Evaluations of k-fold Euler/Zagier sums: a compendium of results for arbitrary k

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Abstract. Euler sums (also called Zagier sums) occur within the context of knot theory and quantum field theory. There are various conjectures related to these sums whose incompletion is a sign that both the mathematics and physics communities do not yet completely understand the field. Here, we assemble results for Euler/Zagier sums (also known as multidimensional zeta/harmonic sums) of arbitrary depth, including sign alternations. Many of our results were obtained empirically and are apparently new. By carefully compiling and examining a huge data base of high precision numerical evaluations, we can claim with some confidence that certain classes of results are exhaustive. While many proofs are lacking, we have sketched derivations of all results that have so far been proved.

1 Introduction

We consider k-fold Euler sums [13, 2, 3] (also called Zagier sums) of arbitrary depth k. These sums occur in a natural way within the context of knot theory and quantum field theory (see [4] for an extended bibliography), carrying on a rich tradition of algebra and number theory as pioneered by Euler. There are various conjectures related to these sums (see e.g. (8) below) whose incompletion is a sign that both the mathematics and physics communities do not yet completely understand the field, whence new results are welcome.

As in [4] we allow for all possible alternations of signs, with $\sigma_j = \pm 1$ in

$$\zeta(s_1, \ldots, s_k; \sigma_1, \ldots, \sigma_k) = \sum_{n_j > n_{j+1} > 0} \prod_{j=1}^{k} \frac{\sigma_j^{n_j}}{n_j^{s_j}}, \tag{1}$$

since alternating Euler sums are essential [7] to the connection [18] of knot theory with quantum field theory [8, 6]. The integral representation

$$\begin{aligned} \zeta(s_1, \ldots, s_k; \sigma_1, \ldots, \sigma_k) &= \prod_{j=1}^{k} \frac{1}{\Gamma(s_j)} \int_1^\infty \frac{dy_j}{y_j} \frac{(\ln y_j)^{s_j-1}}{\prod_{i=1}^{j} \sigma_i y_i - 1}, &(2) \\ &= \prod_{j=1}^{k} \frac{1}{\Gamma(s_j)} \int_0^\infty \frac{u_j^{s_j-1}\, du_j}{\tau_j \exp\left(\sum_{i=1}^{j} u_i\right) - 1} &(3) \end{aligned}$$

generalizes that given in [10] for non-alternating sums. Here,

$$\tau_j := \prod_{i=1}^{j} \sigma_i. \tag{4}$$

For positive integers s_j, each $(\ln y_j)^{s_j-1}/\Gamma(s_j)$ in the integrand of (2) can be written as an iterated integral of the product $x_1^{-1} dx_1 \cdots x_{s_j}^{-1} dx_{s_j}$. Thus, we have the alternative $(s_1 + s_2 + \cdots + s_k)$-dimensional iterated-integral representation

$$\zeta(s_1, \ldots, s_k; \sigma_1, \ldots, \sigma_k) = \int_0^1 \Omega^{s_1-1}\omega_1 \Omega^{s_2-1}\omega_2 \cdots \Omega^{s_k-1}\omega_k, \qquad s_1 > 1, \tag{5}$$

in which the integrand denotes a string of distinct differential 1-forms of type $\Omega = dx/x$ and ω_j is given by

$$\omega_j := \frac{\tau_j\, dx_j}{1 - x_j \tau_j}. \tag{6}$$

Note that (5) shows that Euler sums form a ring, with a product of sums given by ternary reshuffles of the 1-forms dx/x, $dx/(1-x)$, and $dx/(1+x)$, just as products of non-alternating sums involve binary [17, 21] reshuffles of dx/x and $dx/(1-x)$.

We shall combine the strings of exponents and signs into a single string, with s_j in the jth position when $\sigma_j = +1$, and $\overline{s}_j$ in the jth position when $\sigma_j = -1$. We denote n repetitions of a substring by $\{\ldots\}_n$. Finally, we are obliged to point out that the notation (1) is not completely standard. In [10], for example, the argument list is reversed. Unfortunately, both notations have proliferated.

For non-alternating sums, several results are known, notably the duality relation [17]:

$$\zeta(m_1+2,\{1\}_{n_1},\ldots,m_p+2,\{1\}_{n_p}) = \zeta(n_p+2,\{1\}_{m_p},\ldots,n_1+2,\{1\}_{m_1})\,, \tag{7}$$

an explicit evaluation[1] of the self-dual case with $m_j = n_j = 1$, by Zagier [21, 22], (also cited in [10]):

$$\zeta(\{3,1\}_n) \stackrel{?}{=} \frac{2\cdot\pi^{4n}}{(4n+2)!}\,, \tag{8}$$

and the sum rule [14]:

$$\sum_{\substack{n_j>\delta_{j,1}\\ N=\Sigma_j n_j}} \zeta(n_1,n_2,\ldots,n_k) = \zeta(N)\,. \tag{9}$$

These, and other results have been recast in the language of graded commutative rings [16].

We find that (8) is the first member of a class of arbitrary-depth results for self-dual non-alternating sums that evaluate to rational multiples of powers of π^2, and that alternating Euler sums of arbitrary depth have a comparably rich structure.

2 Generating functions and relations

We derived the generating function

$$\sum_{m,n\geq 0} x^{m+1}y^{n+1}\zeta(m+2,\{1\}_n) = 1-\exp\left\{\sum_{k\geq 2}\frac{x^k+y^k-(x+y)^k}{k}\,\zeta(k)\right\}, \tag{10}$$

[1] We mark with $\stackrel{?}{=}$ conjectures for which we have overwhelming evidence, but no proof. For unmarked equalities, we either cite proofs from the literature, or provide a proof sketch in the appendix.

for the non-alternating sums in the $p=1$ case of (7), and the generators

$$\sum_{n\geq 0} x^{sn}\zeta(\{s\}_n) = \prod_{j\geq 1}\left(1+\frac{x^s}{j^s}\right) = \exp\left\{\sum_{k\geq 1}\frac{(-1)^{k-1}x^{sk}\zeta(sk)}{k}\right\}, \tag{11}$$

$$\begin{aligned}\sum_{n\geq 0} x^{sn}\zeta(\{\overline{s}\}_n) &= \prod_{j\geq 1}\left(1+(-1)^j\frac{x^s}{j^s}\right)\\ &= \exp\left\{\sum_{k\geq 1}\left(\frac{2(x/2)^{2sk-s}\zeta(2sk-s)}{2k-1}-\frac{x^{sk}\zeta(sk)}{k}\right)\right\},\end{aligned} \tag{12}$$

with $\Re(s)>1$ in (11), $\Re(s)>0$ in (12), and $\zeta(\{\ldots\}_0)=1$. At $s=1$, generator (12) becomes

$$A(x)\equiv\sum_{n\geq 0}x^n\zeta(\{\overline{1}\}_n) = \frac{2}{B(1+\frac{1}{2}x,\frac{1}{2}-\frac{1}{2}x)}\,. \tag{13}$$

