Competitive Colorings of Oriented Graphs

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Abstract

Nešetřil and Sopena introduced a concept of oriented game chromatic number and developed a general technique for bounding this parameter. In this paper, we combine their technique with concepts introduced by several authors in a series of papers on game chromatic number to show that for every positive integer k, there exists an integer t so that if $\mathcal{C}$ is a topologically closed class of graphs and $\mathcal{C}$ does not contain a complete graph on k vertices, then whenever G is an orientation of a graph from $\mathcal{C}$, the oriented game chromatic number of G is at most t. In particular, oriented planar graphs have bounded oriented game chromatic number. This answers a question raised by Nešetřil and Sopena. We also answer a second question raised by Nešetřil and Sopena by constructing a family of oriented graphs for which oriented game chromatic number is bounded but extended Go number is not.

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1 Introduction

In this paper, we will be discussing graphs without loops or multiple edges and orientations of such graphs. When G is a graph, we will say, for example, xy is an edge in G. In this case, yx is also an edge in G. On the other hand, when G is an oriented graph, we will say, for example, (x, y) is an edge in G when there is an edge in G directed from x to y. In this case, (y, x) will not be an edge of G.

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Throughout this paper, we consider variations of the the following game played by two players Alice and Bob with Alice always playing first. Given a finite graph G and a set C of colors, the players take turns coloring the vertices of G with colors from C so that no two adjacent vertices have the same color. Bob wins if at some time, one of the players has no legal move; otherwise Alice wins and the players eventually create a proper coloring of G. We call this game the (G, C)-coloring game. The *game chromatic number* of G, denoted $\chi_g(G)$, is the least positive integer t such that Alice has a winning strategy for the (G, C)-coloring game when $|C| = t$. The concept of game chromatic number was first introduced by Bodlaender [1]. We refer the reader to [7] for recent results on this parameter and for additional references.

In [10], Nešetřil and Sopena introduced a variation of game chromatic number for oriented graphs. Given a finite oriented graph G and a tournament T, Alice and Bob take turns assigning the vertices of G to vertices in the tournament T. This results in a mapping $\phi : G \longrightarrow T$. When (x, y) is an edge in G, we require that $(\phi(x), \phi(y))$ is an edge in T, i.e., ϕ is a *homomorphism*. We call this game the (G, T)-*coloring game* and refer to the vertices of the tournament T as *colors*.

A moment's reflection reveals that there is one additional restriction which must be placed on the players' moves. Consider two vertices x and y. Suppose that y is colored (mapped to a vertex in T) and that one of the two players is about to color x. Let $M(x, y)$ denote the set of all vertices from G which are midpoints of a directed path of three vertices beginning at one of x and y and ending at the other. If at least one vertex of $M(x, y)$ has already been colored, then x cannot be assigned the same color as y. However, if $M(x, y) \neq \emptyset$ but all vertices in $M(x, y)$ are uncolored at the moment, then assigning x the same color as y is a legal but deadly move—as it is clear that there will be no legal way to color the vertices in $M(x, y)$.

Alice would of course never make such a move, but Bob would certainly do so if it were allowed. Accordingly, to play the (G, T)-coloring game, we add the restriction that Bob is not allowed to assign to vertex x the same color already given to a vertex y when $M(x, y) \neq \emptyset$ and all vertices in $M(x, y)$ are uncolored.

The *oriented game chromatic number* of an oriented graph G, denoted $\mathrm{ogcn}(G)$, is the least positive integer t such that there is a tournament T on t vertices so that Alice has a winning strategy for the (G, T)-coloring game. It is not immediately clear that this parameter is well-defined, i.e., it is not clear that there is *any* tournament T for which Alice has a winning strategy for the (G, T) game. However, Nešetřil and Sopena [10] developed a general technique for showing that for every oriented graph G, there is a tournament T for which Alice has a winning strategy for the (G, T) coloring game. They then used this technique to prove the following results.

Theorem 1.1 *Let P be an oriented path. Then* $\mathrm{ogcn}(P) \leq 7$. *Furthermore, this result is best possible.* □

Theorem 1.2 *Let T be an oriented tree. Then* $\mathrm{ogcn}(P) \leq 19$. □

Theorem 1.3 *There exists an absolute constant c so that if G is an oriented outerplanar graph, then* $\mathrm{ogcn}(G) \leq c$. □

Nešetřil and Sopena [10] raised the question as to whether the preceding theorem holds for oriented planar graphs, and the primary goal of this paper is to settle this question in the affirmative. In fact, we prove a more general result. Recall that a graph H is said to be a *homeomorph* of a graph G when H is obtained from G by inserting vertices on the edges of G. Equivalently, H is obtained by replacing the edges of G by paths. Homeomorphs are also called *subdivisions*. A class $\mathcal{C}$ of graphs is said to be *topologically closed* when the following two conditions are satisfied:

1. If $G \in \mathcal{C}$ and H is a subgraph of G, then $H \in \mathcal{C}$.

2. If H is homeomorph of G and $H \in \mathcal{C}$, then $G \in \mathcal{C}$.

For example, for every integer t, the class $\mathcal{C}_t$ of all graphs of genus at most t is topologically closed. In fact, this class satisfies the following stronger property:

(2') If H is homeomorph of G, then $H \in \mathcal{C}_t$ if and only if $G \in \mathcal{C}_t$.

As a special case, setting $t = 0$, the class of planar graphs is topologically closed.

A class $\mathcal{C}$ of graphs is *minor closed* if $H \in \mathcal{C}$ whenever $G \in \mathcal{C}$ and H is a minor of G. As is well known, whenever a class $\mathcal{C}$ is minor closed, it is also topologically closed, although the converse is not true. It is customary to say that a graph H is a *topological minor* of a graph G when G contains a subgraph which is a homeomorph of H.

