The Robinson-Schensted correspondence, crystal bases, and the quantum straightening at $q = 0$ *

Bernard LECLERC† and Jean-Yves THIBON‡

Submitted: March 14, 1995; Accepted: July 12, 1995

Dedicated to Dominique Foata

Abstract

We show that the quantum straightening for Young tableaux and Young bitableaux reduces in the crystal limit $q \mapsto 0$ to the Robinson-Schensted correspondence.

1 Introduction

Let K be a field of characteristic 0, and $X = (x_{ij})_{1 \le i,\, j \le n}$ a matrix of commutative indeterminates. The ring $K[x_{ij}]$ may be regarded as the algebra $\mathcal{F}[\mathrm{Mat}_n]$ of polynomial functions on the space of $n \times n$ matrices over K. A linear basis of this algebra is given by the bitableaux of Désarménien, Kung, Rota [6], which are defined in the following way.

Given two (semistandard) Young tableaux τ and τ' of the same shape, with columns $c_1, \ldots, c_k$ and $c'_1, \ldots, c'_k$, the Young *bitableau* $(\tau \,|\, \tau')$ is the product of the k minors of X whose row indices belong to c_i and column indices to c'_i, $i = 1, \ldots, k$. For example,

$$\left(\begin{array}{|c|cc} \hline 4 & \multicolumn{2}{c}{} \\ \hline 3 & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\ \hline 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \;\middle|\; \begin{array}{|c|cc} \hline 5 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{7} \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \right)$$

$$:= \begin{vmatrix} x_{11} & x_{12} & x_{15} \\ x_{31} & x_{32} & x_{35} \\ x_{41} & x_{42} & x_{45} \end{vmatrix} \times \begin{vmatrix} x_{13} & x_{14} \\ x_{53} & x_{54} \end{vmatrix} \times \begin{vmatrix} x_{25} & x_{27} \\ x_{65} & x_{67} \end{vmatrix} .$$

More generally, we shall call *tabloid* a sequence of column-shaped Young tableaux, and we shall associate to each pair δ, δ' of tabloids of the same shape a *bitabloid* $(\delta \,|\, \delta')$ defined as the product of minors indexed by the columns of δ and δ'.

*Partially supported by PRC Math-Info and EEC grant n^0 ERBCHRXCT930400

†L.I.T.P., Université Paris 7, 2 place Jussieu, 75251 Paris cedex 05, France

‡Institut Gaspard Monge, Université de Marne-la-Vallée, 2 rue de la Butte-Verte, 93166 Noisy-le-Grand cedex, France

There exists an algorithm due to Désarménien [5] for expanding any polynomial in $K[x_{ij}]$ on the basis of bitableaux. This is the so-called *straightening algorithm* (for bitableaux). In particular, the monomials $x_{i_1 j_1} \cdots x_{i_k j_k}$, which obviously form another linear basis of $K[x_{ij}]$, can be expressed in a unique way as linear combinations of bitableaux. Thus, the straightening of $x_{23}\, x_{11}\, x_{32}$ reads

$$\left(\begin{array}{|c|c|c|}\hline 2&1&3\\\hline\end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline 3&1&2\\\hline\end{array} \right) = \left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \right) - \left(\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\\hline 1&2\\\hline\end{array} \;\middle|\; \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\\hline 1&2\\\hline\end{array} \right)$$

$$+ \left(\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\\hline 1&3\\\hline\end{array} \;\middle|\; \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\\hline 1&2\\\hline\end{array} \right) + \left(\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\\hline 1&2\\\hline\end{array} \;\middle|\; \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\\hline 1&3\\\hline\end{array} \right) - \left(\begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\\hline 1&3\\\hline\end{array} \;\middle|\; \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\\hline 1&3\\\hline\end{array} \right) + \left(\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \;\middle|\; \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \right)$$

On the other hand, the Robinson-Schensted correspondence [26, 27, 28, 17] associates to any word w on the alphabet of symbols $\{1, \ldots, n\}$ a pair $(P(w), Q(w))$ of Young tableaux of the same shape. For example, the image of the word $w = 2\,1\,4\,3\,5\,1\,2$ under this correspondence is the pair

$$\left(\begin{array}{|c|c|c|}\cline{1-1} 4 & \multicolumn{2}{c}{} \\\hline 2&3&5\\\hline 1&1&2\\\hline\end{array} \;,\; \begin{array}{|c|c|c|}\cline{1-1} 6 & \multicolumn{2}{c}{} \\\hline 2&4&7\\\hline 1&3&5\\\hline\end{array} \right)$$