We find, empirically, that cases with alternate alternations of sign are generated by

$$M(x)\equiv\sum_{n\geq 0}\left\{x^{2n}\zeta(\{\overline{1},1\}_n)+x^{2n+1}\zeta(\{\overline{1},1\}_n,\overline{1})\right\} \stackrel{?}{=} \left|A\left(\tfrac{x}{1+i}\right)\right|^2, \tag{14}$$

for real x. This, in turn, generates (8), via the convolution

$$\sum_{n\geq 0}x^{4n}\zeta(\{3,1\}_n) \stackrel{?}{=} M(x)M(-x)\,. \tag{15}$$

With a further alternating summation, the result analogous to (14) is

$$\begin{aligned}T(x) &\equiv 1+\sum_{n\geq 0}\left\{x^{2n+1}\zeta(\overline{1},\{\overline{1},1\}_n)+x^{2n+2}\zeta(\overline{1},\{\overline{1},1\}_n,\overline{1})\right\}\\ &\stackrel{?}{=} M(x)\left\{1-x\,\Im\,\psi\left(1+\tfrac{1}{2}\tfrac{x}{1+i}\right)-x\,\Im\,\psi\left(\tfrac{1}{2}-\tfrac{1}{2}\tfrac{x}{1+i}\right)\right\}\,.\end{aligned} \tag{16}$$

Convolution of (16), in the manner of (15), also generates self-dual non-alternating sums:

$$\sum_{n\geq 0}x^{4n+2}\zeta(2,\{1,3\}_n) \stackrel{?}{=} 1-T(x)T(-x)\,. \tag{17}$$

Moreover, we discovered the remarkable two-parameter self-dual result

$$\zeta(\{2\}_m,\{3,\{2\}_m,1,\{2\}_m\}_n) \stackrel{?}{=} \frac{2(m+1)\cdot\pi^{4(m+1)n+2m}}{(2\{m+1\}\{2n+1\})!}\,, \tag{18}$$

of which the previously known [10] example (8) is the $m = 0$ case. David Bailey (personal communication) has confirmed (18) for $1 \le m, n \le 4$ to 800 decimal places.

Results for sums with unit exponents are generated by

$$
\begin{aligned}
L(x) \equiv \sum_{n\ge 0} x^n \zeta(\overline{1}, \{1\}_n) &= \frac{2^{-x}-1}{x}\,, && (19)\\
\sum_{n\ge 0} x^n \zeta(\overline{1}, \overline{1}, \{1\}_n) &= \sum_{k\ge 1} \frac{2^{-k}}{k(x-k)}\,, && (20)\\
\sum_{n\ge 0} x^n \zeta(\overline{1}, \{1\}_n, \overline{1}) &\stackrel{?}{=} \sum_{k\ge 1} \frac{L(k+x)}{k} + L(x)\log 2\,, && (21)\\
\sum_{m,n\ge 0} x^{m+1} y^{n+1} \zeta(\overline{1}, \{1\}_m \overline{1}, \overline{1}, \{1\}_n) &\stackrel{?}{=} \sum_{k\ge 1} \Big\{ L(k+x) - L(k) \\
&\qquad - \frac{L(k+x-y) - L(k-y)}{2^y} \Big\}. && (22)
\end{aligned}
$$

We also discovered the following reductions to non-alternating sums and unit-exponent alternating sums:

$$
\begin{aligned}
\zeta(\{\overline{2}, 1\}_n) &\stackrel{?}{=} 8^{-n}\zeta(\{2,1\}_n) = 8^{-n}\zeta(\{3\}_n)\,, && (23)\\
\zeta(\overline{1}, \{1\}_m, 2, \{1\}_n) &\stackrel{?}{=} \zeta(\overline{1}, \{1\}_n, \overline{1}, \overline{1}, \{1\}_m) - \zeta(\overline{1}, \{1\}_{m+n+2})\,, && (24)\\
\zeta(\overline{1}, \overline{1}, \{1\}_m, 2, \{1\}_n) &\stackrel{?}{=} \zeta(\overline{1}, \overline{1}, \{1\}_n, \overline{1}, \overline{1}, \{1\}_m) - \zeta(\overline{1}, \overline{1}, \{1\}_{m+n+2}) \\
&\quad + \zeta(\overline{1}, \overline{1}, \{1\}_m)\,\zeta(n+2)\,, && (25)\\
\zeta(\overline{1}, \{1\}_m, 2, 2, \{1\}_n) &\stackrel{?}{=} \zeta(\overline{1}, \{1\}_n, \overline{1}, \overline{1}, \overline{1}, \overline{1}, \{1\}_m) + \zeta(\overline{1}, \{1\}_{m+n+4}) \\
&\quad - \zeta(\overline{1}, \{1\}_{n+2}, \overline{1}, \overline{1}, \{1\}_m) - \zeta(\overline{1}, \{1\}_n, \overline{1}, \overline{1}, \{1\}_{m+2})\,, && (26)\\
\zeta(\overline{1}, \overline{1}, \{1\}_m, 2, 2, \{1\}_n) &\stackrel{?}{=} \zeta(\overline{1}, \overline{1}, \{1\}_n, \overline{1}, \overline{1}, \overline{1}, \overline{1}, \{1\}_m) + \zeta(\overline{1}, \overline{1}, \{1\}_{m+n+4}) \\
&\quad - \zeta(\overline{1}, \overline{1}, \{1\}_{n+2}, \overline{1}, \overline{1}, \{1\}_m) - \zeta(\overline{1}, \overline{1}, \{1\}_n, \overline{1}, \overline{1}, \{1\}_{m+2}) \\
&\quad + \zeta(\overline{1}, \overline{1}, \{1\}_m, 2)\,\zeta(n+2) \\
&\quad - \zeta(\overline{1}, \overline{1}, \{1\}_m)\,\{\zeta(n+4) + \zeta(2, n+2)\}\;, && (27)
\end{aligned}
$$

