Resolving Triple Systems into Regular Configurations

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Abstract

A $\lambda - Triple\ System(v)$, or a λ–$TS(V,\mathcal{B})$, is a pair (V, $\mathcal{B}$) where V is a set and $\mathcal{B}$ is a subset of the 3-subsets of V so that every pair is in **exactly** λ elements of $\mathcal{B}$. A *regular configuration* on p points with regularity ρ on l blocks is a pair (P,$\mathcal{L}$) where $\mathcal{L}$ is a collection of 3-subsets of a (usually small) set P so that every p in P is in **exactly** ρ elements of $\mathcal{L}$, and $|\mathcal{L}| = l$. The Pasch configuration $(\{0,1,2,3,4,5\},\{012,035,245,134\})$ has p=6, l=4, and ρ=2. A λ–$TS(V,\mathcal{B})$, is resolvable into a regular configuration $\mathbb{C}$=(P,$\mathcal{L}$), or $\mathbb{C}$–resolvable, if $\mathcal{B}$ can be partitioned into sets Π_i so that for each i, (V,Π_i) is isomorphic to a set of vertex disjoint copies of (P,$\mathcal{L}$). If the configuration is a single block on three points this corresponds to ordinary resolvability of a Triple System.

In this paper we examine all regular configurations $\mathbb{C}$ on 6 or fewer blocks and construct $\mathbb{C}$–resolvable λ–Triple Systems of order v for many values of v and λ. These conditions are also sufficient for each $\mathbb{C}$ having 4 blocks or fewer. For example for the Pasch configuration $\lambda \equiv 0 \pmod 4$ and $v \equiv 0 \pmod 6$ are necessary and sufficient. *MRSC #05B07*

1 Introduction

The study of the way in which small configurations are germane to analysing the structure of combinatorial objects has progressed from the study of finite geometries

[7] (for example Desargues and Pappus configurations) to using small configurations in the analysis of other designs. The concepts of avoidance of[1, 13] , ubiquity of [16], decomposability into[10] , and bases for[9],small configurations , have all provided insights into the structure of designs.

On the other hand resolvability and λ–resolvability have had a similar but much longer history starting from Euclid's fifth postulate to through the end of the Euler conjecture and to the present.[6]

In this work we shall combine the two ideas into the concept of $\mathbb{C}$–Resolvable triple systems. We start with the following basic definitions:

Definition 1.1 *A λ–Triple System(v), a $\lambda - TS(V, \mathcal{B})$, is a pair (V, $\mathcal{B}$) where V is a v–set and $\mathcal{B}$ is a subset of the 3-subsets of V so that every pair is in* **exactly** *λ elements of $\mathcal{B}$.*

Definition 1.2 *A regular configuration on p points with regularity ρ on b blocks is a pair (P,$\mathcal{L}$) where $\mathcal{L}$ is a collection of 3-subsets of a (usually small) set P so that every p in P is in* **exactly** *ρ elements of $\mathcal{L}$, and $|\mathcal{L}| = l$.*

The Pasch configuration $(\{0, 1, 2, 3, 4, 5\}, \{012, 035, 125, 134\})$ has p=6, l=4, and ρ=2.

Definition 1.3 *A $\mathbb{C}$–parallel (or resolution) class of size $v = pt$ is a set of v points together with a collection of lt lines which is isomorphic to t vertex disjoint copies of $\mathbb{C}$*

Definition 1.4 *A λ-TS(V,$\mathcal{B}$), is resolvable into a regular configuration $\mathbb{C}$= (P,$\mathcal{L}$) if $\mathcal{B}$ can be partitioned into sets Π_i parallel classes $i = 1, 2, \cdots \frac{b}{lt}$, or more simply, a triple system is called $\mathbb{C}$–resolvable iff its blocks can be partitioned into disjoint $\mathbb{C}$–parallel classes.*

If the configuration is a single block on three points this corresponds to ordinary resolvability of a triple system. On the other hand if $\mathbb{C}$ is itself a λ–$TS(k)$, the existence of $\mathbb{C}$-resolvable resolvable $\lambda \times \mu$–$TS(v)$ is equivalent to the existence of a resolvable balanced incomplete block design $RBIBD(v, k, \mu)$. This frames the existence problem for $\mathbb{C}$–resolvable triple systems between the concept of resolvable triple systems and resolvable block designs of other block sizes. Since not much is known about resolvable block designs with $k \geq 7$ perhaps the intermediate problem of $\mathbb{C}$–resolvable triple systems with a small number of lines will shed some light on the general problem.

We shall use $\mathbb{C}$ for a configuration with p for the number of points and l for the number of lines and regularity ρ . Further we define λ_{max} to be the maximal number of lines that any pair occurs in. Similarly rep_{max} will denote the maximal number of times a block is repeated.

Lemma 1.1 *The necessary conditions for a λ–$TS(v)$ to be $\mathbb{C}$–resolvable are*

1. *$v \equiv 0 \pmod p$*

2. *$\lambda(v-1) \equiv 0 \pmod 2$*

3. *$\lambda \geq \lambda_{max}$*

4. *Let $v = tp$ then $\lambda p(pt-1) \equiv 0 \pmod{6l}$*

5. *If $\mathbb{C}$=(P,$\mathcal{L}$) where $\mathcal{L}$ consists of m copies of the set $\mathcal{L}$' then necessary (and sufficient) conditions for $\mathbb{C}$ are those of $\mathbb{C}$' with "λ" replaced by "$m\lambda$"*

Proof: 1 , 2 , 3 and 5 are trivial. The number of blocks in the λ–$TS(v)$ is $\frac{\lambda pt(pt-1)}{6}$ which must be divisible by the number of blocks in a parallel class which is tl.