Also recall that the *density* of a graph G, denoted here by $\mathrm{den}(G)$, is defined by:

$$\mathrm{den}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}.$$

The next theorem is our main result.

Theorem 1.4 *For every positive integer k, there exists an integer t and a tournament T on t vertices so that if $\mathcal{C}$ is a topologically closed class of graphs and $\mathcal{C}$ does not contain a complete graph on k vertices, then whenever G is an orientation of a graph from $\mathcal{C}$, Alice wins the (G, T)-coloring game.*

The reader should note that excluding a clique of a particular size from a topologically closed class is the same as bounding the density. Accordingly, our principal theorem may be restated in the following form, and this is the version that we will actually prove.

Theorem 1.5 *For every positive integer d, there exists an integer t and a tournament T on t vertices so that if $\mathcal{C}$ is a topologically closed class of graphs and each graph in $\mathcal{C}$ has density at most d, then whenever G is an orientation of a graph from $\mathcal{C}$, Alice wins the (G, T)-coloring game.*

For emphasis, we note the immediate corollary.

Corollary 1.6 *There exists a tournament T for which Alice wins the (G, T)-coloring game whenever G is an oriented planar graph.*

The remainder of the paper is organized as follows. In the next section, we discuss coloring numbers and their relation to game coloring problems. In Section 3, we discuss reachability and some related graph parameters. In Section 4, we derive a key lemma and the proof of our main theorem follows in Section 5. In Section 6, we present two examples, the second of which answers another question posed by by Nešetřil and Sopena in [10].

2 Game Coloring Numbers

In the first paper on the subject of game chromatic number, Bodlaender [1] proposed the following "marking" game. Given a graph G and a positive integer t, start with all vertices of G designated as *unmarked*. A move consists of selecting an unmarked vertex and changing its status to *marked*. Once a vertex is marked, it stays marked forever. Alice and Bob alternate turns with Alice having the first move. At each step in the game, the *score* of an unmarked vertex is the number of marked neighbors. Bob wins if there is ever an unmarked vertex whose score is more than t. Alice wins if all vertices are marked and at no point was there an unmarked vertex whose score was more than t. By analogy with the rules of the classic board game "go", Nešetřil and Sopena [10] called this parameter the *Go number* of the graph and denoted it by $\mathrm{Go}(G)$. The *game coloring number* of a graph G, denoted $\mathrm{col}_g(G)$, is one more than the Go number of G. It is easy to see that the game chromatic number of a graph is at most the game coloring number.

In [9], Kierstead and Trotter showed that the game chromatic number of a planar graph is at most 33. This bound was improved to 19 by Zhu [11], [12] using the concept of game coloring number. Most recently, Kierstead used a similar approach to lower the bound to 18.

In [10], Nešetřil and Sopena introduced a variation of the marking game for oriented graphs and then applied this variation to game chromatic problems for oriented graphs. Given an oriented graph G and a positive integer t, start with all vertices of G unmarked. As before, a move consists of selecting an unmarked vertex and changing its status to marked. Once a vertex is marked, it remains marked forever. Alice and Bob alternate turns with Alice having the first move. For each unmarked vertex x, let $B(x)$ denote the set of all marked neighbors of x and let $B_2(x)$ denote the set of all vertices y so that (1) y is marked; (2) G contains a directed path of length 2 with x at one end and y at the other; and (3) for each directed path of length 2 with x at one end and y at the other, the middle point is unmarked.

In this game, the score of an unmarked vertex x is $|B(x) \cup B_2(x)|$. Bob wins if there is ever an unmarked vertex x for which the score of x is more than t, while Alice wins if all vertices are marked and at no point was there an unmarked vertex x whose score was more than t. The *extended Go number* of G, denoted $\mathrm{eGo}(G)$, is the least positive integer t for which Alice has a winning strategy.

Trivially, the extended Go number of an oriented graph on n vertices is at most n. In fact, it is at most $\big(\Delta(G)\big)^2$. With these remarks as background, we can now state the principal result of [10].

Theorem 2.1 *For every integer k, there exists an integer $c(k)$ so that if G is an oriented graph with* $\mathrm{eGo}(G) \le k$*, then* $\mathrm{ogcn}(G) \le c(k)$*.* □

We pause to comment that the argument used by Nešetřil and Sopena to prove Theorem 2.1 is probabilistic. They show that if $c(k)$ is sufficiently large in terms of k, and if T is a random tournament on $c(k)$ vertices, then the probability that Alice wins the (G, T) coloring game is positive. In fact, it suffices to set $c(k) = 100k^2 2^k$.

It makes sense to consider variations of marking games on undirected graphs. For example, given an undirected graph G and a positive integer t, consider the following game. For each unmarked vertex x, let $A(x)$ denote the set of all marked neighbors of x and let $A_2(x)$ denote the set of all vertices y for which (1) y is marked and (2) there is an unmarked vertex z adjacent to both x and y. In this game, the score of an unmarked vertex x is $|A(x) \cup A_2(x)|$. As before, Bob wins if there is ever an unmarked vertex x for which the score of x exceeds t, while Alice wins if all vertices are marked and at no point was there an unmarked vertex x for which the score of x was more than t.

Then define $\mathrm{Go}_2(G)$ as the least number of G for which Alice has a winning strategy.

It is easy to see that if G is an orientation of an undirected graph G', then $\mathrm{eGo}(G) \le \mathrm{Go}_2(G')$. In fact, the basic idea behind the proof of our principal theorem is to show that for every integer d, there exists a constant c_d so that if $\mathcal{C}$ is a topologically closed class of graphs each of which has density at most d, then $\mathrm{Go}_2(G) \le c_d$ for every graph $G \in \mathcal{C}$.