$$
\begin{aligned}
\zeta(\overline{m+1}, \{1\}_n) &\stackrel{?}{=} (-1)^m \sum_{k\le 2^m} \varepsilon_k\, \zeta(\overline{1}, \{1\}_n, S_k)\,, && (28)\\
\zeta(\overline{1}, \overline{m+1}, \{1\}_n) &\stackrel{?}{=} (-1)^m \sum_{k\le 2^m} \varepsilon_k\, \zeta(\overline{1}, \overline{1}, \{1\}_n, S_k)
\end{aligned}
$$

$$- \sum_{p \le m} (-1)^p \zeta(m-p+2, \{1\}_n)\, \zeta(\overline{p})\,, \tag{29}$$

where the last two involve summation over all 2^m unit-exponent substrings of length m, with $\sigma_{k,j}$ as the jth sign of substring S_k, and $\varepsilon_k = \prod_{m/2>i\ge 0} \sigma_{k,m-2i}$, whose effect is to restrict the innermost m summation variables to alternately odd and even integers.

We remark that (11) reduces (23) to zetas, and that (19,22) reduce (24) to zetas and the polylogarithms $Li_n(1/2)$. The $m=1$ case of (28) is reduced to polylogarithms by (19,21). The product terms in (25) and (29) are reduced by (20) and (10); those in (27) involve terms given by (20,25). The analysis of [4] shows that new irreducibles, beyond the polylogarithms from (19–22), result from unit-exponent terms generated by (25,26,27), by (28) when $m \ge 2$, and by (29) when $m \ge 1$.

3 Evaluations at arbitrary depth

From the symmetric generator (10), we obtain

$$\zeta(2, \{1\}_n) = \zeta(n+2)\,, \tag{30}$$

$$\zeta(3, \{1\}_n) = \zeta(n+2,1) = \frac{n+2}{2}\,\zeta(n+3) - \frac{1}{2}\sum_{k=1}^{n} \zeta(k+1)\,\zeta(n+2-k)\,, \tag{31}$$

and, in general, products of up to $\min(m+1, n+1)$ zetas in $\zeta(m+2, \{1\}_n) = \zeta(n+2, \{1\}_m)$, whose symmetry was known from (7). Note that (30) is also implied by (9).

For integer values, $s = m$, generators (11,12) give

$$\sum_{n\ge 0} x^{mn} \zeta(\{m\}_n) = \prod_{j=1}^{m} \frac{1}{\Gamma(1-\omega_m^{2j-1} x)}\,, \tag{32}$$

$$\sum_{n\ge 0} x^{mn} \zeta(\{\overline{m}\}_n) = \prod_{j=1}^{m} \frac{\sqrt{\pi}}{\Gamma(1-\frac{1}{2}\omega_m^{2j-1}x)\,\Gamma(\frac{1}{2}-\frac{1}{2}\omega_m^{2j}x)}\,, \tag{33}$$

with $\omega_m = \exp(i\pi/m)$.

For even integers, $m = 2p$, generators (32,33) give trigonometric products:

$$S_p(x) \equiv \sum_{n\ge 0} x^{2pn} \zeta(\{2p\}_n) = (i\pi x)^{-p} \prod_{j=1}^{p} \sin(\pi \omega_{2p}^{2j-1} x)\,, \tag{34}$$

$$\sum_{n\ge 0} x^{2pn} \zeta(\{\overline{2p}\}_n) = S_p(\tfrac{1}{2}x) \prod_{j=1}^{p} \cos(\tfrac{1}{2}\pi\omega_p^j x)\,, \tag{35}$$

which show that $\zeta(\{2p\}_n)$ and $\zeta(\{\overline{2p}\}_n)$ are rational multiples of π^{2pn}.

The non-alternating result (34) readily yields

$$\begin{aligned}\zeta(\{2\}_n) &= \frac{2\cdot(2\pi)^{2n}}{(2n+1)!}\left(\frac{1}{2}\right)^{2n+1}, &(36)\\ \zeta(\{4\}_n) &= \frac{4\cdot(2\pi)^{4n}}{(4n+2)!}\left(\frac{1}{2}\right)^{2n+1}, &(37)\\ \zeta(\{6\}_n) &= \frac{6\cdot(2\pi)^{6n}}{(6n+3)!}, &(38)\\ \zeta(\{8\}_n) &= \frac{8\cdot(2\pi)^{8n}}{(8n+4)!}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^{4n+2}+\left(1-\frac{1}{\sqrt{2}}\right)^{4n+2}\right\}. &(39)\end{aligned}$$

Comparison of (37) with (8) reveals that Zagier's conjecture can be reformulated as

$$4^n\zeta(\{3,1\}_n) \stackrel{?}{=} \zeta(\{4\}_n) \tag{40}$$

or, in the notation of (5),

$$4^n\int_0^1(\Omega^2\omega^2)^n \stackrel{?}{=} \int_0^1(\Omega^3\omega)^n. \tag{41}$$

Equivalently, from (36), it becomes

$$(2n+1)\zeta(\{3,1\}_n) \stackrel{?}{=} \zeta(\{2,2\}_n) \tag{42}$$

or

$$(2n+1)\int_0^1(\Omega^2\omega^2)^n \stackrel{?}{=} \int_0^1(\Omega\omega)^{2n}, \tag{43}$$

in which, unlike (41), the list of omegas is merely reordered. Comparison of the empirical result (18) with (36,37) reveals that

$$\begin{aligned}\zeta(\{2\}_m,\{3,\{2\}_m,1,\{2\}_m\}_n) &\stackrel{?}{=} \frac{1}{2n+1}\zeta(\{2\}_{2(m+1)n+m}), &(44)\\ \zeta(\{2\}_{2p},\{3,\{2\}_{2p},1,\{2\}_{2p}\}_n) &\stackrel{?}{=} \frac{2p+1}{4^{(2p+1)n+p}}\zeta(\{4\}_{(2p+1)n+p}). &(45)\end{aligned}$$

Result (39) was already known [9]. The next member of the series is rather beautiful:

$$\zeta(\{10\}_n) = \frac{10\cdot(2\pi)^{10n}(L_{10n+5}+1)}{(10n+5)!}, \tag{46}$$

where $L_n = L_{n-1} + L_{n-2}$ is the nth Lucas number, with $L_1 = 1$ and $L_2 = 3$.