■

The solutions to the equation

$$3l = p\rho$$

will be useful in classifying the regular configurations.

2 $\mathbb{C}$–Resolvable Group Divisible Designs

In order to construct the desired triple systems we shall need two auxiliary concepts. We recall the standard definition of a $k - GDD_\lambda(g, n)$.

Definition 2.1 *A $k - GDD_\lambda(g, n)$ is a set V partitioned into n, g-sets G_i called groups together with a collection $\mathcal{B}$ of k-subsets called blocks so that*

1. *every 2-subset (pair) of elements of V which are from different groups are a subset of* **exactly** *λ blocks*

2. *and* **no** *block contains two elements from the same group.*

Definition 2.2 *A resolvable $k - GDD_\lambda(g, n)$ is a $k - GDD_\lambda(g, n)$ where $\mathcal{B}$ can be partitioned into parallel classes i.e each class contains every point exactly once.*

Definition 2.3 *A $k - GDD_\lambda(g, n)$ is $\mathbb{C}$–resolvable when $\mathcal{B}$ can be partitioned into $\mathbb{C}$–parallel classes.*

For this paper, we shall always have $k = 3$ and may omit it from the notation; we may also omit λ when $\lambda = 1$.

The constructions will be based on the following variants of Wilson's Theorem.

Theorem 2.1 (Master by Ingredient) *Let* $(V_M, \mathcal{B}_M)$ *be a resolvable* *3–*$GDD_\lambda(g, n)$, (called the master) *and* $(V_I, \mathcal{B}_I)$ *be a* $\mathbb{C}$*–resolvable* *3–*$GDD_\mu(h, 3)$ (called the ingredient) *then there exists a* $\mathbb{C}$*–resolvable* *3–*$GDD_{\lambda\times\mu}(gh, n)$.

Theorem 2.2 (Filling in groups) *Let* $(V, \mathcal{B})$ *be a* $\mathbb{C}$*–resolvable* *3–*$GDD_\lambda(g, n)$ *and* $(D, \mathcal{B}_D)$ *be* $\mathbb{C}$*–resolvable* λ*–*TS *with* $|D| = g$. *Then there exists a* $\mathbb{C}$*–resolvable* λ*–*$TS(gn)$ *there exists.*

The proofs of the above theorems are routine exercises based on the proofs of the original theorems found in the introductory chapter of [8].

Sometimes we have the fortuitous situation of what we shall call an μ*–auto ingredient configuration.* That is a situation where the configuration $\mathbb{C} = (P, \mathcal{L})$ is a $\mathbb{C}$–parallel class of a $\mathbb{C}$–resolvable 3–$GDD_\mu(g, 3), 3g = |P|$. We give 3 examples:

Example 2.1 *The trivial examples of the* r*–repeated block*

$$\mathbb{C} = (\{1, 2, 3\}, \underbrace{\{123, 123 \cdots 123\}}_{\text{r times}})$$

is a $\mathbb{C}$*–resolvable 3–*$GDD_r(1, 3)$.

Example 2.2 $\mathbb{C}_{4.6.2}$ *or Pasch*

$$P = \{1, 2, 3, 4, 5, 6\} \text{ and } \mathcal{L} = \{125, 146, 326, 345\}$$

This is also a $\mathbb{C}$*–resolvable 3–*$GDD_1(2, 3)$ *with groups* $\{1, 3\},\ \{2, 4\},\ \{5, 6\}$.

Example 2.3 $\mathbb{C}_{4.6.3}$ *or FIFA*

$$P = \{1, 2, 3, 4, 5, 6\} \text{ and } \mathcal{L} = \{125, 126, 346, 345\}$$

This forms one $\mathbb{C}$*–resolvable class of 3–*$GDD_2(2, 3)$ *with groups* $\{1, 3\}$, $\{2, 4\}$, $\{5, 6\}$. *The other is* $\{145, 146, 236, 235\}$.

Corollary 2.1 *If* $\mathbb{C}$ *is an* μ *auto-ingredient configuration* $(P, \mathcal{L})$ *with* $|P| = 3g$ *and there exist a resolvable* λ*–*$TS(w)$ *and a* $\mathbb{C}$*–resolvable* μ*–*$TS(3g)$, *then there exists a* $\mathbb{C}$*–resolvable* $\lambda \times \mu$*–*$TS(gw)$.

3 The regular configurations on 6 or fewer lines

3.1 Enumeration and Necessity

We shall now enumerate all regular configurations on six or fewer lines and give necessary conditions for the existence of a $\mathbb{C}$–resolvable λ–$TS(v)$.

We shall number the configurations by $\mathbb{C}_{l.p.n}$, where l is the number of lines, p the number of points, and n the number of the configuration.

Lemma 3.1 *The enumeration of the regular configurations with $l \leq 3$ lines is as follows*

The case l**=1**

$\mathbb{C}_{1.3.1}$ $P = \{1, 2, 3\}$ *and* $\mathcal{L} = \{123\}$.

A $\mathbb{C}_{1.3.1}$-resolvable λ–$TS(v)$ is just a resolvable triple system for which the necessary conditions are $v \equiv 0 \pmod 3$ and λ even if v is even .