3 Reachability and Related Graph Parameters

Let $\Pi(V)$ denote the set of all linear orders on the vertex set V of a graph G. Then let $L \in \Pi(G)$. For each vertex x of G, let $N^+_{G_L}(x) = \{y \in V : y < x \text{ in } L, xy \in E(G)\}$, and let $d^+_{G_L}(x) = |N^+_{G_L}(x)|$. This quantity is called the the *back degree of x in L* and $N^+_{G_L}(x)$ is the set of *back neighbors* of x in L. Then let $\Delta^+(G_L)$ denote the largest value of $d^+_{G_L}(x)$ taken over all $x \in V$, and let

$$\Delta^+(G) = \min_{L \in \Pi(V)} \Delta^+(G_L).$$

The quantity $\Delta^+(G)$ is called the *back degree* of G, and the quantity $\mathrm{col}(G) = 1 + \Delta^+(G)$ is called the *coloring number* of G.

Clearly, the chromatic number $\chi(G)$ satisfies the inequality $\chi(G) \le \mathrm{col}(G)$, since First Fit will use at most $\mathrm{col}(G)$ colors when the vertices of G are processed according to a linear order L with $1 + \Delta^+(G_L) = \mathrm{col}(G)$.

On the other hand, as the following trivial proposition makes clear, back degree is just a reformulation of density.

Proposition 3.1 *For every positive integer d and every graph G,* $\Delta^+(G) = \lfloor \mathrm{den}(G) \rfloor$*.* □

For example, the density of a planar graph is less than 6 by Euler's formula. Also, a planar graph has back degree at most 5.

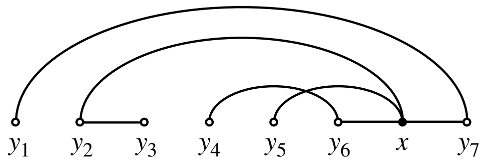

Figure 1: Reachability.

Next, we discuss a variant of back degree which will prove useful in analyzing competitive coloring problems for oriented graphs. Let L be a linear order on the vertex set V of a graph G. Then let $y < x$ in L. We say y is *reachable from x in L* when either (1) xy is an edge of G or (2) there is a vertex z with $y < z$ so that z is adjacent to both x and y. Note that we allow the common neighbor z to come before or after x, but it is not allowed to precede y. We denote the set of all vertices which are reachable from x as $R_{G_L}(x)$. When G and L are fixed, we will just write $R(x)$. In Figure 1, y_1, y_2, y_4, y_5 and y_6 are reachable from x, but y_3 and y_7 are not.

The concept of reachability is closely related to three other graph parameters which have been studied in game chromatic research: arrangeability, admissibility and rank. Arrangeability was introduced by Chen and Schelp [3] in a ramsey-theoretic setting, while admissibility was introduced by the authors in [9] and used to show that the game chromatic number of planar graphs is bounded. The notion of rank was introduced by Kierstead in [7] and used to provide the best bound to date (18) on the game chromatic number of planar graphs.

4 A Technical Lemma

In this section, we develop a technical lemma central to the proof of our principal theorem.

Let L be a linear order on the vertex set V of a graph G, and let E denote the edge set of G. Recall that for each vertex $x \in V$, $R_{G_L}(x)$ denotes the set of vertices which are reachable from x. Then $R_{G_L}(x) = N^+_{G_L}(x) \cup R'_{G_L}(x) \cup R''_{G_L}(x)$ where (1) $R'_{G_L}(x)$ denotes those vertices $y \in R_{G_L}(x)$ for which there is vertex $z \in V$ with $y < z < x$ in L, $xz \in E$ and $yz \in E$, i.e., $y \in N^+_{G_L}(z)$ and $z \in N^+_{G_L}(x)$, and (2) $R''_{G_L}(x)$ denotes those vertices $y \in R_{G_L}(x)$ for which there is vertex $z \in V$ with $y < x < z$ in L, $xz \in E$ and $yz \in E$, i.e., $y < x$ in L and $\{x, y\} \subseteq N^+_{G_L}(z)$.

Then set $R^2_{G_L}(x) = R_{G_L}(x) \cup \{y \in V : y < x \text{ in } L \text{ and there exists a vertex } z \in V \text{ so that } \{x, y\} \subseteq R_{G_L}(z)\}$.

Lemma 4.1 *Let $d \geq 100$ and let $\mathcal{C}$ be a topologically closed class of graphs with each graph in $\mathcal{C}$ having density at most d. Then for every graph $G \in \mathcal{C}$, there exists a linear order L on the vertex set V of G so that*

1. $|\Delta^+(G_L)| \leq d$.
2. $|R_{G_L}(x)| < d^5$ *for every $x \in V$.*

3. $|R^2_{G_L}(x)| < d^{18}$ *for every* $x \in V$.

Proof. Let E denote the edge set of G, and let $|V| = n$. Without loss of generality, $V = \{1, 2, \ldots, n\}$. We construct the desired linear order $L = [x_1, x_2, \ldots, x_n]$ on V in reverse order, beginning with the selection of x_n as a vertex of minimum degree in G. Of course, regardless of how the remainder of the linear order is determined, we know that $d^+_{G_L}(x_n) \le d$.

Now suppose that for some integer i with $1 \le i < n$, we have already selected the vertices in $C_i = \{x_{i+1}, x_{i+2}, \ldots, x_n\}$. Suppose further that these selections have been made so that $d^+_{G_L}(x_j) \le d$ for all $j = i+1, i+2, \ldots, n$.

Let $U_i = V - C_i$ denote the remaining vertices. We now describe the process by which $x_i \in U_i$ is chosen. We first define a probability space Ω_i, where each event in Ω_i is a graph H_i from class $\mathcal{C}$ having U_i as its vertex set. Furthermore, if $u, v \in U_i$ and $uv \in E$, then uv will always be an edge in H_i. Consequently, in what follows, we will concentrate on defining the "bonus" edges in the event H_i. These are edges of the form uv where u and v are distinct vertices of U_i and $uv \notin E$.