In the general case, a Laplace transform of (34) yields

$$\sum_{n\geq 0}(2pn+p)!\left(\frac{z}{(2\pi)^p}\right)^n \zeta(\{2p\}_n) = 2p\sum_{k=1}^{N_p}\frac{z_{p,k}^{1/2}}{z_{p,k}-z}\,, \tag{47}$$

with $N_p \leq 2^p/2p$ poles, whose positions $\{z_{p,k} \mid 1 \leq k \leq N_p\}$ are determined by the Laplace transforms of the 2^p exponentials generated by the product in (34). The pole closest to the origin, at $z_{p,1} = (2\sin(\pi/2p))^{2p}$, gives the first term in

$$\zeta(\{2p\}_n) = \frac{2p\cdot(2\pi)^{2pn}}{(2pn+p)!}\left(\frac{1}{2\sin\frac{\pi}{2p}}\right)^{2pn+p}\left\{1+\sum_{k=2}^{N_p}R_{p,k}^{2pn+p}\right\}, \tag{48}$$

with $R_{p,k} = (z_{p,1}/z_{p,k})^{1/2p}$, and hence $|\,R_{p,k}| < 1$ for $k > 1$. Choices of signs, $\sigma_j = \pm 1$, in

$$\frac{|\,R_{p,k}|}{\sin\frac{\pi}{2p}} = \left|\sum_{j=1}^{p}\sigma_j\omega_p^j\right|, \tag{49}$$

yield all the absolute values, though some choices of sign may not be realized in (48).

Proceeding up to $p = 9$, we derived:

$$\zeta(\{12\}_n) = \frac{12\cdot(2\pi)^{12n}}{(12n+6)!}\left\{\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{12n+6}+\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)^{12n+6}+2^{6n+3}\right\}, \tag{50}$$

$$\zeta(\{14\}_n) = \frac{14\cdot(2\pi)^{14n}}{(14n+7)!}\Re\left(\sum_{k=1}^{3}\frac{1+r_k^{28n+14}}{r_k^{14n+7}}+2\left(\frac{i\sqrt{7}-1}{2}\right)^{14n+7}+1\right), \tag{51}$$

$$\zeta(\{16\}_n) = \frac{16\cdot(2\pi)^{16n}}{(16n+8)!}\sum_{k=1}^{4}\Re\left(\frac{1}{s_k^{16n+8}}+\frac{s_k^{16n+8}}{c_k^{16n+8}}+2\left(\frac{i}{c_k}+c_k+\sqrt{2}\right)^{8n+4}\right), \tag{52}$$

$$\zeta(\{18\}_n) = \frac{18\cdot(2\pi)^{18n}}{(18n+9)!}\sum_{k=1}^{3}\Re\left(\frac{1}{t_k^{18n+9}}+(1+t_k)^{18n+9}+2\,(-\omega_3-t_k)^{18n+9}\right). \tag{53}$$

In (51), $r_k = 2\cos((2k-1)\pi/7)$ are the roots of the cubic equation $r(1+r)(2-r) = 1$. In (52), $s_k = 2\sin((2k-1)\pi/16)$ and $c_k = 2 - s_k^2$, which are the roots of $(2-c^2)^2 = 2$. In (53), $t_k = 2\cos(2^k\pi/9)$ are the roots of $t(3-t^2) = 1$. The method adopted to obtain these results exploited the exactness of the $[N-1\backslash N]$ Padé approximant to (47), for $N \geq N_p$. The roots of its denominator were then used to find $R_{p,k} = 2\sin(\pi/2p)/z_{p,k}^{1/2p}$.

The p-th member of the integer sequence[2]

$$1,1,1,2,3,4,8,12,16,33,62,67,186,316,280,1040,1963,1702,6830,10751,\ldots \tag{54}$$

gives the number of distinct non-zero absolute values of $\sum_{j=1}^{p}\sigma_j\omega_p^j$. Of these possibilities,

$$1,1,1,2,3,3,8,12,9,\ldots \tag{55}$$

are present in (48). Hence, for $p=6$ and $p=9$, some of the choices of signs in (49) are absent. Correspondingly, the values of N_p in the sequence

$$1,1,1,2,3,3,9,16,12,\ldots \tag{56}$$

do not saturate the upper bound $\lfloor 2^p/2p\rfloor$, for $p=6$ and $p=9$.

Explicit results from (35) are much lengthier than those from (34), since the former gives 4^p exponentials, while the latter gives only 2^p. We cite only the first three cases:

$$\zeta(\{\overline{2}\}_n) = \frac{\pi^{2n}}{(2n+1)!}\,\frac{(-1)^{n(n+1)/2}}{2^n}, \tag{57}$$

$$\zeta(\{\overline{4}\}_n) = \frac{\pi^{4n}}{(4n+2)!}\,\frac{(-1)^{n(n+1)/2}}{2^n}\left((1+\sqrt{2})^{2n+1}+(1-\sqrt{2})^{2n+1}\right), \tag{58}$$

$$\zeta(\{\overline{6}\}_n) = \frac{\pi^{6n}}{(6n+3)!}\cdot\frac{3}{2}\Bigg(1+2^{3n+1}(-1)^{n(n+1)/2} \tag{59}$$

$$\times\left\{\left(\frac{1+\sqrt{3}}{2}\right)^{6n+3}+\left(\frac{1-\sqrt{3}}{2}\right)^{6n+3}-1\right\}\Bigg). \tag{60}$$

Comparison of (36) with (57) reveals that

$$\zeta(\{\overline{2}\}_n) = 2^{-n}(-1)^{\lceil n/2\rceil}\zeta(\{2\}_n). \tag{61}$$

Finally, from (12) we obtain

$$\zeta(\{\overline{1}\}_n) = (-1)^n\sum\prod_{k\geq 1}\frac{1}{j_k!}\left(\frac{-Li_k((-1)^k)}{k}\right)^{j_k}, \tag{62}$$

where the sum is over all non-negative integers satisfying $\sum_{k\geq 1}kj_k = n$.

[2]The integer sequence (54) was not identified by Neil Sloane's 'superseeker' utility [19].

From (17), we obtain a self-dual evaluation, more complex than (18):

$$\zeta(2,\{1,3\}_n) \stackrel{?}{=} 4^{-n}\sum_{k=0}^{n}(-1)^k\zeta(\{4\}_{n-k})\Big\{(4k+1)\,\zeta(4k+2) - 4\sum_{j=1}^{k}\zeta(4j-1)\,\zeta(4k-4j+3)\Big\}, \tag{63}$$

with π^2 terms generated by $\zeta(4k+2)$ and by (37). The absence of $\zeta(4k+1)$ is conspicuous.

