The Last Digit of $\binom{2n}{n}$ and $\sum\binom{n}{i}\binom{2n-2i}{n-i}$

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Submitted: June 28, 1996; Accepted: November 11, 1996.
AMS subject classification (1991): Primary 05A10; Secondary 11B65.

Abstract

Let $f_n = \sum_{i=0}^{n} \binom{n}{i}\binom{2n-2i}{n-i}$, $g_n = \sum_{i=1}^{n} \binom{n}{i}\binom{2n-2i}{n-i}$. Let $\{a_k\}_{k=1}$ be the set of all positive integers n, in increasing order, for which $\binom{2n}{n}$ is not divisible by 5, and let $\{b_k\}_{k=1}$ be the set of all positive integers n, in increasing order, for which g_n is not divisible by 5. This note finds simple formulas for a_k, b_k, $\binom{2n}{n}$ *mod* 10, f_n *mod* 10, and g_n *mod* 10.

Definitions

$$f_n = \sum_{i=0}^{n} \binom{n}{i}\binom{2n-2i}{n-i}; \quad g_n = \sum_{i=1}^{n} \binom{n}{i}\binom{2n-2i}{n-i}$$

$\{a_k\}_{k=1}$ is the set of all positive integers n, in increasing order, for which $\binom{2n}{n}$ is not divisible by 5.

$\{b_k\}_{k=1}$ is the set of all positive integers n, in increasing order, for which g_n is not divisible by 5.

u_n is the number of unit digits in the base 5 representation of n .

Theorem 1. *a_k is the number in base 5 whose digits represent the number k in base 3. If $n \geq 1$,*

$$\binom{2n}{n} \; mod\ 10 = \begin{cases} 0 & if\ n \notin \{a_k\} \\ \left.\begin{matrix} 2 \\ 4 \\ 6 \\ 8 \end{matrix}\right\} & if\ n \in \{a_k\}\ and\ u_n\ mod\ 4 = \begin{cases} 1 \\ 2 \\ 0 \\ 3 \end{cases} \end{cases}.$$

Note that if $n \in \{a_k\}$, u_n is odd (even) if and only if n is odd (even).

Proof. From Lucas' theorem [1], we have

$$\binom{2n}{n} \equiv \binom{N_1}{n_1}\binom{N_2}{n_2}\cdots\binom{N_t}{n_t} \quad \text{mod } 5,$$

where $2n = (N_r \cdots N_3N_2N_1)_5$, $n = (n_s \cdots n_3n_2n_1)_5$, and $t = min(r,s)$.

Suppose that for each $i \leq t$, $n_i \leq 2$. Then, for each $i \leq t$, $N_i = 2n_i$. Since $n_i = 0, 1$ or 2, each term of the product $\binom{N_1}{n_1}\binom{N_2}{n_2}\cdots\binom{N_t}{n_t}$ is $1, 2$ or 6. Hence, $\binom{2n}{n}$ is not divisible by 5.

Suppose that for some i, $n_i > 2$. Let i_m be the smallest value of i for which that is true. Then, if n_{i_m} is 3 or 4, N_{i_m} is 1 or 3 (resp.). In either case, $\binom{N_{i_m}}{n_{i_m}} = 0$, and $\binom{2n}{n}$ is divisible by 5.

Thus, $\{a_k\}$ is the set of all positive integers written in base 3, but interpreted as if they were written in base 5. Since $\{a_k\}$ is in increasing order, the first part of the theorem is proved.

Suppose now that $\binom{2n}{n}$ is not divisible by 5. Then each term of the product $\binom{N_1}{n_1}\binom{N_2}{n_2}\cdots\binom{N_t}{n_t}$ is $1, 2$ or 6 (according as $n_i = 0, 1,$ *or* 2). We have, noting that $\binom{2n}{n}$ is even,

$$\left.\begin{array}{c} 2^{u_n}\ mod\ 10 = 6^{\dagger},\ 2,\ 4 \text{ or } 8, \\[2ex] \binom{N_1}{n_1}\binom{N_2}{n_2}\cdots\binom{N_t}{n_t}\ mod\ 10 = 6,\ 2,\ 4 \text{ or } 8, \\[2ex] \binom{2n}{n}\ mod\ 10 = 6,\ 2,\ 4 \text{ or } 8, \end{array}\right\} \text{according as } u_n\ mod\ 4 = 0,\ 1,\ 2 \text{ or } 3.$$

□

Corollary 1.1.

$$a_k = k + 2\sum_{i=1} \left\lfloor \tfrac{k}{3^i} \right\rfloor 5^{i-1}.$$

†Equals 1 if $u_n = 0$; nevertheless, the next line follows since, if $u_n = 0$, at least one n_i must equal 2, making $\binom{2n_i}{n_i} = 6$.

