. Macpherson, The action of an infinite permutation group on the unordered subsets of a set, *Proc. London Math. Soc.* (3) **46** (1983), 471–486.

[12] M. Murphy, Ph.D. thesis, University of St. Andrews, 2002.

[13] J. H. Schmerl, Countable homogeneous partially ordered sets, *Algebra Universalis* **9** (1979), 317–321.

[14] S. Thomas, Reducts of the random graph, *J. Symbolic Logic* **56** (1991), 176–181.