The case l**=2**. *In this case there are two configurations*

$\mathbb{C}_{2.3.1}$ $P = \{1, 2, 3\}$ *and* $\mathcal{L} = \{123, 123\}$

A $\mathbb{C}_{2.3.1}$–resolvable λ–$TS(v)$ is just a resolvable triple system with every block repeated. The necessary conditions are $v \equiv 0 \pmod 3$ and $\lambda \equiv 0 \pmod 2$ if v is odd and $\lambda \equiv 0 \pmod 4$ if v is even.

$\mathbb{C}_{2.6.1}$ $P = \{1, 2, 3, 4, 5, 6\}$ *and* $\mathcal{L} = \{123, 456\}$

A $\mathbb{C}_{2.6.1}$–resolvable λ–$TS(v)$ is just a resolvable triple system with an even number of blocks and the necessary conditions are $v \equiv 0 \pmod 6$ and $\lambda \equiv 0 \pmod 2$.

The case l**=3**

$\mathbb{C}_{3.3.1}$ $P = \{1, 2, 3\}$ *and* $\mathcal{L} = \{123, 123, 123\}$ *A $\mathbb{C}_{3.3.1}$–resolvable λ–$TS(v)$ is just a resolvable triple system with every block repeated 3 times. The necessary conditions are $v \equiv 0 \pmod 3$ and $\lambda \equiv 0 \pmod 3$ if v is odd and $\lambda \equiv 0 \pmod 6$ if v is even.*

$\mathbb{C}_{3.9.1}$ $P = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *and* $\mathcal{L} = \{123, 456, 789\}$

A $\mathbb{C}_{3.9.1}$–resolvable λ–$TS(v)$ is just a resolvable triple system whose number of blocks is divisible by 3. The necessary conditions are $v \equiv 0 \pmod 9$.

Lemma 3.2 *There are six regular configurations with four lines and the necessary conditions for the existence of a $\mathbb{C}_{4.x}$–resolvable λ–$TS(v)$, say $\mathcal{B}_{4.x}$, are as follows:*

$\mathbb{C}_{4.3.1}$

$$P = \{1,2,3\} \textit{ and } \mathcal{L} = \{123, 123, 123, 123\}$$

$v \equiv 0 \pmod 3$ *and* $\lambda \equiv 0 \pmod 4$ *if* v *odd,* $\lambda \equiv 0 \pmod 8$ *if* v *even.*

$\mathbb{C}_{4.4.1}$ **or** $2\mathbb{K}_4$

$$P = \{1,2,3,4\} \textit{ and } \mathcal{L} = \{123, 234, 341, 412\}$$

$v \equiv 4 \pmod{12}, \lambda \equiv 2,4 \pmod 6$ *and* $v \equiv 0 \pmod 4, \lambda \equiv 0 \pmod 6$

$\mathbb{C}_{4.6.1}$

$$P = \{1,2,3,4,5,6\} \textit{ and } \mathcal{L} = \{123, 123, 456, 456\}$$

$v \equiv 0 \pmod 6, \lambda \equiv 0 \pmod 4$

$\mathbb{C}_{4.6.2}$ **or Pasch**

$$P = \{1,2,3,4,5,6\} \textit{ and } \mathcal{L} = \{125, 146, 326, 345\}$$

$v \equiv 0 \pmod 6, \lambda \equiv 0 \pmod 4$

$\mathbb{C}_{4.6.3}$ **or FIFA**

$$P = \{1,2,3,4,5,6\} \textit{ and } \mathcal{L} = \{125, 126, 346, 345\}$$

$v \equiv 0 \pmod 6, \lambda \equiv 0 \pmod 4$

$\mathbb{C}_{4.12.1}$

$$P = \{1,2,3,4,5,6,7,8,9,A,B,C\} \textit{ and } \mathcal{L} = \{123, 456, 789, ABC\}$$

$v \equiv 0 \pmod{12}$, $\lambda \equiv 0 \pmod 2$

Lemma 3.3 *There are four regular configurations with five lines and the necessary conditions for the existence of a* $\mathbb{C}_{5.x}$*–resolvable* λ*–*$TS(v)$*, say* $\mathcal{B}_{5.x}$*, are as follows:*

$\mathbb{C}_{5.3.1}$

$$P = \{1,2,3\} \textit{ and } \mathcal{L} = \{123, 123, 123, 123, 123\}$$

$v \equiv 0 \pmod 6, \lambda \equiv 0 \pmod{10}$ *and* $v \equiv 3 \pmod 6, \lambda \equiv 0 \pmod 5$

$\mathbb{C}_{5.5.1}$

$$P = \{1, 2, 3, 4, 5\} \textit{ and } \mathcal{L} = \{123, 123, 145, 245, 345\}$$

$v \equiv 0 \pmod 5, \lambda \equiv 0 \pmod 6$;
$v \equiv 10 \pmod{15}, \lambda \equiv 2, 4 \pmod 6,\ \lambda \geq 3$;
$v \equiv 5 \pmod{10}, \lambda \equiv 3 \pmod 6$;
$v \equiv 10 \pmod{15}, \lambda \equiv 1, 5 \pmod 6,\ \lambda \geq 3$.

$\mathbb{C}_{5.5.2}$

$$P = \{1, 2, 3, 4, 5\} \textit{ and } \mathcal{L} = \{123, 124, 135, 245, 345\}$$

$v \equiv 0 \pmod 5, \lambda \equiv 0 \pmod 6$; $v \equiv 10 \pmod{15}, \lambda \equiv 2, 4 \pmod 6$; $v \equiv 5 \pmod{10}, \lambda \equiv 3 \pmod 6$;
$v \equiv 10 \pmod{15}, \lambda \equiv 1, 5 \pmod 6,\ \lambda \geq 2$.