Explicit results generated by (19–22) involve the polylogarithms

$$A_n \equiv Li_n(1/2) = \sum_{k=1}^{\infty}\frac{1}{2^k k^n}, \quad P_n \equiv \frac{(\ln 2)^n}{n!}, \quad Z_n \equiv (-1)^n\zeta(n), \tag{64}$$

in terms of which we obtain

$$\zeta(\overline{1},\{1\}_n) = (-1)^{n+1}P_{n+1}, \tag{65}$$

$$\zeta(\overline{1},\overline{1},\{1\}_n) = -A_{n+2}, \tag{66}$$

$$\zeta(\overline{1},\{1\}_n,\overline{1}) \stackrel{?}{=} -Z_{n+2} + (-1)^n\sum_{k=1}^{n+2}A_kP_{n+2-k}, \tag{67}$$

$$\zeta(\overline{1},\{1\}_m,\overline{1},\overline{1},\{1\}_n) \stackrel{?}{=} (-1)^m\sum_{k=1}^{m+2}\binom{n+k}{n+1}A_{k+n+1}P_{m+2-k} + (-1)^n\sum_{k=1}^{n+2}\binom{m+k}{m+1}Z_{k+m+1}P_{n+2-k}. \tag{68}$$

We also have

$$\zeta(\overline{2},\{1\}_n) = -Z_{n+2} + 2(-1)^{n+1}P_{n+2} + (-1)^n\sum_{k=0}^{n+2}A_kP_{n+2-k}, \tag{69}$$

which shows that (67) and the $m=1$ case of (28) are equivalent. The complexity of the proof of (69), outlined in the Appendix, may serve as an indication of the difficulty of proving (28) in general.

4 Evaluations at specific depths

Several thousand evaluations, obtained in the work for [4] with the aid of MPPSLQ [1] and REDUCE [15], were inspected, in a search for further, comparably simple, results. These include analytical results for all 1457 sums with weight $w = \sum_j s_j \le 7$, for all 3698 double sums with weight $w \le 44$, and for all 1092 non-alternating sums with depth $k \le 4$ and weight $w \le 14$. To these we adjoined more than 2000 strategically selected high-precision numerical evaluations of self-dual sums with $s_j \le 3$ and weights up to $w = 40$, which enabled the discovery and validation of the remarkable generalization of (8) that is given in (18). The reader will find a detailed discussion of our scheme for computing these high-precision numerical evaluations in section 4 of [4]. For other approaches, see [12] and [11] in which Euler-Maclaurin based techniques are eschewed in favour of transformation to explicitly convergent sums.

It was found that precisely 11 of the 64 convergent depth-7 sums with unit exponents are reducible to the polylogarithms (64) and their products. They are given by the 6 results (13,14,16,65,66,67) and 5 instances of (68). Combining these with 5 instances of (24) and the $m = 1$ case of (28), we exhaust the weight-7 reducible alternating sums with depth $k \ge 5$. We computed, to high precision, all 2046 self-dual non-alternating sums comprising up to 10 'atomic' substrings of the form $\{m+2, \{1\}_n\}$, with $m, n = 0, 1$, as in (18,63), and hence having weight $w = 2k \le 40$. Precisely 25 of these are rational multiples of powers of π^2. They are exhausted by (18). Moreover, (10,18,63) were found to exhaust all zeta-reducible cases of non-alternating sums with $w = 2k = 10$, of self-dual sums with $w = 12$, and of self-dual sums with $s_j \le 3$ and $8 \le w \le 16$. At $w = 16$, computation and MPPSLQ analysis of 34 self-dual sums, to 300 significant figures, took about 0.5 CPUhour/sum on a DEC AlphaStation 600 5/333 at the Open University. Such exhaustion of reducible cases by our results (10–29) suggests that they are, like our database, reasonably comprehensive.

Among many MPPSLQ results at specific depths, the following are rather distinctive:

$$\zeta(2,1,\overline{2},\overline{2}) \overset{?}{=} \tfrac{39}{128}\,\zeta(4)\,\zeta(3) - \tfrac{193}{64}\,\zeta(5)\,\zeta(2) + \tfrac{593}{128}\,\zeta(7)\,, \tag{70}$$

$$\zeta(\overline{2},\overline{2},1,2) \overset{?}{=} \tfrac{9}{128}\,\zeta(4)\,\zeta(3) + \tfrac{447}{128}\,\zeta(5)\,\zeta(2) - \tfrac{1537}{256}\,\zeta(7)\,, \tag{71}$$

$$\begin{aligned}\zeta(\{4,1,1\}_2) &\overset{?}{=} \tfrac{3\pi^4}{16}\,\{\zeta(6,2) - 4\zeta(5)\,\zeta(3)\} - \tfrac{41\pi^6}{5040}\left\{\zeta^2(3) - \tfrac{77023\pi^6}{14414400}\right\} \\ &\quad + \tfrac{397}{8}\zeta(9)\,\zeta(3) + \zeta^4(3)\,,\end{aligned} \tag{72}$$

$$\begin{aligned}\zeta(2,2,1,2,3,2) &\overset{?}{=} \tfrac{75\pi^2}{32}\left\{\zeta(8,2) - 2\zeta(7)\,\zeta(3) + \tfrac{34}{225}\zeta^2(5) + \tfrac{4528801\pi^{10}}{61297236000}\right\} \\ &\quad - \tfrac{825}{8}\zeta(7)\,\zeta(5)\,,\end{aligned} \tag{73}$$

$$\begin{aligned}\zeta(\{\overline{3},1\}_2) &\overset{?}{=} -7\left(\alpha(5) - \tfrac{39}{64}\zeta(5) + \tfrac{1}{8}\zeta(4)\ln 2\right)\zeta(3) + \left(2\alpha(4) - \tfrac{1}{4}\zeta(4)\right)^2 \\ &\quad + 2\left(\alpha(4) - \tfrac{15}{16}\zeta(4) + \tfrac{7}{8}\zeta(3)\ln 2\right)^2 - \tfrac{1}{32}\zeta(8)\,,\end{aligned} \tag{74}$$

with $\alpha(n) \equiv A_n + (-1)^n(P_n - \frac{\pi^2}{12} P_{n-2})$, as in [4]. Note that the alternating sums (70,71) are pure zeta, yet we were unable to find generalizations of them; only from (12,23) have we obtained arbitrary-depth pure-zeta alternating results. Note also that the self-dual sums (72) and (73), with $w = 2k = 12$, contain non-zeta [2] irreducibles, $\zeta(6,2)$ and $\zeta(8,2)$, yet their kinship with distinct reducible classes, generated by (15) and (17), manifests itself in the unusual circumstance that they share only π^{12} as a common term. Finally, note that the polylogarithmic complexity of (74) contrasts greatly with the zeta-reducibility of (23), via (11), yet its kinship with (23) is reflected by the absence of 12 of the 21 terms [4] that occur in alternating sums with $w = 2k = 8$. In each of (70–74) one senses, from the relatively small number of terms, a degree of proximity to an arbitrary-depth reduction.