For each $j = i+1, i+2, \ldots, n$, we use the short form $N^+(x_j) = N^+_{G_L}(x_j)$ to denote the back neighbors of x_j. Note that elements of $N^+(x_j)$ can belong to U_i or C_i. Also, we let $R''(x_j)$ denote the set of all vertices y in $U_i \cup C_i$ so that (1) if $y = x_m \in C_i$, then $m < j$, and (2) there exists an integer $k > j$ so that $\{x_j, y\} \subseteq N^+(x_k)$.

Next, we define a "random" labelling of the elements of C_i and then use this labelling to determine the random graph H_i. We begin by selecting for each $x_j \in C_i$ a random digit from $\{1, 2, 3\}$ with all three digits being equally likely. Choices for distinct elements of C_i are independent.

Suppose that the digit for x_j is 1. If $N^+(x_j) = \emptyset$, label x_j with the pair $(1, \emptyset)$. If $N^+(x_j) \ne \emptyset$, choose an element $y \in N^+(x_j)$ at random, with all elements equally likely, and label x_j with the pair $(1, y)$.

Now suppose that the digit for x_j is 2. If $|N^+(x_j)| \le 1$, label x_j with the pair $(2, \emptyset)$. If $|N^+(x_j)| \ge 2$, choose a 2-element subset $\{y, z\}$ of $N^+(x_j)$ at random, with all 2-element subsets equally likely and label x_j with the pair $(2, \{y, z\})$.

Now suppose the digit for x_j is 3. If $|R''(x_j)| < 2$, assign x_j the label $(3, \emptyset)$. If $|R''(x_j)| \ge 2$, choose a distinct pair $y, z \in R''(x_j)$ at random, with all pairs equally likely, and label x_j with the pair $(3, \{y, z\})$.

Next, we describe how the labels assigned to the vertices in C_i are used to determine the random graph H_i. Start with the graph G. Then let $x_j \in C_i$. When x_j is labelled $(1, \emptyset)$, delete all edges of the form $x_j y$ where $y \in N^+(x_j)$ (if there are any). If x_j is labelled $(1, u)$, delete all edges of the form $x_j y$ where $y \in N^+(x_j)$ *except* the edge $x_j u$. If $u \in C_i$ and the digit of u is 2 or 3, delete the edge $x_j u$.

When x_j is labelled $(2, \emptyset)$, delete all edges $x_j y$ where $y \in N^+(x_j)$. If x_j is labelled $(2, \{y, z\})$, delete all edges of the form $x_j u$ where $u \in N^+(x_j)$ *except* $x_j y$ and $x_j z$. If the digit for y is 2, delete $x_j y$. If the digit for z is 2, delete the edge $x_j z$. If the digit of y is 3 and z is not a member of the second coordinate of the label of y, delete the edge $x_j y$. If the digit of z is 3 and y is not a member of the second coordinate of the label of z, delete the edge $x_j z$.

If the first coordinate of the label for x_j is 3, delete all edges of the form $x_j u$ where $u \in N^+(x_j)$.

For each pair $y, z \in V$, if there are two or more vertices labelled $(2, \{y, z\})$, delete all but the L-least one. If there are two or more vertices labelled $(3, \{y, z\})$, delete all but the L-least one. For each vertex x_j labelled $(1, u)$ with $u \in V$, contract the edge $x_j u$. After all such contractions have been made, let G' denote the resulting graph. It is easy to see that the vertices in U_i induce the same subgraph of G as they do in G'. Furthermore, all vertices in C_i whose digit is 1 have either been collapsed to a vertex in U_i or to an isolated component of G'.

Now let G'' denote the graph obtained from G' by deleting the vertices in U_i. Then each component of G'' is a path of at most 3 vertices. If a component has 2 vertices, one vertex has digit 2 and the other has digit 3. If a component has 3 vertices, then the two endpoints have digit 2 while the middle point has digit 3. Any edge linking a component path P with a vertex in U_i is incident with a point of P having digit 2. A component consisting of just one vertex whose digit is 2 can be linked to at most 2 vertices of U_i. A component path consisting of two vertices can be linked to at most one vertex in U_i. A component path of three vertices can be linked to at most two vertices in U_i and this occurs only if one endpoint is linked to one vertex in U_i and the other is linked to a second vertex in U_i.

Finally, we obtain the graph H_i from G' by:

1. Deleting all component paths of G'' which are not linked to two distinct vertices of U_i.

2. Deleting all component paths of G'' linked to distinct vertices u and v of U_i when $uv \in E$.

3. Contracting the edges on a component path P of G'' to form a "bonus" edge uv when P is linked to u and v in U_i and $uv \notin E$.

Evidently, H_i is a topological minor of G, so that $H_i \in \mathcal{C}$. Thus $\mathrm{den}(H_i) \leq d$. For each $x \in U_i$, let X_x be the random variable defined by setting X_x to be the degree of the vertex x in the random graph H_i. For each ordered pair (x, y) of distinct vertices in U_i, let X_{xy} be the random variable defined by setting $X_{xy} = 1$ when xy is an edge of H_i; otherwise, $X_{xy} = 0$. Then $X_x = \sum_{y \neq x} X_{xy}$.

Now let $E(X_x)$ be the expected value of X_x. Then $E(X_x) = \sum_{y \neq x} E(X_{xy})$. Also, since $H_i \in \mathcal{C}$, we know that $\sum_{x \in U_i} X_x \leq di$. Therefore $\sum_{x \in U_i} E(X_x) \leq di$. It follows that there exists a vertex $x \in U_i$ for which $E(X_x) \leq d$. Alice chooses such a vertex x from U_i and designates x to be x_i. Note that all edges in E_0 of the form ux where $u \in U$ also belong to H_i, so we know that $d^+_{G_L}(x_i) \leq d$.