$\mathbb{C}_{5.15.1}$

$$P = \{1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$$

and

$$\mathcal{L} = \{123, 456, 789, ABC, DEF\}$$

$v \equiv 15 \pmod{30}$, *any* λ, *and* $v \equiv 0 \pmod{30}, \lambda \equiv 0 \pmod 2$

In order to distinguish the isomorphism classes for $\mathbb{C}_{6.6.x}$ and $\mathbb{C}_{6.9.x}$, we shall use the invariants of number of repeated blocks, number of repeated pairs and the maximal number of disjoint blocks in the configuration.

Lemma 3.4 *There are 18 regular configurations with six lines and the necessary conditions for the existence of a $\mathbb{C}_{6.x}$–resolvable λ–$TS(v)$, say $\mathcal{B}_{6.x}$, are as follows:*

$\mathbb{C}_{6.3.1}$

$$P = \{1, 2, 3\} \textit{ and } \mathcal{L} = \{123, 123, 123, 123, 123, 123\}$$

$$v \equiv 0 \pmod 3, \lambda \equiv 0 \pmod 6$$

$\mathbb{C}_{6.6.1}$

$$P = \{1, 2, 3, 4, 5, 6\} \textit{ and } \mathcal{L} = \{123, 123, 123, 456, 456, 456\}$$

$$v \equiv 0 \pmod 6,\ \lambda \equiv 0 \pmod 6$$

$\mathbb{C}_{6.6.2}$

$$P = \{1,2,3,4,5,6\} \textit{ and } \mathcal{L} = \{123,123,134,256,456,456\}$$

$$v \equiv 0 \pmod 6,\ \lambda \equiv 0 \pmod 6$$

$\mathbb{C}_{6.6.3}$

$$P = \{1,2,3,4,5,6\} \textit{ and } \mathcal{L} = \{123,124,135,236,456,456\}$$

$$v \equiv 0 \pmod 6,\ \lambda \equiv 0 \pmod 6$$

$\mathbb{C}_{6.6.4}$

$$P = \{1,2,3,4,5,6\} \textit{ and } \mathcal{L} = \{123,124,134,256,356,456\}$$

$$v \equiv 0 \pmod 6,\ \lambda \equiv 0 \pmod 6$$

$\mathbb{C}_{6.6.5}$

$$P = \{1,2,3,4,5,6\} \textit{ and } \mathcal{L} = \{123,124,135,246,356,456\}$$

$$v \equiv 0 \pmod 6,\ \lambda \equiv 0 \pmod 6$$

$\mathbb{C}_{6.6.6}$

$$P = \{1,2,3,4,5,6\} \textit{ and } \mathcal{L} = \{123,124,135,346,256,456\}$$

$$v \equiv 0 \pmod 6,\ \lambda \equiv 0 \pmod 6$$

$\mathbb{C}_{6.6.7}$

$$P = \{1,2,3,4,5,6\} \textit{ and } \mathcal{L} = \{123,124,156,256,345,346\}$$

$$v \equiv 0 \pmod 6,\ \lambda \equiv 0 \pmod 6$$

$\mathbb{C}_{6.9.1}$

$$P = \{1,2,3,4,5,6,7,8,9\} \textit{ and } \mathcal{L} = \{123,123,456,456,789,789\}$$

$v \equiv 0 \pmod 9,\ \lambda \equiv 0 \pmod 4$ *and*
$v \equiv 9 \pmod{18},\ \lambda \equiv 2 \pmod 4$

$\mathbb{C}_{6.9.2}$

$$P = \{1,2,3,4,5,6,7,8,9\} \textit{ and } \mathcal{L} = \{123,123,456,457,689,789\}$$

$v \equiv 9s \pmod{36}$ $\lambda \equiv 0 \pmod 4, s = 0,2;$
$\lambda \equiv 0 \pmod 2, s = 3; \lambda \geq 2, s = 1$

$\mathbb{C}_{6.9.3}$

$$P = \{1,2,3,4,5,6,7,8,9\} \textit{ and } \mathcal{L} = \{123,124,356,457,689,789\}$$

$v \equiv 9s \pmod{36}$ $\lambda \equiv 0 \pmod 4, s = 0,2;$
$\lambda \equiv 0 \pmod 2, s = 3; \lambda \geq 2, s = 1$

$\mathbb{C}_{6.9.4}$

$$P = \{1,2,3,4,5,6,7,8,9\} \textit{ and } \mathcal{L} = \{123,124,367,489,567,589\}$$

$v \equiv 9s \pmod{36}$ $\lambda \equiv 0 \pmod 4, s = 0,2;$
$\lambda \equiv 0 \pmod 2, s = 3; \lambda \geq 2, s = 1$

$\mathbb{C}_{6.9.5}$

$$P = \{1,2,3,4,5,6,7,8,9\} \textit{ and } \mathcal{L} = \{123,124,367,489,568,579\}$$

$v \equiv 9s \pmod{36}$ $\lambda \equiv 0 \pmod 4, s = 0,2;$
$\lambda \equiv 0 \pmod 2, s = 3; \lambda \geq 2, s = 1$

$\mathbb{C}_{6.9.6}$

$$P = \{1,2,3,4,5,6,7,8,9\} \textit{ and } \mathcal{L} = \{123,145,246,379,578,689\}$$

$v \equiv 9s \pmod{36}$ $\lambda \equiv 0 \pmod 4, s = 0,2;$
$\lambda \equiv 0 \pmod 2, s = 3; \textit{any } \lambda,\ s = 1$