It is conjectured that, at any depth $k > 1$, Euler sums of weight w are reducible to a rational linear combination of lesser-depth sums (and their products) whenever w and k are of opposite parity. It is also conjectured that the lowest-weight irreducible depth-k alternating sum occurs at weight $k+2$ and entails $Li_{k+2}(1/2)$ [4]. The critical weight w_k, at which depth-k non-alternating sums first fail to be reducible to non-alternating sums of lesser depth, is more problematic. In [2] it was found that $w_2 = 8$; in [3] that $w_3 = 11$; in [4] that $w_4 = 12$. Reducibility was proved below these critical weights; reducibility at them was shown to be incredible, by lattice methods [1]. There is likewise good support for $w_5 = 15$ and $w_6 = 18$. It is conjectured [5] that $w_k = 3k$, for all $k \geq 4$. It appears that a large majority of non-alternating sums are irreducible whenever w and k are of the same parity and $w \geq w_k$. Additionally, R. Girgensohn (personal communication) has outlined a proof that, in the notation of (1),

$$\zeta(s_1, \ldots, s_k; \sigma_1, \ldots, \sigma_k) + (-1)^k \zeta(s_k, \ldots, s_1; \sigma_k, \ldots, \sigma_1)$$

is reducible for every $k > 1$.

For depths 2, 3 and 4, we have the following more specific remarks:

Depth 2. Whenever $s + t$ is odd, we have

$$\zeta(s,t;\sigma,\tau) = \tfrac{1}{2}\left(-\lambda_{s+t} + (1+(-1)^s)\zeta(s;\sigma)\zeta(t;\tau) + \mu_{s+t}\right) - \sum_{0<k<(s+t)/2} \lambda_{2k}\mu_{s+t-2k}, \quad (75)$$

where $\lambda_r = \zeta(r;\sigma\tau)$ and $\mu_r = (-1)^s \left(\binom{r-1}{s-1}\zeta(r;\sigma) + \binom{r-1}{t-1}\zeta(r;\tau)\right)$. This compact formula summarizes the evaluations given in [3]. Recently, a shorter proof has been given by R. Girgensohn (personal communication). A conjectured minimal **Q**-basis for all depth-2

Euler sums is formed by [4]: the depth-1 sums, $\ln 2$, π^2, $\{\zeta(2a+1) \mid a > 0\}$, and the depth-2 sums $\{\zeta(\overline{2a+1}, \overline{2b+1}) \mid a > b \geq 0\}$. All 3698 convergent double sums with weights $w \leq 44$ have been proved [4] to be expressible in this basis, using identities derived in [2] and augmented in [4]. A conjectured minimal $\mathbf{Q}$-basis for non-alternating depth-2 Euler sums is formed by π^2, $\{\zeta(2a+1) \mid a > 0\}$ and $\{\zeta(2a+1, 2b+1) \mid a \geq 2b > 0\}$, which is likewise proven to be sufficient up to weight 44. It is conjectured that the proven result [2]

$$\zeta(4,2) = \zeta^2(3) - \frac{4\pi^6}{2835}, \tag{76}$$

is the *sole* case of an even-weight reduction of a non-alternating sum $\zeta(a,b)$ with $a > b > 1$.

Depth 3. In [3], it is proved that non-alternating Euler sums of depth 3 and weight w are reducible to a rational linear combination of lesser depth sums when w is even or $w \leq 10$. It is conjectured that most depth-3 non-alternating sums of odd weight exceeding 10 are irreducible. The only reductions that have been found at odd weights in the range 17 to 33 are the cases $\zeta(a,a,a)$ and $\zeta(a,1,1)$. A conjectured $\mathbf{Q}$-basis for all depth-3 non-alternating sums is the set of lesser-depth non-alternating sums along with the set $\{\zeta(2a+1, 2b+1, 2c+1) \mid a \geq b \geq c > 0, a > c\}$.

Depth 4. It is proved [5] that every depth-4 non-alternating Euler sum with weight less than 12 is reducible to non-alternating sums of lesser depth. It is conjectured that a depth-4 non-alternating Euler sum with even weight exceeding 14 is reducible if and only if it is of one of the following forms: $\zeta(a,b,a,1)$, $\zeta(a,a,1,b)$, $\zeta(a,1,b,b)$, or $\zeta(a,b,b,a)$, with $a = b$, or $b = 1$, permitted. (It is proven and will be shown in a subsequent paper that these forms reduce.)

For more on questions of reducibility, see [4, 5].

5 Conclusions

Euler sums of arbitrary depth are a rich source of fascinating identities, with (16) and (18) serving as spectacular examples. Many of our results were discovered empirically; to date, we have not proven conjectures (14–18, 21–29) and their corollaries. The evidence in their favour is, however, overwhelming. The reader may consult the appendix for sketched derivations of results that have been proved.