This completes the description of the linear order L. The next step in the argument is to provide additional information on the special properties of L.

Let $x \in V$. Fixing the linear order L and the graph G, we will continue to use the short form $N^+(x) = N^+_{G_L}(x)$ for the set of back neighbors of x in L, but now we will also use the short form $R(x) = R_{G_L}(x)$ for the set of vertices reachable from x in L. We also set $R'(x) = R'_{G_L}(x)$, $R''(x) = R''_{G_L}(x)$, and $R^2(x) = R^2_{G_L}(x)$, so that $R(x) = N^+(x) \cup R'(x) \cup R''(x)$.

Claim 1. For every $x \in V$, $|R'(x)| \le d^2$.

Proof. The claim follows immediately from the fact that $|N^+(x)| \le d$ for every $x \in V$. $\triangle$

Claim 2. For every $x \in V$, $|R(x)| < d^5$

Proof. In view of Claim 1 and the fact that $|N^+(x)| \le d$, it suffices to show that $|R''(x)| < d^4$. We argue by contradiction. Suppose there exists a vertex $x \in V$ with $|R''(x)| \ge d^4$. Choose d^4 distinct elements $y_1, y_2, \ldots, y_{d^4}$ from $R''(x)$. Then for each $j = 1, 2, \ldots, d^4$, choose an element z_j so that $\{x, y_j\} \subseteq B(z_j)$. Now suppose that $x = x_i$ and consider the step at which $x = x_i$ was selected in the construction of L. At this point, $\{x\} \cup R''(x) \subseteq U_i$. It follows that xy_j is an edge in the random graph H_i whenever z_j is assigned the label $(2, \{x, y_j\})$. However, the probability that this label is assigned to z_j is at least $\frac{2}{3d(d-1)} > d^{-3}$. Thus

$$E(X_x) = \sum_{y \ne x} E(X_{xy}) \ge \sum_{j=1}^{d^4} E(X_{xy_j}) > d^4 \frac{1}{d^3} = d.$$

The contradiction shows that $|R''(x)| < d^4$ so that $|R(x)| < d^5$ as claimed. $\triangle$

For any $y \in R^2(x) - R(x)$, there is some z for which both x and y are reachable from z. Accordingly, $R^2(x) - R(x)$ is covered by the union of the following nine subsets;

1. $S_1 = \{y \in V : y < x$ in L and there exists $z \in V$ with $x \in N^+(z)$ and $y \in N^+(z)\}$.
2. $S_2 = \{y \in V : y < x$ in L and there exists $z \in V$ with $x \in N^+(z)$ and $y \in R'(z)\}$.
3. $S_3 = \{y \in V : y < x$ in L and there exists $z \in V$ with $x \in N^+(z)$ and $y \in R''(z)\}$.
4. $S_4 = \{y \in V : y < x$ in L and there exists $z \in V$ with $x \in R'(z)$ and $y \in N^+(z)\}$.
5. $S_5 = \{y \in V : y < x$ in L and there exists $z \in V$ with $x \in R'(z)$ and $y \in R'(z)\}$.
6. $S_6 = \{y \in V : y < x$ in L and there exists $z \in V$ with $x \in R'(z)$ and $y \in R''(z)\}$.
7. $S_7 = \{y \in V : y < x$ in L and there exists $z \in V$ with $x \in R''(z)$ and $y \in N^+(z)\}$.
8. $S_8 = \{y \in V : y < x$ in L and there exists $z \in V$ with $x \in R''(z)$ and $y \in R'(z)\}$.
9. $S_9 = \{y \in V : y < x$ in L and there exists $z \in V$ with $x \in R''(z)$ and $y \in R''(z)\}$.

These nine sets are illustrated in Figure 2, where we have displaced points vertically if there is some ambiguity as to their order in L. It is easy to see that $S_1 \subset R(x)$ and $S_3 = S_4$.

From Claim 2, we know that $|R(x)| < d^5$. To complete the proof of our lemma and show that $|R^2(x)| \le d^{18}$, we need only verify the following subclaims.

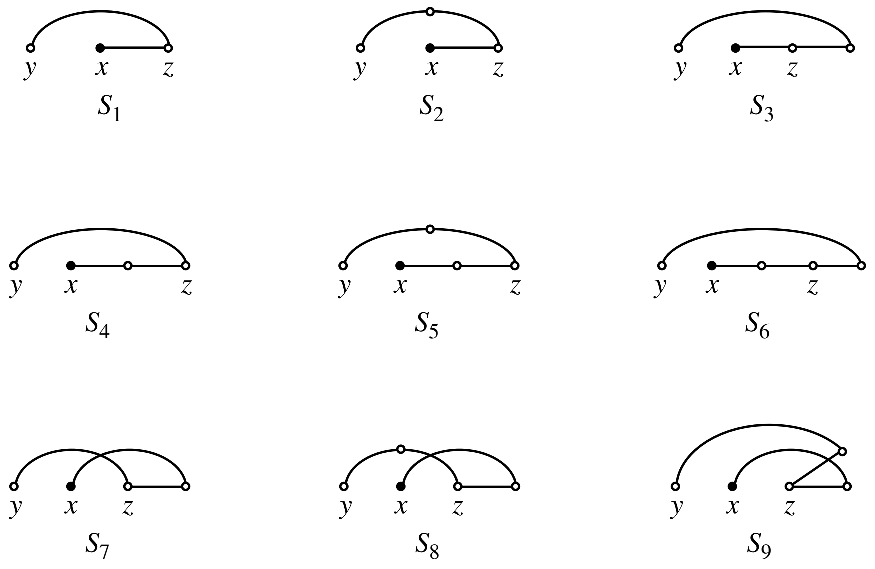

Figure 2: A Covering of $R^2(x) - R(x)$.