$\mathbb{C}_{6.9.7}$

$$P = \{1,2,3,4,5,6,7,8,9\} \textit{ and } \mathcal{L} = \{123,145,267,367,489,589\}$$

$v \equiv 9s \pmod{36}$ $\lambda \equiv 0 \pmod 4, s = 0,2;$
$\lambda \equiv 0 \pmod 2, s = 3; \lambda \geq 2, s = 1$

$\mathbb{C}_{6.9.8}$

$$P = \{1,2,3,4,5,6,7,8,9\} \textit{ and } \mathcal{L} = \{123,145,267,389,468,579\}$$

$v \equiv 9s \pmod{36}$ $\lambda \equiv 0 \pmod 4, s = 0,2;$
$\lambda \equiv 0 \pmod 2, s = 3; \textit{any } \lambda,\ s = 1$

$\mathbb{C}_{6.9.9}$

$$P = \{1,2,3,4,5,6,7,8,9\} \textit{ and } \mathcal{L} = \{123,123,456,478,579,689\}$$

$v \equiv 9s \pmod{36}$ $\lambda \equiv 0 \pmod 4, s = 0, 2;$
$\lambda \equiv 0 \pmod 2, s = 3; \lambda \geq 2, s = 1$

$\mathbb{C}_{6.18.1}$

$$P = \{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r\} \textit{ and}$$
$$\mathcal{L} = \{abc, def, ghi, jkl, mno, pqr\}$$

$v \equiv 0 \pmod{18}$, $\lambda \equiv 0 \pmod 2$

3.2 Necessary and Sufficient conditions for all $l \leq 4$ and some $l = 5, 6$

Theorem 3.1 *The necessary conditions for the following $\mathbb{C}$–resolvable designs to exist are sufficient with the addition of $v \neq 6$, $v \neq 6$ and $\lambda \equiv 2 \pmod 4$, $v \neq 6$ and $\lambda \equiv 6 \pmod{12}$ to those marked respectively with a "*", "**", "***":*

Configuration	*Note*	*Configuration*	*Note*	*Configuration*	*Note*
$\mathcal{B}_{1.3.1}$ *	*1*	$\mathcal{B}_{2.3.1}$ *	*3*	$\mathcal{B}_{2.6.1}$ **	*2,3*
$\mathcal{B}_{3.3.1}$ *	*1*	$\mathcal{B}_{3.9.1}$	*2*	$\mathcal{B}_{4.3.1}$ *	*2*
$\mathcal{B}_{4.4.1}$	*2*	$\mathcal{B}_{4.6.1}$	*2,3*	$\mathcal{B}_{4.12.1}$	*1*
$\mathcal{B}_{5.3.1}$ *	*3*	$\mathcal{B}_{5.15.1}$	*1*	$\mathcal{B}_{6.3.1}$ *	*3*
$\mathcal{B}_{6.6.1}$ ***	*2,3*	$\mathcal{B}_{6.9.1}$	*2,3*	$\mathcal{B}_{6.18.1}$	*1*

Proof: The desired $\mathbb{C}$–resolvable design is equivalent to the existence of a RBIBD whose number of blocks is a multiple of the number of blocks in the former and whose λ is a divisor of the former because

1. A parallel class of the RBIBD can be partitioned to form a $\mathbb{C}$–resolvable parallel class.

2. Some multiple of each of the RBIBD can be partitioned into copies of $\mathbb{C}$.

3. A $\mathbb{C}$ parallel class is just an RBIBD parallel class with each block repeated μ times.

The "Note" indicates which reason(s) should be used for the given configuration.

■

Theorem 3.2 *The necessary conditions for the existence of a $\mathcal{B}_{4.6.2}$ and a $\mathcal{B}_{4.6.3}$ are sufficient except possibly if $v = 12$.*

Proof: It is well-known that a 3–$RGDD(3, n)$ (or a Kirkman triple system of order $3n$) exists if and only if $n \equiv 1 \pmod 2$ and also that a 3–$RGDD_2(3, n)$ exists for all integers $n \neq 2$. We use the master by ingredient construction using for a master a 3–$RGDD(3, n)$ if $v \equiv 6 \pmod{12}$ and a 3–$RGDD_2(3, n)$ if $v \equiv 0 \pmod{12}$, $v \geq 24$. For auto-ingredient use example 2.2 (taken 4 times in the first case and 2 times in the second one) for the Pasch and example 2.3 (taken twice in the first case and 1 time in the second one) for the FIFA.

In order to fill in groups we need $\mathbb{C}_{4.6.2 \text{ and } .3}$ resolvable designs:

$\mathcal{B}_{4.6.2}$, $V = Z_5 \cup \{\infty\}$, $\lambda = 4$. The 5 $\mathbb{C}$–parallel classes are:
$\big\{\{\infty, 1+i, 3+i\},\ \{\infty, 2+i, 4+i\},\ \{0+i, 1+i, 2+i\},$
$\{0+i, 3+i, 4+i\}\ mod\ 5\big\}, i \in Z_5$

$\mathcal{B}_{4.6.3}$, $V = Z_5 \cup \{\infty\}$, $\lambda = 4$. The 5 $\mathbb{C}$–parallel classes are
$\big\{\{\infty, 0+i, 1+i\},\ \{\infty, 2+i, 4+i\},\ \{0+i, 1+i, 4+i\},$
$\{0+i, 2+i, 4+i\}\ mod\ 5\big\}, i \in Z_5$

Theorem 3.3 *If there is a $RBIBD(v, 5, \mu)$ then there is a $\mathcal{B}_{5.5.x}$ for the following values of x and lambda: $x = 1$ and $\lambda \equiv 0 \pmod{6\mu}$; $x = 2$ and $\lambda \equiv 0 \pmod{3\mu}$.*

Proof: The existence of a RBIBD$(v, 5, \mu)$ is known in many cases, see [6] for a survey of the results. The proof follows from the existence of a $\mathcal{B}_{5.5.x}$,and using one parallel class of blocks as the groups to create the master RGDD needed. $x = 1$ and $\lambda = 6$, $x = 2$ and $\lambda = 3$.

