HYPERGEOMETRIC SERIES ACCELERATION VIA THE WZ METHOD

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Submitted: Sept 5, 1996. Accepted: Sept 12, 1996

Dedicated to Herb Wilf on his one million-first birthday

Abstract. Based on the WZ method, some series acceleration formulas are given. These formulas allow us to write down an infinite family of parametrized identities from any given identity of WZ type. Further, this family, in the case of the Zeta function, gives rise to many accelerated expressions for $\zeta(3)$.

AMS Subject Classification: Primary 05A

We recall [Z] that a discrete function A(n,k) is called *Hypergeometric* (or *Closed Form* (CF)) *in two variables* when the ratios $A(n+1,k)/A(n,k)$ and $A(n,k+1)/A(n,k)$ are both rational functions. A discrete 1-form $\omega = F(n,k)\delta k + G(n,k)\delta n$ is a *WZ 1-form* if the pair (F,G) of CF functions satisfies $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$.

We use: N and K for the *forward shift operators* on n and k, respectively. $\Delta_n := N-1$, $\Delta_k := K-1$.

Consider the WZ 1-form $\omega = F(n,k)\delta k + G(n,k)\delta n$. Then, we define the sequence ω_s, $s=1,2,3,\dots$ of new WZ 1-forms: $\omega_s := F_s\delta k + G_s\delta n$; where

$$F_s(n,k) = F(sn,k) \qquad \text{and} \qquad G_s(n,k) = \sum_{i=0}^{s-1} G(sn+i,k).$$

Proposition: ω_s is WZ, for all s.

Proof: (a) ω_s is closed:

$$\begin{aligned}
\Delta_n F_s &= F(s(n+1),k) - F(sn,k) \\
&= \sum_{i=0}^{s-1}\Big(F(sn+i+1,k) - F(sn+i,k)\Big) \\
&= \sum_{i=0}^{s-1}\Big(G(sn+i,k+1) - G(sn+i,k)\Big) \\
&= \sum_{i=0}^{s-1} G(sn+i,k+1) - \sum_{i=0}^{s-1} G(sn+i,k) \\
&= \Delta_k G_s.
\end{aligned}$$

Note that since ω is a WZ, it has the form ([Z], p.590):

$$(*) \qquad \omega = f(n,k)\big(P(n,k)\delta k + Q(n,k)\delta n\big)$$

for some CF f and some polynomials P and Q.

(b) ω_s has the form $(*)$:

Indeed, ω_s can be rewritten as:

$$\begin{aligned}\omega_s &= f(sn,k)\left(P(sn,k)\delta k + \sum_{i=0}^{s-1}\frac{f(sn+i,k)}{f(sn,k)}Q(sn+i,k)\delta n\right)\\ &= f(sn,k)\big(P(sn,k)\delta k + R(n,k)\delta n\big);\end{aligned}$$

where R(n,k) is a rational function and f(sn,k) is still CF. Hence after pulling out a common denominator, we see that ω_s too has the form $(*)$. This proves the Proposition. $\square$

Theorem 1: ([Z], Theorem 7, p.596) For any WZ pair (F,G)

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=1}^{\infty}\left(F(n,n-1)+G(n-1,n-1)\right) - \lim_{n\to\infty}\sum_{k=0}^{n-1}F(n,k),$$

whenever both side converge.

Formula 1:

$$(1) \qquad \sum_{n=0}^{\infty} G(n,0) = \sum_{n=0}^{\infty}\left(F(s(n+1),n) + \sum_{i=0}^{s-1} G(sn+i,n)\right) - \lim_{n\to\infty}\sum_{k=0}^{n-1}F(sn,k).$$

Proof: Apply Theorem 1 above on ω_s. Alternatively, integrate ω along the boundary contour $\partial\Omega_s$ of the region $\Omega_s = \{(n,k) : sn \geq k\}$. $\square$

Formula 2: We also have that

$$(2) \qquad \sum_{k=0}^{\infty} F(0,k) - \lim_{n\to\infty}\sum_{k=0}^{n}F(n,k) = \sum_{n=0}^{\infty}G(n,0) - \lim_{k\to\infty}\sum_{n=0}^{k}G(n,k),$$

whenever both side converge.

Proof: Integrate ω along the boundary contour $\partial\Omega_0$ of the region $\Omega_0 = \{(n,k) : n \geq 0, k \geq 0\}$. $\square$

Remark: By shear symmetry, a formulation similar to (2) can be given in 'k'. And a combination leads to:

Formula 3: For $\omega_{s,t} = F_{s,t}\delta k + G_{s,t}\delta n$; where

$$F_{s,t}(n,k) = \sum_{j=0}^{t-1} F(sn, tk+j) \qquad \text{and} \qquad G_{s,t}(n,k) = \sum_{i=0}^{s-1} G(sn+i, tk), \qquad \text{we have}$$

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{t-1} F(s(n+1), tn+j) + \sum_{i=0}^{s-1} G(sn+i, tn)\right) - \lim_{n\to\infty}\sum_{k=0}^{n-1} F_{s,t}(n,k). \tag{3}$$

Analogous statements hold in several variables. To wit:
for the WZ 1-form in 3 variables, $\omega_{s,t,r} := F_{s,t,r}\delta k + G_{s,t,r}\delta n + H_{s,t,r}\delta a$; where

$$F_{s,t,r}(n,k,a) = \sum_{j=0}^{t-1} F(sn, tk+j, ra), \qquad G_{s,t,r}(n,k,a) = \sum_{i=0}^{s-1} G(sn+i, tk, ra) \qquad \text{and}$$

$$H_{s,t,r}(n,k,a) = \sum_{u=0}^{r-1} H(sn, tk, ra+u),$$

Formula 4:

$$\sum_{n=0}^{\infty} H(0,0,n) = \sum_{n=0}^{\infty}\left(\sum_{u=0}^{r-1} H(s(n+1), t(n+1), rn+u) + \sum_{j=0}^{t-1} F(s(n+1), tn+j, rn) + \sum_{i=0}^{s-1} G(sn+i, tn, rn)\right)$$

$$- \lim_{a\to\infty}\sum_{k=0}^{a+1} F_{s,t,r}(a+1,k,a) - \lim_{a\to\infty}\sum_{n=0}^{a+1} G_{s,t,r}(n,a+1,a).$$

In [A], formula (1) was used to give a list of series acceleration for $\zeta(3)$ (where $F(n,k)$ is given and its companion G(n,k) is produced by the amazing Maple Package `EKHAD` accompanying [PWZ]). A small Maple Package `accel` applying (3) is available at `http://www.math.temple.edu/~[tewodros, zeilberg]`.

For example: with $F(n,k) = (-1)^k \frac{n!^6(2n-k-1)!k!^3}{2(n+k+1)!^2(2n)!^3}$, s=1 and t=1 `accel` produces the following pretty formula:

$$\zeta(3) = \sum_{n=0}^{\infty}(-1)^n \frac{n!^{10}(205n^2+250n+77)}{64(2n+1)!^5}. \tag{**}$$

Greg Fee and Simon Plouffe used (**) in their evaluation of $\zeta(3)$ to *520,000 digits* (available at `http://www.cecm.sfu.ca/projects/ISC/records.html`).

ACKNOWLEDGMENT: We would like to express our gratitude to Professor Herbert Wilf for his valuable comments and suggestions.

References

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`The package EKHAD is available by the www at http://www.math.temple.edu/~zeilberg/programs.html`

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