Both the straightening algorithm and the Robinson-Schensted algorithm are strongly connected with the representation theory of GL_n. Indeed, the straightening algorithm allows to compute the action of GL_n in (polynomial) irreducible representations, while the Robinson-Schensted correspondence was devised by Robinson to obtain a proof of the Littlewood-Richardson rule for decomposing into irreducibles the tensor product of two irreducible representations. The problem that we want to investigate is whether there exists any relation between the straightening algorithm and the Schensted algorithm. It turns out that to answer this question, one has to replace the algebra $\mathcal{F}[\mathrm{Mat}_n]$ by its quantum analogue $\mathcal{F}_q[\mathrm{Mat}_n]$ [25]. This is the associative algebra over $K(q)$ generated by n^2 letters t_{ij}, $i, j = 1, \ldots, n$ subject to the relations

$$\begin{aligned} t_{ik}\, t_{il} &= q^{-1}\, t_{il}\, t_{ik}\,, && (1)\\ t_{ik}\, t_{jk} &= q^{-1}\, t_{jk}\, t_{ik}\,, && (2)\\ t_{il}\, t_{jk} &= t_{jk}\, t_{il}\,, && (3)\\ t_{ik}\, t_{jl} - t_{jl}\, t_{ik} &= (q^{-1} - q)\, t_{il}\, t_{jk}\,, && (4) \end{aligned}$$

for $1 \le i < j \le n$ and $1 \le k < l \le n$. The *quantum determinant* of $T = (t_{ij})$ is the element

$$\det{}_q = \det{}_q T := \sum_{w \in S_n} (-q)^{-\ell(w)}\, t_{1w_1} \ldots t_{nw_n}$$

of $\mathcal{F}_q[\mathrm{Mat}_n]$. Here S_n is the symmetric group on $\{1,\dots,n\}$ and $\ell(w)$ denotes the length of the permutation w. The quantum determinant of T belongs to the center of $\mathcal{F}_q[\mathrm{Mat}_n]$. More generally, for $I=(i_1,\dots,i_k)$ and $J=(j_1,\dots,j_k)$ one defines the *quantum minor* of the submatrix T_{IJ} as

$$\det{}_q\, T_{IJ} := \sum_{w\in S_k} (-q)^{-\ell(w)}\, t_{i_1 j_{w_1}} \,\dots\, t_{i_k j_{w_k}} \ ,$$

and the *quantum bitableau* $(\tau|\tau')$ (resp. *quantum bitabloid* $(\delta\,|\,\delta')$) as the product of the quantum minors indexed by the columns of the Young tableaux τ and τ' (resp. of the tabloids δ and δ'). As proved by Huang, Zhang [10], the set of quantum bitableaux is again a linear basis of $\mathcal{F}_q[\mathrm{Mat}\,_n]$. For example, the expansion of the monomial $t_{23}\,t_{11}\,t_{32}$ on this basis is

$$\left(\begin{array}{|c|c|c|}\hline 2&1&3\\\hline\end{array}\;\middle|\;\begin{array}{|c|c|c|}\hline 3&1&2\\\hline\end{array}\right) = q^3\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\;\middle|\;\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right) - q^3\left(\begin{array}{|c|c}\cline{1-1} 3&\\\hline 1&\multicolumn{1}{c|}{2}\\\hline\end{array}\;\middle|\;\begin{array}{|c|c}\cline{1-1} 3&\\\hline 1&\multicolumn{1}{c|}{2}\\\hline\end{array}\right)$$

$$+(1-q^2+q^4)\left(\begin{array}{|c|c}\cline{1-1} 2&\\\hline 1&\multicolumn{1}{c|}{3}\\\hline\end{array}\;\middle|\;\begin{array}{|c|c}\cline{1-1} 3&\\\hline 1&\multicolumn{1}{c|}{2}\\\hline\end{array}\right) + q^4\left(\begin{array}{|c|c}\cline{1-1} 3&\\\hline 1&\multicolumn{1}{c|}{2}\\\hline\end{array}\;\middle|\;\begin{array}{|c|c}\cline{1-1} 2&\\\hline 1&\multicolumn{1}{c|}{3}\\\hline\end{array}\right) - q^5\left(\begin{array}{|c|c}\cline{1-1} 2&\\\hline 1&\multicolumn{1}{c|}{3}\\\hline\end{array}\;\middle|\;\begin{array}{|c|c}\cline{1-1} 2&\\\hline 1&\multicolumn{1}{c|}{3}\\\hline\end{array}\right)$$

$$+\ q^5\left(\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\;\middle|\;\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\right)$$

In the present example the coefficients of the expansion are polynomials in q, and only one of them has a nonzero constant term. In other words, denote by $\mathcal{B}$ the linear basis of quantum bitableaux in $\mathcal{F}_q[\mathrm{Mat}\,_n]$, and let $\mathcal{L}$ be the lattice generated over $K[q]$ by the elements of $\mathcal{B}$. Then

$$\left(\begin{array}{|c|c|c|}\hline 2&1&3\\\hline\end{array}\;\middle|\;\begin{array}{|c|c|c|}\hline 3&1&2\\\hline\end{array}\right) \equiv \left(\begin{array}{|c|c}\cline{1-1} 2&\\\hline 1&\multicolumn{1}{c|}{3}\\\hline\end{array}\;\middle|\;\begin{array}{|c|c}\cline{1-1} 3&\\\hline 1&\multicolumn{1}{c|}{2}\\\hline\end{array}\right) \mod q\mathcal{L}$$

This is an illustration of the following

Theorem 1.1 *Let $u=i_1\cdots i_k$ and $v=j_1\cdots j_k$ be two words on $\{1,\dots,n\}$, and denote by $(P(u),\,Q(u))$, $(P(v),\,Q(v))$ their images under the Robinson-Schensted correspondence. Then,*

$$t_{i_1j_1}\cdots t_{i_kj_k} \equiv \begin{cases} (P(u)|P(v)) & \mod q\mathcal{L} \quad \text{if } Q(u)=Q(v) \\ \qquad 0 & \mod q\mathcal{L} \quad \text{otherwise} \end{cases}\ .$$

More generally, as will be proved in Section 5, any quantum bitabloid lies in the $K[q]$-linear span $\mathcal{L}$ of quantum bitableaux, and is congruent modulo $q\mathcal{L}$ to a unique quantum bitableau (or to 0) described by means of the Robinson-Schensted correspondence.