$\mathcal{B}_{6.5.1}$, $V = Z_5$, $\lambda = 6$. The 4 $\mathbb{C}$–parallel classes are:

$$\{032, 032, 014, 214, 314\},$$

$$\{012, 012, 034, 134, 234\},$$

$$\{123, 123, 104, 204, 304\},$$

$$\{013, 013, 024, 124, 324\}.$$

$\mathcal{B}_{6.5.2}$, $V = Z_5$, $\lambda = 3$. The 2 $\mathbb{C}$–parallel classes are:

$$\{032, 034, 021, 341, 241\},$$

$$\{041, 042, 013, 423, 123\}.$$

3.3 Further sufficient conditions for $l = 6$

In this ,section we examine some sufficient conditions which fall short of the necessary conditions. In each case there is a range of uncertainty which further woek may narrow.

Theorem 3.4 *For each* $v \equiv 6 \pmod{12}$, $\lambda \equiv 0 \pmod 6$ *and* $v \equiv 0 \pmod{12}$, $v \geq 24$, $\lambda \equiv 0 \pmod{12}$, *there exists a* $\mathcal{B}_{6.6.3}$.

Proof: We proceed as in Theorem 3.2 by using the same master, the following 6-auto ingredient configuration and $\mathcal{B}_{6.6.3}$ with $v = 6$, $\lambda = 6$.

A 6-auto ingredient configuration $\mathbb{C}_{6.6.3}$:

$V = Z_6$. The groups are: $\{0, 3\}$, $\{1, 4\}$, $\{2, 5\}$. The 4 $\mathbb{C}$–parallel classes are:

$$\{012, 015, 024, 123, 534, 534\},$$

$$\{312, 315, 324, 120, 045, 045\},$$

$$\{315, 312, 354, 150, 024, 024\},$$

$$\{015, 012, 054, 153, 234, 234\}.$$

$\mathcal{B}_{6.6.3}$, $V = Z_5 \cup \{\infty\}$, $\lambda = 6$. The 5 $\mathbb{C}$–parallel classes are:

$\big\{\{\infty, 0+i, 1+i\}, \{\infty, 0+i, 2+i\}, \{\infty, 1+i, 4+i\},$
$\{0+i, 1+i, 3+i\}, \{2+i, 3+i, 4+i\},$
$\{2+i, 3+i, 4+i\}\ mod\ 5\big\}, i \in Z_5$

■

Theorem 3.5 *For each* $v \equiv 0 \pmod 6$, $\lambda \equiv 0 \pmod{12}$, $v \neq 24$, *there exists a* $\mathcal{B}_{6.6.2}$ *and a* $\mathcal{B}_{6.6.5}$.

Proof: A 6-auto ingredient configuration $\mathbb{C}_{6.6.2}$:

$V = Z_6$. The groups are: $\{0, 3\}$, $\{1, 4\}$, $\{2, 5\}$. The 4 $\mathbb{C}$–parallel classes are:

$$\{012, 012, 024, 135, 435, 435\},$$

$$\{312, 312, 324, 105, 405, 405\},$$

$$\{015, 015, 054, 123, 423, 423\},$$

$$\{315, 315, 354, 102, 402, 402\}.$$

$\mathcal{B}_{6.6.2}$, $V = Z_5 \cup \{\infty\}$, $\lambda = 12$. The 10 $\mathbb{C}$–parallel classes are:
$\big\{\{\infty, 0+i, 1+i\},\ \{\infty, 0+i, 1+i\},\ \{\infty, 1+i, 3+i\},$
$\{0+i, 2+i, 4+i\},\ \{2+i, 3+i, 4+i\},$
$\{2+i, 3+i, 4+i\}\ mod\ 5\big\}, i \in Z_5$

$\big\{\{\infty, 0+i, 2+i\},\ \{\infty, 0+i, 2+i\},\ \{\infty, 2+i, 3+i\},$
$\{0+i, 1+i, 4+i\},\ \{1+i, 3+i, 4+i\},$
$\{1+i, 3+i, 4+i\}\ mod\ 5\big\}, i \in Z_5$

A 6-auto ingredient configuration $\mathbb{C}_{6.6.5}$:
$V = Z_6$. The groups are: $\{0,3\}$, $\{1,4\}$, $\{2,5\}$. The 4 $\mathbb{C}$–parallel classes are:

$$\{012, 015, 024, 153, 243, 543\},$$
$$\{045, 042, 051, 423, 513, 213\},$$
$$\{342, 345, 321, 450, 210, 510\},$$
$$\{042, 045, 021, 453, 213, 513\}.$$