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6 Appendix: Some Proof Sketches

The integral representation (2) may be derived using the well-known identity

$$n^{-s}\Gamma(s) = \int_1^\infty (\log y)^{s-1} y^{-n-1}\, dy. \tag{77}$$

Thus, the LHS of (2) may be written as

$$\zeta(s_1,\ldots,s_k;\sigma_1,\ldots,\sigma_k) = \sum \prod_{j=1}^k \int_1^\infty \frac{dy_j}{y_j} \frac{(\log y_j)^{s_j-1}}{\Gamma(s_j)} \left(\frac{\sigma_j}{y_j}\right)^{n_j}, \tag{78}$$

where the sum is over all positive integers $n_1 > n_2 > \cdots > n_k > 0$. Now make the change of summation variables $m_k = n_k$, and $m_j = n_j - n_{j+1}$ for $j = 1, 2, \ldots, k-1$. Then each m_j runs independently over the positive integers, and (78) becomes

$$\begin{aligned}\zeta(s_1,\ldots,s_k;\sigma_1,\ldots,\sigma_k) &= \prod_{j=1}^k \int_1^\infty \frac{dy_j}{y_j} \frac{(\log y_j)^{s_j-1}}{\Gamma(s_j)} \sum_{m_j\ge 1} \left(\prod_{i=1}^j \frac{\sigma_i}{y_i}\right)^{m_j} \\ &= \prod_{j=1}^k \frac{1}{\Gamma(s_j)} \int_1^\infty \frac{dy_j}{y_j} \frac{(\log y_j)^{s_j-1}}{\prod_{i=1}^j y_i/\sigma_i - 1},\end{aligned} \tag{79}$$

after summing the geometric series. Since each $\sigma_i = \pm 1$, this is the same as (2).

In the introduction, we briefly indicated how the iterated-integral representation (5) arises from the non-iterated multiple integral representation (2). We present a direct derivation below. Yet another approach is taken in [17], but there only the non-alternating case is considered. With Ω and ω_j as in the introduction, put $\Omega_n := x^n\Omega = x^n\, dx/x$. We begin with the self-evident integral representation

$$\frac{y^n}{n^k} = \int_0^y \Omega^{k-1}\Omega_n, \tag{80}$$

valid for positive integers n and k. It follows that for positive integers n, p, r, and k,

$$\frac{y^{n+p}}{(n+p)^r n^k} = \frac{1}{n^k}\int_0^y \Omega^{r-1}\Omega_{n+p} = \int_0^y \Omega^{r-1}\left(\frac{x^n}{n^k}\right) x^p \frac{dx}{x}. \tag{81}$$

Now substitute (80) for x^n/n^k, obtaining

$$\frac{y^n}{n^r(n+p)^k} = \int_0^y \Omega^{r-1} \int_0^x \Omega^{k-1}\Omega_n\, x^p \frac{dx}{x} = \int_0^y \Omega^{r-1}\Omega_p\Omega^{k-1}\Omega_n. \tag{82}$$

In general, for positive integers m_j, s_j, we have

$$\frac{y^{m_1}}{\prod_{j=1}^k (m_1+m_2+\cdots+m_j)^{s_j}} = \int_0^y \prod_{j=1}^k \Omega^{s_j-1}\Omega_{m_j}. \tag{83}$$

But, recalling the definition (4) of τ_j from the introduction, we have

$$\zeta(s_1,\ldots,s_k;\sigma_1,\ldots,\sigma_k) = \sum_{n_j>n_{j+1}} \prod_{j=1}^k \frac{\sigma_j^{n_j}}{{n_j}^{s_j}} = \sum_{m_j\ge 1} \prod_{j=1}^k \frac{\tau_j^{m_j}}{(m_1+m_2+\cdots+m_j)^{s_j}}. \tag{84}$$

Thus, from (83),

$$\zeta(s_1,\ldots,s_k;\sigma_1,\ldots,\sigma_k) = \sum_{m_j\ge 1} \int_0^1 \prod_{j=1}^k \Omega^{s_j-1}\tau_j^{m_j}\Omega_{m_j} = \int_0^1 \prod_{j=1}^k \Omega^{s_j-1}\omega_j, \tag{85}$$

by summing the k geometric series and recalling the definition (6) of ω_j from the introduction.

A general property of iterated integrals [17] such as (5) or (85) is that the string in the integrand can be reversed if the integration limits are exchanged and the appropriate sign factor is taken into account. If in addition, the integration variables x_j are all replaced by their complement $1-x_j$, this has the effect of switching Ω and ω. Thus,

$$\begin{aligned}
\zeta(m_1+2,\{1\}_{n_1},\ldots,m_p+2,\{1\}_{n_p}) &= \int_0^1 \Omega^{m_1+1}\omega^{n_1+1}\cdots\Omega^{m_p+1}\omega^{n_p+1} \\
&= \int_0^1 \Omega^{n_p+1}\omega^{m_p+1}\cdots\Omega^{m_1+1}\omega^{n_1+1} \\
&= \zeta(n_p+2,\{1\}_{m_p},\ldots,n_1+2,\{1\}_{m_1}),
\end{aligned} \tag{86}$$

which proves the duality relation (7).

To prove (10), we write the left side as

$$xy \sum_{m\ge 0}\sum_{k\ge 1} \frac{x^m}{k^{m+2}} \prod_{j=1}^{k-1}\left(1+\frac{y}{j}\right). \tag{87}$$

After summing on m, what remains is an instance of the hypergeometric series with first term omitted:

$$1 - {}_2F_1(-x, y; 1-x) = 1 - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}, \quad \Re(x+y) < 1. \tag{88}$$

To complete the proof, write Γ in the form $\exp(\int \Gamma'/\Gamma)$ and employ the Maclaurin series representation for Γ'/Γ.

For (11), write

$$F(x) := \sum_{n\ge 0} x^{sn}\zeta(\{s\}_n) \tag{89}$$

and note that

$$F(x) = \prod_{j\ge 1}\left(1 + \frac{x^s}{j^s}\right) \tag{90}$$

follows directly from the definition (1). Taking the logarithmic derivative, we have

$$\frac{F'}{F}(x) = \sum_{j\ge 1} \frac{sx^{s-1}/j^s}{(1 + x^s/j^s)}. \tag{91}$$

Now expand the denominator of (91) in powers of x^s/j^s and interchange summation order, obtaining

$$\frac{F'}{F}(x) = \sum_{k\ge 1}(-1)^{k-1} s x^{sk-1}\zeta(sk). \tag{92}$$

Finally, integrate, exponentiate, and check that the result agrees with (90) at $x = 0$.

The proof of (12) is analogous, with

$$G(x) := \prod_{j\ge 1}\left(1 + (-1)^j \frac{x^s}{j^s}\right) \tag{93}$$

replacing $F(x)$ in (90) above. Note that the special case (13) is example 1, page 259 of [20].