As a corollary we obtain an unexpected characterization of the *plactic congruence* on words [17, 20], defined by

$$u \sim v \quad \Longleftrightarrow \quad P(u) = P(v)\,.$$

Corollary 1.2 *With the same notations as above,*

$$u \sim v \quad \Longleftrightarrow \quad t_{i_1 i_1} \cdots t_{i_k i_k} \equiv t_{j_1 j_1} \cdots t_{j_k j_k} \bmod q\mathcal{L}\,.$$

At this point, it is important to recall that the existence of a connection between the Robinson-Schensted correspondence and the representation theory of the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ at $q = 0$ was first discovered by Date, Jimbo, Miwa [4]. The results presented here are in fact of the same kind as those of [4], namely, it is shown in [4] that if $V_{(1)}$ denotes the basic representation of $U_q(\mathfrak{gl}_n)$, the transition matrix in $V_{(1)}^{\otimes k}$ from the basis of monomial tensors to the Gelfand-Zetlin basis specializes when $q = 0$ to a permutation matrix given by the Robinson-Schensted map. Similarly, Theorem 1.1 states that the transition matrix in the $U_q(\mathfrak{gl}_n)$-module $\mathcal{F}_q[\mathrm{Mat}_n]$ from the basis of monomials

$$B_\tau = \{t_{i_1 j_1} \cdots t_{i_k j_k} \mid Q(i_1 \cdots i_k) = Q(j_1 \cdots j_k) = \tau_\nu \text{ for some } \nu\}$$

to the basis of quantum bitableaux is equal at $q = 0$ to a permutation matrix also computed from the Robinson-Schensted algorithm. Here, τ_ν denotes for each partition ν a fixed standard Young tableau of shape ν.

The work of Date, Jimbo, Miwa, provided the starting point from which Kashiwara developed his theory of *crystal bases* for quantized enveloping algebras [13, 14]. We shall use crystal bases as the main tool for proving Theorem 1.1.

The paper is organized as follows. In Section 2, we collect the necessary material about $U_q(\mathfrak{gl}_n)$, $U_q(\mathfrak{sl}_n)$ and their representation theory. In Section 3, we review the definition and basic properties of Kashiwara's crystal bases at $q = 0$. These two sections are intended to provide a leisurely introduction to quantum groups and crystal bases, in order to make the paper more self-contained and more accessible. The new results appear in Section 4 and Section 5. In Section 4, we formulate and prove a version of Theorem 1.1 for the case of (single) tableaux, that is, we work in the subring of $\mathcal{F}_q[\mathrm{Mat}_n]$ generated by the quantum minors taken on the initial rows of T. Finally in Section 5 we prove Theorem 1.1, and its above-mentioned generalization.

2 $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{sl}_n)$

A general reference for this Section and the following one is the nice exposition [3]. We first recall the definition of the quantized enveloping algebras $U_q(\mathfrak{gl}_n)$ [12] and $U_q(\mathfrak{sl}_n)$ [11, 7]. $U_q(\mathfrak{gl}_n)$ is the associative algebra over $K(q)$ generated by the $4n-2$ symbols e_i, f_i, $i = 1, \ldots, n-1$ and q^{ϵ_i}, $q^{-\epsilon_i}$, $i = 1, \ldots, n$, subject to the relations

$$q^{\epsilon_i} q^{-\epsilon_i} = q^{-\epsilon_i} q^{\epsilon_i} = 1\,, \quad [q^{\epsilon_i}, q^{\epsilon_j}] = 0\,, \tag{5}$$

$$q^{\epsilon_i} e_j q^{-\epsilon_i} = \begin{cases} q e_j & \text{for } i = j \\ q^{-1} e_j & \text{for } i = j+1 \\ e_j & \text{otherwise} \end{cases}, \tag{6}$$

$$q^{\epsilon_i} f_j q^{-\epsilon_i} = \begin{cases} q^{-1} f_j & \text{for } i = j \\ q f_j & \text{for } i = j+1 \\ f_j & \text{otherwise} \end{cases}, \tag{7}$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{\epsilon_i} q^{-\epsilon_{i+1}} - q^{-\epsilon_i} q^{\epsilon_{i+1}}}{q - q^{-1}}, \tag{8}$$

$$[e_i, e_j] = [f_i, f_j] = 0 \quad \text{for } |i-j| > 1, \tag{9}$$

$$e_j e_i^2 - (q+q^{-1}) e_i e_j e_i + e_i^2 e_j = f_j f_i^2 - (q+q^{-1}) f_i f_j f_i + f_i^2 f_j = 0 \text{ for } |i-j| = 1. \tag{10}$$

The subalgebra of $U_q(\mathfrak{gl}_n)$ generated by e_i, f_i, and

$$q^{h_i} = q^{\epsilon_i} q^{-\epsilon_{i+1}}, \quad q^{-h_i} = q^{-\epsilon_i} q^{\epsilon_{i+1}}, \quad i = 1, \ldots, n-1, \tag{11}$$

is denoted by $U_q(\mathfrak{sl}_n)$.