Proof. Let $k = (\cdots d_3 d_2 d_1)_3$, and consider $a_k = \sum_{i=1} d_i 5^{i-1}$.

$$\begin{array}{ccccc} d_1 & = & k & - & 3\left\lfloor \frac{k}{3} \right\rfloor \\ d_2 & = & \left\lfloor \frac{k}{3} \right\rfloor & - & 3\left\lfloor \frac{k}{3^2} \right\rfloor \\ d_3 & = & \left\lfloor \frac{k}{3^2} \right\rfloor & - & 3\left\lfloor \frac{k}{3^3} \right\rfloor \\ \vdots & & \vdots & & \vdots \end{array}$$

Therefore,

$$\sum_{i=1} d_i 5^{i-1} = \sum_{i=1} \left(\left\lfloor \frac{k}{3^{i-1}} \right\rfloor - 3 \left\lfloor \frac{k}{3^i} \right\rfloor \right) 5^{i-1}.$$

Since $\left\lfloor \frac{k}{3^i} \right\rfloor 5^i - 3 \left\lfloor \frac{k}{3^i} \right\rfloor 5^{i-1} = 2 \left\lfloor \frac{k}{3^i} \right\rfloor 5^{i-1}$, the corollary is proved.

□

Corollary 1.2. *Let μ_k be the largest integer t such that $k/3^t$ is an integer. Then,*

$$a_k - a_{k-1} = \frac{5^{\mu_k} + 1}{2}, \text{ and } a_k = 1 + \sum_{i=2}^{k} \frac{5^{\mu_i} + 1}{2}.$$

$\mu_k = m$ if and only if $k \in \{j3^m\}$, where j is a positive integer and $j \bmod 3 \neq 0$.

Proof. If $\mu_k > 0$, then

$$k = (\cdots d_{\mu_k+1} 0 \cdots 0)_3; \quad d_{\mu_k+1} \geq 1; \quad \text{and } k - 1 = (\cdots (d_{\mu_k+1} - 1) 2 \cdots 2)_3.$$

Hence,

$$a_k - a_{k-1} = 5^{\mu_k} - 2[5^{\mu_k - 1} + 5^{\mu_k - 2} + \cdots + 1] = \frac{5^{\mu_k} + 1}{2}.$$

If $\mu_k = 0$, then

$$k = (\cdots d_1)_3; \quad d_1 \geq 1; \quad \text{and } k - 1 = (\cdots (d_1 - 1))_3.$$

Hence,

$$a_k - a_{k-1} = 1 = \frac{5^{\mu_k} + 1}{2}.$$

The remaining parts of the corollary follow immediately.

□

Corollary 1.3. *If $k > 1$,*

$$a_k = \begin{cases} 5a_{\frac{k}{3}} & \text{if } k \bmod 3 = 0, \\ a_{k-1} + 1 & \text{if } k \bmod 3 \neq 0. \end{cases}$$

Proof. If $k \bmod 3 = 0$, then $k = (\cdots d_2 0)_3$ and $\frac{k}{3} = (\cdots d_2)_3$. Hence, $a_k = 5a_{\frac{k}{3}}$.
If $k \bmod 3 \neq 0$, then $\mu_k = 0$ and from Corollary 1.2, we have $a_k - a_{k-1} = 1$. □

Theorem 2. *b_k is the number in base 5 whose digits represent the number $2k-1$ in base 3, i.e. $b_k = a_{2k-1}$. Furthermore, $g_n \bmod 10$ can only take on the values 1,5 or 9, as follows:*

$$g_n \bmod 10 = \begin{cases} 5 & \text{if } n \notin \{b_k\} \\ \left.\begin{matrix} 1 \\ 9 \end{matrix}\right\} & \text{if } n \in \{b_k\} \text{ and } u_n \bmod 4 = \begin{cases} 1 \\ 3 \end{cases} \end{cases}.$$

Proof. Let $F(z) = \sum_n f_n z^n = \sum_n z^n \sum_i \binom{n}{i}\binom{2n-2i}{n-i}$.