Subclaim a. $|S_2| < d^7$.

Subclaim b. $|S_3| < d^5$.

Subclaim c. $|S_5| < d^8$.

Subclaim d. $|S_6| < d^6$.

Subclaim e. $|S_7| < d^5$.

Subclaim f. $|S_8| < d^8$.

Subclaim g. $|S_9| < d^{17}$.

The arguments for these subclaims are quite similar and each continues in the same spirit as the proof of Claim 2. So we provide the details for Subclaims a, c and g, leaving the remaining four subclaims for the reader. The reader should note that the proof for Subclaim c requires the result for Subclaim b. Also, the proof of Subclaim f requires the result for Subclaim e.

Proof of Subclaim a. Suppose to the contrary that $|S_2| \geq d^7$. Choose d^7 distinct elements $y_1, y_2, \ldots, y_{d^7}$ from S_2. For each $j = 1, 2, \ldots, d^7$, choose an element z_j so that $x \in N^+(z_j)$ and $y_j \in R'(z_j)$. Then choose an element $w_j \in N^+(z_j)$ so that $y_j \in N^+(w_j)$.

Since $\Delta^+(G_L) \leq d$, we may assume that after relabelling $w_1, w_2, \ldots, w_{d^6}$ are distinct. Any w_j preceding x in L belongs to $R(x)$ and we know by Claim 2 that $|R(x)| < d^5$. So after another relabelling, we may assume that $w_1, w_2, \ldots, w_{d^5}$ all come after x in L.

Then consider the step in the construction of L at which Alice selects $x = x_i$. At that moment, w_j and z_j are elements of C_i for each $j = 1, 2, \ldots, d^5$. Furthermore, xy_j is an edge in the random graph H_i whenever z_j is labelled $(2, \{x, w_j\})$ and w_j is labelled $(1, y_j)$. For any value of j, the probability that both these labels are assigned is greater than d^{-4}.

It follows that follows that $E(X_x) > d^5 \frac{1}{d^4} = d$. The contradiction shows that $|S_2| < d^7$, as claimed.

Proof of Subclaim c. Suppose to the contrary that $|S_5| \geq d^8$. Choose d^8 distinct elements $y_1, y_2, \ldots, y_{d^8}$ from S_5. For each $j = 1, 2, \ldots, d^8$, choose an element z_j so that $\{x, y_j\} \subseteq R'(z_j)$. Then choose elements v_j and w_j so that $\{v_j, w_j\} \subseteq N^+(z_j)$, $x \in N^+(v_j)$ and $y_j \in N^+(w_j)$.

Since $\Delta^+(G_L) \leq d$, we may assume that after relabelling, $w_1, w_2, \ldots, w_{d^7}$ are distinct. Now any w_j which precedes x in L belongs to S_3 and assuming Subclaim b has been verified, we know that there are less than d^5 such vertices. After another relabelling, we may then assume that $w_1, w_2, \ldots, w_{d^6}$ are distinct and come after x in L.

Then consider the step at which Alice selects $x = x_i$. At that moment, for each $j \leq d^6$, xy_j is an edge in the random graph H_i whenever w_j is labelled $(1, y_j)$, v_j is labelled $(1, x)$ and z_j is labelled $(2, \{v_j, w_j\})$. The probability that all three of these labels are assigned is greater than d^{-5}. However, this implies that $E(X_x) > d^6 \frac{1}{d^5} = d$. The contradiction shows that $|S_5| < d^8$ as claimed.

Proof of Subclaim g. Suppose the subclaim is false and that $|S_9| \geq d^{17}$. Choose d^{17} distinct elements $y_1, y_2, \ldots, y_{d^{17}}$ from S_9. For each $j = 1, 2, \ldots, d^{17}$, choose z_j so that $\{x, y_j\} \subseteq R''(z_j)$. Then choose vertices v_j and w_j so that $\{x, z_j\} \subseteq N^+(v_j)$ and $\{z_j, y_j\} \subseteq N^+(w_j)$. If $w_j = v_j$, then $y_j \in S_1 = R(x)$, and since $|R(x)| < d^5$, we may assume that after relabelling $v_j \neq w_j$ for $j = 1, 2, \ldots, d^{16}$.

Then consider the step at which Alice selects $x = x_i$. At that moment, xy_j is an edge in the random graph H_i whenever z_j is labelled $(3, \{x, y_j\})$, v_j is labelled $(2, \{x, z_j\})$ and w_j is labelled $(2, \{y_j, z_j\})$. Since $|R(z_j)| < d^5$, the probability that all three are labelled in this manner is greater than d^{-15}. But this now implies that $E(X_x) > d$. The contradiction shows that $|S_9| < d^{17}$ as claimed. Also, assuming that the reader has verified the remaining four Subclaims, the proof of our lemma is complete. $\square$

5 Proof of the Main Theorem

In this section, we present the proof of the main theorem. Let $\mathcal{C}$ be a topologically closed class of graphs with $\mathrm{den}(G) \leq d$ for every graph $G \in \mathcal{C}$. In view of Nešetřil and Sopena's Theorem 2.1 and the fact that $\mathrm{eGo}(G) \leq \mathrm{Go}_2(G)$, it suffices to show that $\mathrm{Go}_2(G) \leq d^{22}$ for every $G \in \mathcal{C}$. To accomplish this, we consider the marking game discussed in Section 2 and describe a winning strategy for Alice when $t \geq d^{22}$.

Using the technical lemma of the preceding section, Alice chooses a linear order L on the vertex set V of G so that

1. $\Delta^+(G_L) \leq d$.
2. $|R_{G_L}(x)| < d^5$ for every $x \in V$.
3. $|R^2_{G_L}(x)| < d^{18}$ for every $x \in V$.