The representation theories of $U_q(\mathfrak{gl}_n)$ and $U_q(\mathfrak{sl}_n)$ are closely parallel to those of their classical counterparts $U(\mathfrak{gl}_n)$ and $U(\mathfrak{sl}_n)$. Let M be a $U_q(\mathfrak{gl}_n)$-module and $\mu = (\mu_1, \ldots, \mu_n)$ be a n-tuple of nonnegative integers. The subspace

$$M_\mu = \{ v \in M \mid q^{\epsilon_i} v = q^{\mu_i} v, \quad i = 1, \ldots, n \}$$

is called a *weight space* and its elements are called *weight vectors* (of weight μ). Relations (6) (7) show that

$$e_i M_\mu \subset M_{\mu+\alpha_i}, \qquad f_i M_\mu \subset M_{\mu-\alpha_i},$$

where $\mu+\alpha_i = (\mu_1, \ldots, \mu_i+1, \mu_{i+1}-1, \ldots, \mu_n)$ and $\mu-\alpha_i = (\mu_1, \ldots, \mu_i-1, \mu_{i+1}+1, \ldots, \mu_n)$. Thus, ordering the weights in the usual way by setting

$$\mu \le \lambda \iff \sum_{i=1}^{k} \mu_i \le \sum_{i=1}^{k} \lambda_i, \quad k = 1, \ldots, n,$$

we see that the e_i's act as raising operators and the f_i's as lowering operators. A weight vector is said to be a *highest weight vector* if it is annihilated by the e_i's. M is called a *highest weight module* if it contains a highest weight vector v such that $M = U_q(\mathfrak{gl}_n)\, v$. If v is of weight λ, it follows that $\dim M_\lambda = 1$ and $M = \oplus_{\mu \le \lambda} M_\mu$. One then shows that there exists for each partition λ of length $\le n$ a unique highest weight finite-dimensional irreducible $U_q(\mathfrak{gl}_n)$-module V_λ, with highest weight λ.

Example 2.1 The basic representation $V = V_{(1)}$ of $U_q(\mathfrak{gl}_n)$ is the n-dimensional vector space over $K(q)$ with basis $\{v_i, \ 1 \le i \le n\}$, on which the action of $U_q(\mathfrak{gl}_n)$ is as follows:

$$q^{\epsilon_i} v_j = q^{\delta_{ij}} v_j, \quad e_i v_j = \delta_{i+1\,j} v_i, \quad f_i v_j = \delta_{ij} v_{i+1}.$$

Example 2.2 More generally, the $U_q(\mathfrak{gl}_n)$-module $V_{(1^k)}$ is a $\binom{n}{k}$-dimensional vector space with basis $\{v_c\}$ labelled by the subsets c of $\{1, \ldots, n\}$ with k elements (*i.e.* by the Young tableaux of shape (1^k) over $\{1, \ldots, n\}$). The action of $U_q(\mathfrak{gl}_n)$ on this basis is given by

$$q^{\epsilon_i} v_c = \begin{cases} v_c & \text{if } i \notin c \\ q v_c & \text{otherwise} \end{cases},$$

$$e_i v_c = \begin{cases} 0 & \text{if } i+1 \notin c \text{ or } i \in c \\ v_d & \text{otherwise, where } d = (c \setminus \{i+1\}) \cup \{i\} \end{cases},$$

$$f_i v_c = \begin{cases} 0 & \text{if } i+1 \in c \text{ or } i \notin c \\ v_d & \text{otherwise, where } d = (c \setminus \{i\}) \cup \{i+1\} \end{cases}.$$

We see that the action of the lowering operators f_i does not depend on q, and can be recorded on a colored graph whose vertices are the column-shaped Young tableaux c and whose arrows are given by:

$$c \xrightarrow{i} d \quad \Longleftrightarrow \quad f_i v_c = v_d \, .$$

Thus for $k = 2,\ n = 4$, one has the following graph:

$$\begin{tikzcd}
\begin{array}{|c|}\hline 2\\1\\\hline\end{array} \arrow[r, "2"] & \begin{array}{|c|}\hline 3\\1\\\hline\end{array} \arrow[r, "3"] \arrow[d, "1"] & \begin{array}{|c|}\hline 4\\1\\\hline\end{array} \arrow[d, "1"] \\
 & \begin{array}{|c|}\hline 3\\2\\\hline\end{array} \arrow[r, "3"] & \begin{array}{|c|}\hline 4\\2\\\hline\end{array} \arrow[d, "2"] \\
 & & \begin{array}{|c|}\hline 4\\3\\\hline\end{array}
\end{tikzcd}$$

This is one of the simplest examples of crystal graphs (*cf.* Section 3).