$\mathcal{B}_{6.6.5}$, $V = Z_5 \cup \{\infty\}$, $\lambda = 12$ The 10 $\mathbb{C}$–parallel classes are:
$\big\{\{\infty, 0+i, 1+i\},\ \{\infty, 0+i, 2+i\},\ \{\infty, 1+i, 3+i\},$
$\{0+i, 2+i, 4+i\},\ \{1+i, 3+i, 4+i\},$
$\{2+i, 3+i, 4+i\}\ mod\ 5\big\}, i \in Z_5$
$\big\{\{\infty, 0+i, 4+i\},\ \{\infty, 0+i, 2+i\},\ \{\infty, 4+i, 3+i\},$
$\{0+i, 2+i, 1+i\},\ \{4+i, 3+i, 1+i\},$
$\{2+i, 3+i, 1+i\}\ mod\ 5\big\}, i \in Z_5$

■

Theorem 3.6 *Let λ be even. The necessary conditions for a $\mathbb{C}$–resolvable $\mathcal{B}_{6.9.x}$, $x = 2, 3, 4, 8$, to exist are sufficient with the addition of $v \neq 18$.*

Proof: A 2-auto ingredient configuration $\mathbb{C}_{6.9.2}$:
$V = Z_9$. The groups are: $\{1,3,0\}$, $\{2,5,7\}$, $\{4,6,8\}$. The 3 $\mathbb{C}$–parallel classes are:

$$\{176, 176, 432, 435, 280, 580\},$$
$$\{378, 378, 415, 412, 560, 260\},$$
$$\{470, 470, 182, 185, 236, 536\}.$$

$\mathcal{B}_{6.9.2}$, $V = Z_9$, $\lambda = 2$. The 4 $\mathbb{C}$–parallel classes are:

$$\{123, 123, 705, 708, 564, 864\},$$
$$\{247, 247, 180, 186, 053, 653\},$$
$$\{258, 258, 673, 671, 340, 140\},$$
$$\{260, 260, 387, 384, 715, 415\}.$$

A 2-auto ingredient configuration $\mathbb{C}_{6.9.3}$:

$V = Z_9$. The groups are: $\{0,1,3\}$, $\{2,5,7\}$, $\{4,6,8\}$. The 3 $\mathbb{C}$–parallel classes are:

$$\{124, 128, 470, 873, 563, 560\},$$
$$\{541, 543, 176, 378, 620, 820\},$$
$$\{580, 581, 047, 176, 423, 623\}.$$

$\mathcal{B}_{6.9.3}$, $V = Z_9$, $\lambda = 2$. The 4 $\mathbb{C}$–parallel classes are:

$$\{130, 134, 085, 486, 627, 527\},$$
$$\{120, 124, 056, 457, 638, 738\},$$
$$\{178, 176, 802, 604, 453, 253\},$$
$$\{704, 703, 428, 326, 851, 651\}.$$

A 2-auto ingredient configuration $\mathbb{C}_{6.9.4}$:

$V = Z_9$. The groups are: $\{1,2,3\}$, $\{4,6,8\}$, $\{0,5,7\}$. The 3 $\mathbb{C}$–parallel classes are:

$$\{145, 147, 526, 738, 026, 038\},$$
$$\{167, 160, 728, 034, 528, 534\},$$
$$\{365, 367, 518, 724, 018, 024\}.$$

$\mathcal{B}_{6.9.4}$, $V = Z_9$, $\lambda = 2$. The 4 $\mathbb{C}$–parallel classes are:

$$\{374, 378, 124, 125, 608, 605\},$$
$$\{167, 163, 287, 280, 453, 450\},$$
$$\{301, 302, 481, 486, 752, 756\},$$
$$\{701, 704, 851, 853, 623, 264\}.$$

A 2-auto ingredient configuration $\mathbb{C}_{6.9.8}$:

$V = Z_9$. The groups are: $\{1,2,3\}$, $\{4,5,6\}$, $\{0,7,8\}$. The 3 $\mathbb{C}$–parallel classes are:

$$\{147, 160, 428, 735, 638, 025\},$$
$$\{160, 158, 627, 034, 537, 824\},$$
$$\{158, 147, 520, 836, 430, 726\}.$$

For a $\mathcal{B}_{6.9.8}$ with $\lambda = 2$ take two copies of the following Kirkman triple system of order 9:

$\mathcal{B}_{6.9.8}$, $V = Z_9$, $\lambda = 1$. The 2 $\mathbb{C}$–parallel classes are:

$$\{023, 067, 245, 318, 658, 741\},$$
$$\{162, 150, 287, 634, 537, 048\}.$$

■

4 Conclusions

$\mathcal{B}_{6.6.4}$, $V = Z_5 \cup \ \{\infty\}$, $\lambda = 12$. The 10 $\mathbb{C}$–parallel classes are:

$$\Big\{\{\infty, 0+i, 1+i\},\ \{\infty, 0+i, 2+i\},\ \{\infty, 1+i, 2+i\},$$
$$\{0+i, 3+i, 4+i\},\ \{1+i, 3+i, 4+i\},$$
$$\{2+i, 3+i, 4+i\}\ mod\ 5\Big\}, i \in Z_5$$
$$\Big\{\{\infty, 0+i, 3+i\},\ \{\infty, 0+i, 2+i\},\ \{\infty, 3+i, 2+i\},$$
$$\{0+i, 1+i, 4+i\},\ \{3+i, 1+i, 4+i\},$$
$$\{2+i, 1+i, 4+i\}\ mod\ 5\Big\}, i \in Z_5$$