Letting t=n-i, we have

$$\begin{aligned} F(z) &= \sum_n z^n \sum_t \binom{n}{t}\binom{2t}{t} \\ &= \sum_t \binom{2t}{t} \sum_n \binom{n}{t} z^n \\ &= \frac{1}{1-z} \sum_t \binom{2t}{t} \left(\frac{z}{1-z}\right)^t \qquad \text{(see [2])} \\ &= \frac{1}{1-z} \frac{1}{\sqrt{1-\frac{4z}{1-z}}} = \frac{1}{\sqrt{1-z}} \frac{1}{\sqrt{1-5z}} \qquad \text{(see [2])} \\ &= [1 + (\frac{1}{4})\binom{2}{1} z + (\frac{1}{4})^2 \binom{4}{2} z^2 + \cdots][1 + (\frac{1}{4})\binom{2}{1} 5z + (\frac{1}{4})^2 \binom{4}{2} 5^2 z^2 + \cdots]. \end{aligned}$$

Hence,

$$f_n = \frac{1}{4^n} \sum_{i=0} \binom{2i}{i}\binom{2n-2i}{n-i} 5^i,$$

and

$$g_n = \frac{1}{4^n}\sum_{i=0}\binom{2i}{i}\binom{2n-2i}{n-i}5^i - \binom{2n}{n},$$

$$= \frac{\sum_{i=1}\binom{2i}{i}\binom{2n-2i}{n-i}5^i - (4^n-1)\binom{2n}{n}}{4^n}.$$

Thus we see that g_n is divisible by 5 if and only if $(4^n-1)\binom{2n}{n}$ is divisible by 5. And since g_n is odd, g_n mod $10 = 5$ if and only if g_n is divisible by 5. $4^n - 1$ is divisible by 5 if and only if n is even. Therefore, g_n mod $10 \neq 5$ if and only if n is odd and $n \in \{a_k\}$. Hence, $b_k = a_{2k-1}$, from which it follows that b_k is the number in base 5 whose digits represent the number 2k-1 in base 3.

Suppose that g_n *mod* $10 \neq 5$. Then $\binom{2n}{n}$ *mod* $10 = c$, where (since $n \in \{a_k\}$ and n and u_n are odd) c is 2 or 8, according as u_n mod $4 = 1$ or 3. Thus, for some non-negative integers j and k, $4^n - 1 = 10j + 3$ and $\binom{2n}{n} = 10k + c$. Since $\binom{2i}{i}$ is even when $i \geq 1$, for some non-negative integer q we have

$$4^n g_n = 10q - (10j+3)(10k+c).$$

Since g_n is odd, and 4^n *mod* $10 = 4$, we have

If c=2, g_n *mod* $10 = 1$;
if c=8, g_n *mod* $10 = 9$.

□

Corollary 2.1.

$$b_k = 2k - 1 + 2\sum_{i=1}\left\lfloor \tfrac{2k-1}{3^i}\right\rfloor 5^{i-1}.$$

Proof. This follows from Corollary 1.1, since $b_k = a_{2k-1}$.

□

Corollary 2.2. *Let ν_k be the largest integer t for which $\frac{(k-1)(2k-1)}{3^t}$ is an integer. Then,*

$$b_k - b_{k-1} = \frac{5^{\nu_k}+3}{2}, \text{ and } b_k = 1 + \sum_{i=2}^{k} \frac{5^{\nu_i}+3}{2}.$$

If $m \geq 1$, $\nu_k = m$ if and only if $k \in \left\{ \left\lceil \frac{j3^m+1}{2} \right\rceil \right\}$, where j is a positive integer and j mod $3 \neq 0$; if $m = 0$, $\nu_k = m$ if and only if $k \in \{3j\}$, where j is a positive integer.