In order to construct more interesting $U_q(\mathfrak{gl}_n)$-modules, we use the tensor product operation. Given two $U_q(\mathfrak{gl}_n)$-modules M, N, we can define a structure of $U_q(\mathfrak{gl}_n)$-module on $M \otimes N$ by putting

$$q^{\epsilon_i}(u \otimes v) = q^{\epsilon_i} u \otimes q^{\epsilon_i} v, \tag{12}$$

$$e_i(u \otimes v) = e_i u \otimes v + q^{-h_i} u \otimes e_i v, \tag{13}$$

$$f_i(u \otimes v) = f_i u \otimes q^{h_i} v + u \otimes f_i v. \tag{14}$$

Indeed, the formulas

$$\Delta q^{\epsilon_i} = q^{\epsilon_i} \otimes q^{\epsilon_i}, \quad \Delta e_i = e_i \otimes 1 + q^{-h_i} \otimes e_i, \quad \Delta f_i = f_i \otimes q^{h_i} + 1 \otimes f_i,$$

make $U_q(\mathfrak{gl}_n)$ into a bialgebra. One shows that the decomposition into irreducible components of the tensor product of two irreducible $U_q(\mathfrak{gl}_n)$-modules is given by

$$V_\lambda \otimes V_\mu \simeq \bigoplus_\nu c^\nu_{\lambda\,\mu}\, V_\nu\,, \tag{15}$$

where the $c^\nu_{\lambda\,\mu}$ are the classical Littlewood-Richardson numbers. In particular, it follows that

$$V^{\otimes k} \simeq \bigoplus_{\nu \vdash k} f_\nu\, V_\nu\,, \tag{16}$$

where f_ν denotes the number of standard Young tableaux of shape ν.

Example 2.3 The n^2-dimensional $U_q(\mathfrak{gl}_n)$-module $V^{\otimes 2}$ decomposes into the q-symmetric square $V_{(2)}$ and the q-alternating square $V_{(1,1)}$. For $n = 2$ this decomposition is described by the following diagram:

$$0 \xleftarrow{e_1} v_1 \otimes v_1 \xrightarrow{f_1} v_1 \otimes v_2 + q v_2 \otimes v_1 \xrightarrow{f_1} (q + q^{-1})\, v_2 \otimes v_2 \xrightarrow{f_1} 0 \quad \simeq V_{(2)}$$

$$0 \xleftarrow{e_1} v_2 \otimes v_1 - q v_1 \otimes v_2 \xrightarrow{f_1} 0 \quad \simeq V_{(1,1)}$$

The algebra $\mathcal{F}_q[\mathrm{Mat}\,_n]$ defined in Section 1 is also endowed with a natural structure of $U_q(\mathfrak{gl}_n)$-module via the action defined by

$$q^{\epsilon_i}\, t_{kl} = q^{\delta_{il}}\, t_{kl}, \quad e_i\, t_{kl} = \delta_{i+1\,l}\, t_{k\,l-1}, \quad f_i\, t_{kl} = \delta_{il}\, t_{k\,l+1}, \tag{17}$$

and the Leibniz formulas

$$\begin{aligned} q^{\epsilon_i}(PQ) &= (q^{\epsilon_i} P)\,.\,(q^{\epsilon_i} Q), & (18)\\ e_i(PQ) &= (e_i P)\,.\,Q + (q^{-h_i} P)\,.\,(e_i Q), & (19)\\ f_i(PQ) &= (f_i P)\,.\,(q^{h_i} Q) + P\,.\,(f_i Q), & (20)\end{aligned}$$

for P, Q in $\mathcal{F}_q[\mathrm{Mat}\,_n]$. This provides a very convenient realization of the irreducible modules V_λ as natural subspaces of $\mathcal{F}_q[\mathrm{Mat}\,_n]$. To describe it, we introduce some notations. We shall write y_λ for the unique Young tableau of shape and weight λ. This is the so-called *Yamanouchi tableau* of shape λ. Let τ be any Young tableau of shape λ. The quantum bitableau $(y_\lambda \,|\, \tau)$ will be simply denoted by (τ) and will be called a *quantum tableau.* This is a product of quantum minors taken on the first rows of the matrix T. *Quantum tabloids* are defined similarly. Finally, denote by $\mathcal{T}_\lambda$ the subspace of $\mathcal{F}_q[\mathrm{Mat}\,_n]$ spanned by quantum tableaux (τ) of shape λ. Then one can show [19, 23] the following q-analogue of a classical result of Deruyts (see [9]).

Theorem 2.4 *The subspace $\mathcal{T}_\lambda$ is invariant under the action of $U_q(\mathfrak{gl}_n)$ on $\mathcal{F}_q[\mathrm{Mat}\,_n]$, and is isomorphic as a $U_q(\mathfrak{gl}_n)$-module to the simple module V_λ.*

The action of $U_q(\mathfrak{gl}_n)$ on $\mathcal{T}_\lambda$ is computed by means of the q-straightening formula. Namely, for column-shaped quantum tableaux one checks easily that the action coincides with the one previously described in Example 2.2. For general quantum tableaux we use Leibniz formulas (18) (19) (20), and when necessary we use the q-straightening algorithm (*cf.* Section 4) for converting the quantum tabloids of the right-hand side into a linear combination of quantum tableaux.