$\mathcal{B}_{6.6.6}$, $V = Z_5 \cup \ \{\infty\}$, $\lambda = 6$. The 5 $\mathbb{C}$–parallel classes are:

$$\Big\{\{\infty, 0+i, 1+i\},\ \{\infty, 0+i, 4+i\},\ \{\infty, 1+i, 3+i\},$$
$$\{1+i, 4+i, 2+i\},\ \{0+i, 3+i, 2+i\},$$
$$\{2+i, 3+i, 4+i\}\ mod\ 5\Big\}, i \in Z_5$$

$\mathcal{B}_{6.6.7}$, $V = Z_5 \cup \ \{\infty\}$, $\lambda = 12$. The 10 $\mathbb{C}$–parallel classes are:

$$\Big\{\{\infty, 0+i, 1+i\},\ \{\infty, 0+i, 3+i\},\ \{\infty, 4+i, 2+i\},$$
$$\{0+i, 2+i, 4+i\},\ \{1+i, 3+i, 2+i\},$$
$$\{1+i, 3+i, 4+i\}\ mod\ 5\Big\}, i \in Z_5$$
$$\Big\{\{\infty, 0+i, 1+i\},\ \{\infty, 1+i, 4+i\},\ \{\infty, 3+i, 2+i\},$$
$$\{1+i, 2+i, 3+i\},\ \{0+i, 2+i, 4+i\},$$
$$\{0+i, 3+i, 4+i\}\ mod\ 5\Big\}, i \in Z_5$$

$\mathcal{B}_{6.9.2}$, $V = Z_9$, $\lambda = 3$. The 6 $\mathbb{C}$–parallel classes are:

$$\{123, 123, 470, 478, 650, 658\},$$
$$\{268, 268, 174, 170, 354, 350\},$$
$$\{348, 348, 157, 152, 607, 602\},$$
$$\{108, 108, 452, 456, 372, 376\},$$
$$\{146, 146, 278, 275, 308, 305\},$$
$$\{240, 240, 361, 367, 581, 587\}.$$

$\mathcal{B}_{6.9.9}$, $V = Z_9$, $\lambda = 2$. The 4 $\mathbb{C}$–parallel classes are:

$$\{123, 123, 465, 478, 570, 680\},$$
$$\{158, 158, 246, 270, 347, 360\},$$
$$\{140, 140, 256, 287, 357, 368\},$$
$$\{167, 167, 245, 280, 348, 350\}.$$

Definition 4.1 *A configuration* $\mathbb{C}$= $(P,\mathcal{L})$ *is strongly 3-colorable if and only the vertices of* P *can be colored such that each* $l \in \mathcal{L}$ *receives one vertex of each color.[2]*

Definition 4.2 *A coloring of a configuration* $\mathbb{C}$= $(P,\mathcal{L})$ *is equitable if and only if all color classes have the same size.[3]*

Definition 4.3 *A regular configuration is uniform if it has a strong equitable coloring.*

For example $\mathbb{C}_{6.9.2}$ and $\mathbb{C}_{6.9.9}$ is uniform, $\mathbb{C}_{5.5.1}$ and $\mathbb{C}_{5.5.2}$ is not uniform.

Let $\mathbb{C}$ be a uniform configuration, and let (V,$\mathcal{B}$) be a $\mathbb{C}$–resolvable λ–TS. We will suppose that the blocks of $\mathcal{B}$ are colored by inheriting the colors of $\mathbb{C}$. Further we will always write the blocks as $\{a_1, a_2, a_3\}$ where the color of $a_i = i$.

Theorem 4.1 *Let* $\mathbb{C}$ *be uniform configuration. Suppose there exist: a* $\mathbb{C}$*–resolvable* λ*–TS(v), (V,* $\mathcal{B}$*); a* $\mathbb{C}$*–resolvable* λ*–TS(w); two orthogonal quasigroups of order* w*,* $(Z_w, \cdot)$ *and* $(Z_w, \circ)$*. Then there is a* $\mathbb{C}$*–resolvable* λ*–TS(vw).*

Proof: For each $\alpha \in Z_w$ let $T_\alpha = \{(i,j,i\circ j) \mid i,j \in Z_w \;\; i\cdot j = \alpha\}$ be a transversal of $(Z_w, \circ)$. Let W=(V$\times Z_w$) $\cup\, T$ and construct a $\mathbb{C}$–resolvable λ–TS(vw), (W,$\mathcal{D}$) in the following way:

For each $\mathbb{C}$–parallel class $\mathcal{B}_x$ of (V, $\mathcal{B}$), construct the following w $\mathbb{C}$–parallel classes of (W,$\mathcal{D}$),

$$\mathcal{B}_x^\alpha = \big\{\{a_i, b_j, c_{i\circ j}\} \mid \{a,b,c\} \in \mathcal{B}_x \;\; and \;\; (i,j,i\circ j) \in T_\alpha\big\}.$$

For each $a \in$V let $(a \times Z_w, \mathcal{E}_a)$ be a $\mathbb{C}$–resolvable λ–TS(w). Clearly $\cup_{a\in V}\, \mathcal{E}_a$ is a $\mathbb{C}$–parallel class of (W,$\mathcal{D}$).

■

Corollary 4.1 *For each* v*=*9^n *there is a* $\mathcal{B}_{6,9,2}$ *with* $\lambda = 3$ *and a* $\mathcal{B}_{6,9,9}$ *with* $\lambda = 2$*.*

Proof: The proof follows from Theorem 4.1 and the existence of above $\mathcal{B}_{6,9,2}$ with $v = 9$, $\lambda = 3$ and $\mathcal{B}_{6,9,9}$ with $v = 9$, $\lambda = 2$.

■

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