To aid in her decisions, Alice maintains a list of the vertices which have been marked and uses L to maintain a dynamic partition of the vertex set into two disjoint sets called *active* and *inactive*. She updates this partition immediately after each move she makes. After the updates are made, all marked vertices will be active, but unmarked vertices can be either active or inactive. However, once a vertex becomes active, it remains active forever.

At the start of the game, all vertices are unmarked and all are inactive. On Alice's first move, she marks the first (least) element of L. Of course, she immediately updates the partition by moving this vertex from inactive to active.

Now it is Bob's turn. Bob marks a second vertex of G. We denote this vertex as y_0. At this point, Alice employs a sequence of "tie-breaking" rules to determine her next move.

Let W denote the set of unmarked vertices—this definition is of course dynamic. Without loss of generality, $W \neq \emptyset$. Now let $R_W(y_0)$ denote the set of unmarked vertices which are reachable from y_0. If $R_W(y_0) = \emptyset$, then Alice marks the L-least element of W. She then updates the partition to reflect the fact that the vertices marked by Bob and Alice on their last turns are now active and marked.

Now suppose $R_W(y_0) \neq \emptyset$. Let y_1 denote the L-least element of $R_W(y_0)$. At this point, we describe Alice's strategy recursively. Suppose for some $j \geq 1$, Alice has defined a vertex and called it y_j. If y_j is active, Alice marks y_j. If y_j is inactive, she considers the set $R_W(y_j)$ of unmarked vertices which are reachable from y_j. If $R_W(y_j) = \emptyset$, then Alice marks y_j. If $R_W(y_j) \neq \emptyset$, let y_{j+1} denote the L-least element of $R_W(y_j)$.

This recursive procedure eventually results in the selection of some vertex y_t, where $t \geq 1$, which Alice then marks. Note that prior to marking y_t, it could have been either active or inactive. On the other hand, if $t > 1$, then $y_1, y_2, \ldots, y_{t-1}$ were all inactive. However, Alice now updates her records as follows. First, she adds y_0 and y_t to the list of marked vertices. If $t > 1$, she also changes the status of $y_1, y_2, \ldots, y_{t-1}$ from inactive to active. Of course, for the moment, $y_1, y_2, \ldots, y_{t-1}$ are still unmarked.

The remainder of the proof is devoted to showing that Alice's strategy actually works. Let x be an unmarked vertex. Recall that $A(x)$ denotes the set of all marked neighbors of x, while $A_2(x)$ denotes the set of all marked vertices y so that there exists an unmarked vertex z adjacent to both x and y. We now show that $|A(x) \cup A_2(x)| < d^{22}$. Let $A'(x) = \{y \in V : y \text{ is active and either } x \in R(y) \text{ or } y \in R(x)\}$.

Claim 1. At no point will there ever be an unmarked vertex x for which $|A'(x)| \geq d^{20}$.

Proof. We argue by contradiction. Suppose the claim is false. Consider a point in the game where there is an unmarked vertex x for which $|A'(x)| \geq d^{20}$. Since $|R(x)| < d^5$, we may assume that $A'(x)$ contains d^{19} distinct elements $y_1, y_2, \ldots, y_{d^{19}}$ with $x \in R(y_k)$ and y_k active for each $k = 1, 2, \ldots, d^{19}$. Without loss of generality, we may assume that these elements have been labelled in the order which they became active; in particular, $y_{d^{19}}$ was the last of these elements to become active. Then, for each $k = 1, 2, \ldots, d^{19} - 1$, there is a vertex $z_k \neq y_k$ with $z_k \in R(y_k)$ and $z_k \leq x$ in L so that either Alice marked z_k or Alice changed the status of z_k from inactive to active. Note that either $z_k = x$

or $z_k \in R^2(x)$. Also note that the elements in $\{z_1, z_2, \ldots, z_{d^{19}-1}\}$ need not be distinct. However, it is clear that there is at most one value of k for which $x = z_k$. Moreover, if $1 \le k_1 < k_2 < d^{19}$, and $z = z_{k_1} = z_{k_2}$, then $z \ne z_k$ for every $k = 1, 2, \ldots, d^{19} - 1$ except $k = k_1$ and $k = k_2$. In other words, for each $z \in V$, there are at most two values of k with $1 \le k < d^{19}$ so that $z = z_k$. It follows that $|R^2(x)| \ge d^{18}$. The contradiction completes the proof of the claim. △

We already know that $|R(x)| < d^5$. Furthermore, all marked vertices are active. Set

$$A''(x) = \big(A(x) \cup A_2(x)\big) - R(x) - A'(x).$$

To complete the proof of our theorem, it suffices to show that $|A''(x)| < d^{21}$. Suppose this inequality fails. Choose d^{21} distinct elements $y_1, y_2, \ldots, y_{d^{21}}$ from $A''(x)$. Note that for each $j = 1, 2, \ldots, d^{21}$, there is an unmarked vertex z_j adjacent to both x and y_j. Furthermore, we must have $z_j < x$ in L, else $y_j \in A'(x)$. However, since $\Delta^+(G_L) \le d$, it follows that there is an unmarked vertex $z \in N^+_{G_L}(x)$ so that at least d^{20} elements of $A''(x)$ are adjacent to z. However, all these elements belong to $A'(z)$ which then contradicts Claim 1 above. The contradiction completes the proof. □

6 Two Examples

In this section, we first present an example which shows that the game chromatic number of a class of graphs may be bounded even when the Go number is not.