Example 2.5 We choose $n = 3$ and $\lambda = (2,1)$. $\mathcal{T}_{(2,1)}$ is 8-dimensional and one has for instance

$$f_1 \begin{array}{ll} \boxed{3} & \\ \boxed{1} & \boxed{1} \end{array} = \begin{array}{ll} \boxed{3} & \\ \boxed{1} & \boxed{2} \end{array} + q \begin{array}{ll} \boxed{3} & \\ \boxed{2} & \boxed{1} \end{array} = (1+q^2) \begin{array}{ll} \boxed{3} & \\ \boxed{1} & \boxed{2} \end{array} - q^3 \begin{array}{ll} \boxed{2} & \\ \boxed{1} & \boxed{3} \end{array}$$

In this example, the quantum tableaux (τ) have been written for short τ (without brackets). This small abuse of notation will be used freely in the sequel.

This realization of V_λ is, up to some minor changes of convention, the same as the one described in [2] via the q-Young symmetrizers of the Hecke algebra of type A.

We denote by $\mathcal{F}_q[GL_n/B]$ the subspace of $\mathcal{F}_q[\mathrm{Mat}_n]$ spanned by the quantum tableaux. It follows from the q-straightening formula that this is in fact a subalgebra of $\mathcal{F}_q[\mathrm{Mat}_n]$. It coincides for $q = 1$ with the ring of polynomial functions on the flag variety GL_n/B, hence the notation. The quantum deformation $\mathcal{F}_q[GL_n/B]$ has been studied by Lakshmibai, Reshetikhin [19] and Taft, Towber [29]. As a $U_q(\mathfrak{gl}_n)$-module it decomposes into:

$$\mathcal{F}_q[GL_n/B] \simeq \bigoplus_{\ell(\lambda)\le n} V_\lambda \,. \tag{21}$$

Returning to $\mathcal{F}_q[\mathrm{Mat}_n]$ and its linear basis formed by quantum bitableaux, we note that the action of $U_q(\mathfrak{gl}_n)$ defined by (17) involves only the column indices of the variables t_{ij}. The defining relations (1) (2) (3) (4) being invariant under transposition of the matrix T, we see that we have another action of $U_q(\mathfrak{gl}_n)$ given by

$$(q^{\epsilon_i})^\dagger t_{kl} = q^{\delta_{ik}}\, t_{kl}, \quad e_i^\dagger t_{kl} = \delta_{i+1\,k}\, t_{k-1\,l}, \quad f_i^\dagger t_{kl} = \delta_{ik}\, t_{k+1\,l}, \tag{22}$$

(where the symbol $\dagger$ has been added to distinguish this action from the previous one), and the Leibniz formulas (18) (19) (20). These two actions obviously commute with each other, so that $\mathcal{F}_q[\mathrm{Mat}_n]$ is now endowed with the structure of a (left) bimodule over $U_q(\mathfrak{gl}_n)$. The quantum version of the Peter-Weyl theorem provides the decomposition [23]

$$\mathcal{F}_q[\mathrm{Mat}_n] \simeq \bigoplus_{\ell(\lambda)\le n} V_\lambda \otimes V_\lambda \,. \tag{23}$$

Here, the irreducible bimodule $V_\lambda \otimes V_\lambda$ is generated by applying all possible products of lowering operators $f_i^\dagger$, f_j to the highest weight vector $(y_\lambda|y_\lambda)$.

We end this Section by noting that every $U_q(\mathfrak{gl}_n)$-module M can be regarded by restriction as a $U_q(\mathfrak{sl}_n)$-module (that we still denote by M). In particular, the V_λ are also irreducible under $U_q(\mathfrak{sl}_n)$. However we point out that, as $U_q(\mathfrak{sl}_n)$-modules,

$$V_\lambda \simeq V_\mu \iff \lambda_i - \lambda_{i+1} = \mu_i - \mu_{i+1}\,,\ i = 1,\ldots,n-1\,.$$

Example 2.6 The $U_q(\mathfrak{sl}_2)$-modules $V_{(l)}$ will be very important in the sequel and we describe them precisely. For $l \ge 0$, $V_{(l)}$ is a $(l+1)$-dimensional vector space over $K(q)$ with basis $\{u_k,\ 0 \le k \le l\}$, on which the action of $U_q(\mathfrak{sl}_2)$ is as follows:

$$q^{h_1}\, u_k = q^{l-2k}\, u_k\,, \quad e_1\, u_k = [l-k+1]\, u_{k-1}\,, \quad f_1\, u_k = [k+1]\, u_{k+1}\,.$$

In these formulas $[m]$ denotes the q-integer $(q^m - q^{-m})/(q - q^{-1})$, and we understand $u_{-1} = u_{l+1} = 0$. Setting $[m]! = [m][m-1]\cdots[1]$ and $f_1^{(m)} = f_1^m/[m]!$, we see that the basis $\{u_k\}$ is characterized by $u_k = f_1^{(k)}\, u_0$. Also, we note that the weight spaces being one-dimensional, there is up to normalization a unique basis of $V_{(l)}$ whose elements are weight vectors. The basis $\{u_k\}$ may therefore be regarded as canonical. This will provide the starting point for defining the crystal basis of a $U_q(\mathfrak{gl}_n)$-module.

3 Crystal bases

In this Section, we give, following [15], a self-contained introduction to crystal bases of $U_q(\mathfrak{gl}_n)$-modules.