Although we currently have no proof of (14), from

$$A\Big(\frac{x}{1+i}\Big)A\Big(\frac{x}{1-i}\Big) = \prod_{j\ge 1}\left(1 + \frac{(-1)^j x}{j\sqrt{2}}e^{i\pi/4}\right)\left(1 + \frac{(-1)^j x}{j\sqrt{2}}e^{-i\pi/4}\right) \tag{94}$$

$$= \prod_{j\ge 1}\left(1 + \frac{(-1)^j x}{j} + \frac{x^2}{2j^2}\right), \tag{95}$$

it follows that

$$\left(\frac{1}{2i}\right)^n \sum_{k=0}^{2n} i^k \zeta(\{\overline{1}\}_k)\zeta(\{\overline{1}\}_{2n-k}) = \sum_{2p+q=2n} \zeta(\{\overline{1}\}_q) 2^{-p} \zeta(\{2\}_p). \tag{96}$$

Similarly, from

$$\prod_{j\geq 1}\left(1+\frac{(-1)^j x^3}{j^3}\right) = \prod_{j\geq 1}\left(1+\frac{(-1)^j x}{j}\right)\left(1-\frac{(-1)^j x}{j}+\frac{x^2}{j^2}\right), \tag{97}$$

it follows that

$$\zeta(\{\overline{3}\}_n) = \sum_{2p+q+r=3n} (-1)^q \zeta(\{\overline{1}\}_q)\zeta(\{\overline{1}\}_r)\zeta(\{2\}_p). \tag{98}$$

To prove (19), take $t=-1$ in

$$\sum_{m\geq 1} \frac{t^m}{m} \prod_{j=1}^{m-1}\left(1+\frac{x}{j}\right) = t\sum_{m\geq 0}(-t)^m \binom{-x-1}{m} \int_0^1 u^m\,du = \frac{(1-t)^{-x}-1}{x}. \tag{99}$$

For (20), consider

$$S := \sum_{m\geq 1} \frac{(-1)^m}{m} \sum_{k=1}^{m-1} \frac{(-1)^k}{k} \prod_{j=1}^{k-1}\left(1+\frac{x}{j}\right), \tag{100}$$

the generating function for $\zeta(\overline{1},\overline{1},\{1\}_n)$. Since the inner sum of (100) is the generating function for $\zeta(\overline{1},\{1\}_n)$, we may write, in view of (99),

$$S = \int_0^{-1} \frac{(1+u)^{-x}-1}{x(1-u)}\,du = \frac{1}{2}\int_0^1 \frac{1-u^{-x}}{x(1-u/2)}\,du = \sum_{k\geq 1} 2^{-k}\left(\frac{1}{k}-\frac{1}{k-x}\right), \tag{101}$$

which is the right side of (20).

We factored the generating function (11) into linear factors and then applied the infinite product representation for the Gamma function to arrive at (32). In the same way, we arrived at (33) from (12). The same procedure is done, in greater generality and with more details provided, in [20], pp. 238–239. Equations (34) and (35) arise from applying the reflection formula for the Gamma function to (32) and (33) respectively. Evaluations (36) through (39), and (46) were derived from (34) using the addition formulae to combine products of sine functions into sums of trigonometric functions. Likewise, evaluations (57) through (60) were derived from (35).

Finally, the promised proof outline of (69) is given. Note that in terms of generating functions, it is equivalent to prove that

$$\sum_{n\geq 0} t^{n+2}(-1)^{n+1}\zeta(\overline{2},\{1\}_n) = -t\left(\psi(1-t)+\gamma\right)+2\cdot 2^t - A(t)2^t - 1, \tag{102}$$

where

$$\begin{aligned} A(t) &:= \sum_{k\geq 0} t^k A(k) = 1+\sum_{k\geq 1}\frac{t^k(-1)^{k-1}}{(k-1)!}\int_0^1 \log^{k-1}(1-u)\frac{du}{1+u} && (103)\\ &= 1+t\int_0^1 \frac{(1-u)^{-t}}{1+u}\,du, && (104)\end{aligned}$$

ψ denotes the logarithmic derivative of the gamma function, and γ is Euler's constant. Since

$$\psi(1-t)+\gamma = \int_0^1 \frac{1-u^{-t}}{1-u}\,du, \quad t<1, \tag{105}$$

we need to show that

$$\int_0^1 \left(\frac{1-u}{2}\right)^{-t}\frac{du}{1+u}+\int_0^1 \frac{1-u^{-t}}{1-u}\,du+\sum_{n\geq 0}(-t)^{n+1}\zeta(\overline{2},\{1\}_n) = \frac{2^t-1}{t}, \tag{106}$$

for $|t|<1$ say. But

$$\begin{aligned} \sum_{n\geq 0}(-t)^{n+1}\zeta(\overline{2},\{1\}_n) &= -t\sum_{m\geq 0}\frac{(-1)^{m-1}}{(m+1)^2}\prod_{j=1}^{m}(1-t/j)\\ &= t\sum_{m\geq 0}\binom{t-1}{m}\int_0^\infty ye^{-(m+1)y}\,dy\\ &= t\int_0^\infty (1+e^{-y})^{t-1}ye^{-y}\,dy\\ &= -t\int_0^1 (1+u)^{t-1}\log u\,du\\ &= \int_0^1 \frac{(1+u)^t-1}{u}\,du.\end{aligned} \tag{107}$$

Therefore, we need only show that

$$\int_0^1 \left(\frac{1-u}{2}\right)^{-t} \frac{du}{1+u} + \int_0^1 \frac{1-u^{-t}}{1-u}\,du + \int_0^1 \frac{(1+u)^t-1}{u}\,du = \frac{2^t-1}{t} \tag{108}$$

for suitable t. With help from David Borwein, we let $v = (1-u)/2$ in the first integral, and let $v = 1/(1+u)$ in the third integral. Then the left side of (108) becomes

$$\begin{aligned}
& \int_0^{1/2} \frac{v^{-t}}{1-v}\,dv + \int_0^{1/2} \frac{1-v^{-t}}{1-v}\,dv + \int_{1/2}^1 \frac{1-v^{-t}}{1-v}\,dv + \int_{1/2}^1 \frac{v^{-t}-1}{v(1-v)}\,dv \\
= & \int_0^{1/2} \frac{dv}{1-v} + \int_{1/2}^1 \frac{v(1-v^{-t})+v^{-t}-1}{v(1-v)}\,dv \\
= & \int_{1/2}^1 \frac{v^{-t}(1-v)+v-1+1-v}{v(1-v)}\,dv \\
= & \int_{1/2}^1 v^{-t-1}\,dv \\
= & \frac{2^t-1}{t},
\end{aligned} \tag{109}$$

as required.

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