Example 6.1 *Let G_n be the graph obtained by subdividing every edge of the complete balanced bipartite graph $K_{n,n}$. Then the acyclic chromatic number of G_n is 3, the game chromatic number of G_n is 4, but the Go number of G_n tends to infinity with n.*

Proof. Label the two parts of $K_{n,n}$ as A and B, and for each $a \in A$ and $b \in B$, let $s_{a,b}$ denote the vertex inserted on the edge ab. First note that we can obtain an acyclic 3-coloring of G_n by letting A, B, and $S = \{s_{a,b} : a \in A, b \in B\}$ be color classes. Alice can win the coloring game using the set $[4]$ of colors by ensuring that any uncolored vertex in A can be colored with 1 or 2, any uncolored vertex in B can be colored with 3 or 4, and any uncolored vertex in S can be colored with some color. Until $A \cup B$ is completely colored, Alice will only color vertices from $A \cup B$. Suppose Bob has just colored a vertex $s_{a,b}$ with α. If $\alpha \in \{1, 2\}$ and a is still uncolored, then Alice will color a with $3 - \alpha$. If $\alpha \in \{3, 4\}$ and b is still uncolored, then Alice will color b with $7 - \alpha$. Otherwise Alice will be able to color any uncolored vertex in A with 1 or 2 and any uncolored vertex in B with 3 or 4. After $A \cup B$ is completely colored, Alice can color S greedily. It is easy to check that these upper bounds are tight.

It remains to show that the Go number of $G = G_n$ tends to infinity with n. Let $n = 4^{2k+1}$. We shall describe a strategy for Bob that works in $2k$ rounds. At the end of the i-th round, $i \in [2k]$, G will contain an induced copy G^i of $G_{4^{2k+1-i}}$ such that none of the vertices of G^i has been marked. Let the vertices of G^i be $A' \cup B' \cup S'$ where $A' \subset A$,

$B' \subset B$, and $S' \subset S$. In addition we require that there exist two integers s and t with $i = s + t$ such that every vertex of A' has at least s marked neighbors in G and every vertex of B' has at least t marked neighbors in G. Then the score of the game will be greater than k. The first round is trivial since $G^0 = G$ works with $s = 0 = t$. So suppose that Bob has completed the i-th round and has obtained G^i, s, and t. Let $A_1 \subset A'$ have cardinality $\frac{|A'|}{2}$ and let f be a one-to-one correspondence between A' and B'. During the $(i+1)$-st round Bob marks the vertices in the set $\{s_{a,f(a)} : a \in A_1\}$ one at a time. While Bob is doing this, Alice can mark at most $\frac{|A'|}{2}$ vertices. So there exists a set D consisting of $\frac{|A'|}{4} = 4^{2k-i}$ unmarked vertices such that either $D \subset A_1$ or $D \subset f(A_1)$. Without loss of generality, assume that $D \subset f(A_1)$. There also exists a set $C \subset A - A_1$ such that $|C| = 4^{2k-i}$ and neither a nor $s_{a,b}$ is marked for all $a \in C$ and $b \in D$. Then the graph induced by $C \cup D \cup \{s_{a,b} : a \in C \text{ and } b \in D\}$ satisfies the requirements on G^i with $t := t + 1$. □

Our second example shows that the oriented game chromatic number of a class of oriented graphs may be bounded even when the extended Go number is not. This examples answers a second question of Nešetřil and Sopena [10].

Example 6.2 *Let G_n be the graph in Example 6.1. Consider G_n as an oriented graph with edges $(a, s_{a,b})$ and $(s_{a,b}, b)$ for each $a \in A, b \in B$. Then let H_n be the oriented graph obtained by subdividing each edge of G_n, inserting $r_{a,b}$ on the edge $(a, s_{a,b})$, and inserting $t_{a,b}$ on the edge $(s_{a,b}, b)$. For each $a \in A$ and $b \in B$, we have a directed path $a, r_{a,b}, s_{a,b}, t_{a,b}, b$ from a to b in H_n. Then the oriented game chromatic number of H_n is bounded, but the extended Go number of H_n tends to infinity with n.*

Proof. Bob can drive the extended Go number to infinity by playing in rounds and coloring vertices of S, essentially as in Example 6.1. It remains to check that the oriented game chromatic number of H_n is bounded. Let $T = ([k], F)$ be a tournament such that:

1. for all colors β there exists an even color α such that $(\alpha, \beta) \in F$;

2. for all colors β there exists an odd color γ such that $(\beta, \gamma) \in F$; and

3. for all distinct colors α and γ there exists a color β such that $(\alpha, \beta), (\beta, \gamma) \in F$.

An easy probabilistic argument shows that such a tournament exists. We claim that Alice can win the oriented coloring game on (H_n, T). Much as in Example 6.1, Alice tries to color the vertices in A with even colors and the vertices in B with odd colors. To accomplish this, she must be moderately careful. Until $A \cup B$ is completely colored, she only colors vertices in $A \cup B$. If Bob colors a vertex of the form $r_{a,b}$ and a is uncolored, she immediately colors a with an even color. Similarly, if Bob colors a vertex of the form $t_{a,b}$ and b is uncolored, she immediately colors b with an odd color. If Bob colors a vertex of the form $s_{a,b}$ with an even color and a is uncolored, she immediately colors a with an even color. If Bob colors a vertex of the form $s_{a,b}$ with an odd color and b is uncolored, she immediately colors b with an odd color. By Conditions 1 and 2, this is clearly possible.

It remains to show that Alice, and therefore Bob, will always be able to color an uncolored vertex $u \in X$. Since Alice will color $A \cup B$ first, we may assume that the unique inneighbor of u is colored, say with α. If the unique outneighbor v' of u is also colored, then Alice is safe by Condition 3. Otherwise, if the unique outneighbor v'' of v' is uncolored, then Alice is safe by condition 2. This leaves only the possibility that v' is uncolored, but v'' is colored, say with γ. If $\alpha = \gamma$, Alice is still safe by Condition 2. Otherwise, apply Condition 3 to α and γ to obtain a color β such that $(\alpha, \beta), (\beta, \gamma) \in F$. Then $\beta \neq \gamma$ and so Alice can color u with β. $\square$

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