It follows from relations (6) (7) (8) (11) that for any $i = 1, \ldots, n-1$, the subalgebra U_i generated by $e_i,\ f_i,\ q^{h_i},\ q^{-h_i}$ is isomorphic to $U_q(\mathfrak{sl}_2)$. Hence a $U_q(\mathfrak{gl}_n)$-module M can be regarded by restriction to U_i as a $U_q(\mathfrak{sl}_2)$-module. We shall assume from now on that the weight spaces M_μ are finite-dimensional, that $M = \oplus_\mu M_\mu$, and that for any i, M decomposes into a direct sum of finite-dimensional U_i-modules. Such modules M are said to be *integrable*. It follows from the representation theory of $U_q(\mathfrak{sl}_2)$ that for any i, the integrable module M is a direct sum of irreducible U_i-modules $V_{(l)}$.

Example 3.1 Let M denote the $U_q(\mathfrak{gl}_3)$-module $V_{(2,1)}$ in the realization given by Theorem 2.4. As a U_1-module, M decomposes into 4 irreducible components, as shown by the following diagram:

$$0 \xleftarrow{\ e_1\ } \begin{array}{|c|c|}\hline 2 \\ \hline 1 & 1 \\ \hline\end{array} \xrightarrow{\ f_1\ } \begin{array}{|c|c|}\hline 2 \\ \hline 1 & 2 \\ \hline\end{array} \xrightarrow{\ f_1\ } 0$$

$$0 \xleftarrow{\ e_1\ } \begin{array}{|c|c|}\hline 3 \\ \hline 1 & 1 \\ \hline\end{array} \xrightarrow{\ f_1\ } (1+q^2)\begin{array}{|c|c|}\hline 3 \\ \hline 1 & 2 \\ \hline\end{array} - q^3 \begin{array}{|c|c|}\hline 2 \\ \hline 1 & 3 \\ \hline\end{array} \xrightarrow{\ f_1/[2]\ } \begin{array}{|c|c|}\hline 3 \\ \hline 2 & 2 \\ \hline\end{array} \xrightarrow{\ f_1\ } 0$$

$$0 \xleftarrow{\ e_1\ } \begin{array}{|c|c|}\hline 2 \\ \hline 1 & 3 \\ \hline\end{array} \xrightarrow{\ f_1\ } 0$$

$$0 \xleftarrow{\ e_1\ } \begin{array}{|c|c|}\hline 3 \\ \hline 1 & 3 \\ \hline\end{array} \xrightarrow{\ f_1\ } \begin{array}{|c|c|}\hline 3 \\ \hline 2 & 3 \\ \hline\end{array} \xrightarrow{\ f_1\ } 0$$

On the other hand, as a U_2-module, M decomposes into:

$$0 \xleftarrow{e_2} \begin{smallmatrix}2&\\1&1\end{smallmatrix} \xrightarrow{f_2} \begin{smallmatrix}3&\\1&1\end{smallmatrix} \xrightarrow{f_2} 0$$

$$0 \xleftarrow{e_2} \begin{smallmatrix}2&\\1&2\end{smallmatrix} \xrightarrow{f_2} \begin{smallmatrix}2&\\1&3\end{smallmatrix} + q \begin{smallmatrix}3&\\1&2\end{smallmatrix} \xrightarrow{f_2/[2]} \begin{smallmatrix}3&\\1&3\end{smallmatrix} \xrightarrow{f_2} 0$$

$$0 \xleftarrow{e_2} \begin{smallmatrix}3&\\1&2\end{smallmatrix} - q \begin{smallmatrix}2&\\1&3\end{smallmatrix} \xrightarrow{f_2} 0$$

$$0 \xleftarrow{e_2} \begin{smallmatrix}3&\\2&2\end{smallmatrix} \xrightarrow{f_2} \begin{smallmatrix}3&\\2&3\end{smallmatrix} \xrightarrow{f_2} 0$$

We observe that the U_1-decomposition leads to the basis

$$B_1 = \left((1+q^2) \begin{smallmatrix}3&\\1&2\end{smallmatrix} - q^3 \begin{smallmatrix}2&\\1&3\end{smallmatrix} \;;\; \begin{smallmatrix}2&\\1&3\end{smallmatrix} \right)$$

for the 2-dimensional weight space $M_{(1,1,1)}$, while the U_2-decomposition leads to

$$B_2 = \left(\begin{smallmatrix}2&\\1&3\end{smallmatrix} + q \begin{smallmatrix}3&\\1&2\end{smallmatrix} \;;\; \begin{smallmatrix}3&\\1&2\end{smallmatrix} - q \begin{smallmatrix}2&\\1&3\end{smallmatrix} \right)$$

These two bases are different and therefore one cannot find a basis B of weight vectors in M compatible with both decompositions. However, as noted by Kashiwara, 'at $q = 0$' the bases B_1 and B_2 coincide. The next definitions will allow us to state this in a more formal way.