Proof.

$$b_k = a_{2k-1},$$
$$b_k - b_{k-1} = (a_{2k-1} - a_{2k-2}) + (a_{2k-2} - a_{2k-3}),$$
$$b_k - b_{k-1} = \frac{5^{\mu_{2k-1}}+1}{2} + \frac{5^{\mu_{2k-2}}+1}{2},$$

where μ_k is the largest integer t such that $k/3^t$ is an integer.

Note that ν_k is also the largest integer t for which $\frac{(2k-1)(2k-2)}{3^t}$ is an integer. Then we must have one of the following cases:

$$\mu_{2k-1} = 0 \text{ and } \mu_{2k-2} = 0, \text{ or}$$
$$\mu_{2k-1} = \nu_k \text{ and } \mu_{2k-2} = 0, \text{ or}$$
$$\mu_{2k-1} = 0 \text{ and } \mu_{2k-2} = \nu_k.$$

In any of these cases,

$$b_k - b_{k-1} = \frac{5^{\nu_k}+3}{2}.$$

If $m \geq 1$, at most one of $(k-1)$ and $(2k-1)$ is divisible by 3^m. $\nu_k = m$ if and only if either $(k-1)$ or $(2k-1)$ is divisible by 3^m but not by 3^{m+1}. Suppose $m \geq 1$ and j mod $3 \neq 0$.

If j is odd, $\left\lceil \frac{j3^m+1}{2} \right\rceil = \frac{j3^m+1}{2}$; if $k = \frac{j3^m+1}{2}$, $2k-1 = j3^m$, and $\nu_k = m$.

If j is even, $\left\lceil \frac{j3^m+1}{2} \right\rceil = \frac{j3^m+2}{2}$; if $k = \frac{j3^m+2}{2}$, $k-1 = \frac{j3^m}{2}$, and $\nu_k = m$.

It is straightforward to show the converse, that if $\nu_k = m \geq 1$, $k \in \left\{ \left\lceil \frac{j3^m+1}{2} \right\rceil \right\}$.

If $m = 0$, $\nu_k = m$ if and only if neither $(k-1)$ or $(2k-1)$ is a multiple of 3. This occurs when $2k$ (and therefore k) is a multiple of 3.

□

Corollary 2.3. *If* $k \geq 1$,

$$\begin{aligned} b_{3k} &= b_{3k-1} + 2, \\ b_{3k+1} &= 5b_{k+1} - 4, && k \ mod\ 3 = 0, \\ &= b_{3k} + 4, && k \ mod\ 3 \neq 0, \\ b_{3k+2} &= 5b_{k+1}. \end{aligned}$$

Proof.

$$\begin{aligned} b_{3k} &= a_{6k-1} = a_{6k-2} + 1 = a_{6k-3} + 2 = b_{3k-1} + 2, \\ b_{3k+2} &= a_{6k+3} = 5a_{2k+1} = 5b_{k+1}, \\ b_{3k+1} &= a_{6k+1} = a_{6k} + 1 = 5a_{2k} + 1, \text{ and} \end{aligned}$$

if k mod $3 = 0$,

$$b_{3k+1} = 5(a_{2k+1} - 1) + 1 = 5b_{k+1} - 4;$$

if k mod $3 \neq 0$,

$$b_{3k+1} = 5(a_{2k-1} + 1) + 1 = 5b_k + 6 = b_{3k-1} + 6 = (b_{3k} - 2) + 6 = b_{3k} + 4.$$

□

Theorem 3.

$$f_n \ mod\ 10 = \left\{ \begin{array}{ll} 5 & if\ n \notin \{a_k\} \\ \left.\begin{array}{l} 1 \\ 3 \\ 7 \\ 9 \end{array}\right\} & if\ n \in \{a_k\}\ and\ u_n\ mod\ 4 = \left\{ \begin{array}{l} 0 \\ 1 \\ 3 \\ 2 \end{array}\right. \end{array}\right. .$$

Proof. Since $f_n = \binom{2n}{n} + g_n$, the corollary can be proved easily by combining the results of Theorem 1 and Theorem 2. □

References

[1] I. Vardi, Computational Recreations in Mathematica, Addison-Welsey, California, 1991, p.70 (4.4).

[2] H.S. Wilf, generatingfunctionology (1st ed.), Academic Press, New York, 1990, p.50 (2.5.7, 2.5.11).