Consider the simple $U_q(\mathfrak{sl}_2)$-module $V_{(l)}$ with basis $\{u_k\}$ (*cf.* Example 2.6). Kashiwara [13, 14] introduces the endomorphisms $\tilde{e},\ \tilde{f}$ of $V_{(l)}$ defined by

$$\tilde{e}\, u_k = u_{k-1}\,, \quad \tilde{f}\, u_k = u_{k+1}\,, \quad k = 0, \ldots, l,$$

where $u_{-1} = u_{l+1} = 0$. More generally, if M is a direct sum of modules $V_{(l)}$, that is, if there exists an isomorphism of $U_q(\mathfrak{sl}_2)$-modules $\phi : M \xrightarrow{\sim} \oplus V_{(l)}^{\oplus \alpha_l}$, one defines endomorphisms $\tilde{e},\ \tilde{f}$ of M by means of ϕ in the obvious way, and one checks easily that they do not depend on the choice of ϕ. In particular, if M is an integrable $U_q(\mathfrak{gl}_n)$-module, regarding M as a U_i-module we define operators $\tilde{e}_i,\ \tilde{f}_i$ on M for $i = 1, \ldots, n-1$.

Example 3.2 We keep the notations of Example 3.1. We have

$$\begin{smallmatrix}2&\\1&3\end{smallmatrix} + q \begin{smallmatrix}3&\\1&2\end{smallmatrix} \xrightarrow{\tilde{f}_2} \begin{smallmatrix}3&\\1&3\end{smallmatrix}$$

$$\begin{smallmatrix}3&\\1&2\end{smallmatrix} - q \begin{smallmatrix}2&\\1&3\end{smallmatrix} \xrightarrow{\tilde{f}_2} 0$$

Hence,

$$\begin{array}{|c|c|} \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} \xrightarrow{\quad \tilde{f}_2 \quad} \frac{1}{1+q^2} \begin{array}{|c|c|} \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array} \xrightarrow{\quad \tilde{f}_2 \quad} \frac{q}{1+q^2} \begin{array}{|c|c|} \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array}$$

Since we want to let q tend to 0, we introduce the subring $\mathcal{A}$ of $K(q)$ consisting of rational functions without pole at $q = 0$. A *crystal lattice* of M is a free $\mathcal{A}$-module L such that $M = K(q) \otimes_{\mathcal{A}} L$, $L = \oplus_\mu L_\mu$ where $L_\mu = L \cap M_\mu$, and

$$\tilde{e}_i L \subset L\,, \quad \tilde{f}_i L \subset L\,, \quad i = 1, \ldots, n-1\,. \tag{24}$$

In other words, L spans M over $K(q)$, L is compatible with the weight space decomposition of M and is stable under the operators $\tilde{e}_i$ and $\tilde{f}_i$. It follows that $\tilde{e}_i$, $\tilde{f}_i$ induce endomorphisms of the K-vector space L/qL that we shall still denote by $\tilde{e}_i$, $\tilde{f}_i$. Now Kashiwara defines a *crystal basis* of M (at $q = 0$) to be a pair (L, B) where L is a crystal lattice in M and B is a basis of L/qL such that $B = \sqcup B_\mu$ where $B_\mu = B \cap (L_\mu / qL_\mu)$, and

$$\tilde{e}_i B \subset B \sqcup \{0\}\,, \quad \tilde{f}_i B \subset B \sqcup \{0\}\,, \quad i = 1, \ldots, n-1\,, \tag{25}$$

$$\tilde{e}_i v = u \quad \Longleftrightarrow \quad \tilde{f}_i u = v, \quad u, v \in B, \quad i = 1, \ldots, n-1\,. \tag{26}$$

Example 3.3 We continue Examples 3.1 and 3.2. Denote by $\mathcal{B}$ the basis of quantum tableaux in M, and let L be the $\mathcal{A}$-lattice in M spanned by the elements of $\mathcal{B}$. Let B be the projection of $\mathcal{B}$ in L/qL. Then, Example 3.1 shows that (L, B) is a crystal basis of M.

Kashiwara has proven the following existence and uniqueness result for crystal bases [13, 14].

Theorem 3.4 *Any integrable $U_q(\mathfrak{gl}_n)$-module M has a crystal basis (L, B). Moreover, if (L', B') is another crystal basis of M, then there exists a $U_q(\mathfrak{gl}_n)$-automorphism of M sending L on L' and inducing an isomorphism of vector spaces from L/qL to L'/qL' which sends B on B'. In particular, if $M = V_\lambda$ is irreducible, its crystal basis $(L(\lambda), B(\lambda))$ is unique up to an overall scalar multiple. It is given by*

$$L(\lambda) = \sum_{1 \le i_1, i_2, \ldots, i_r \le n-1} \mathcal{A}\, \tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_r}\, u_\lambda, \tag{27}$$

$$B(\lambda) = \{\tilde{f}_{i_1} \tilde{f}_{i_2} \cdots \tilde{f}_{i_r}\, u_\lambda \bmod qL(\lambda) \,|\, 1 \le i_1, \ldots, i_r \le n\} \backslash \{0\}, \tag{28}$$

where u_λ is a highest weight vector of V_λ.

It follows that one can associate to each integrable $U_q(\mathfrak{gl}_n)$-module M a well-defined colored graph $\Gamma(M)$ whose vertices are labelled by the elements of B and whose edges describe the action of the operators $\tilde{f}_i$:

$$u \xrightarrow{i} v \iff \tilde{f}_i u = v\,.$$

$\Gamma(M)$ is called the *crystal graph* of M.

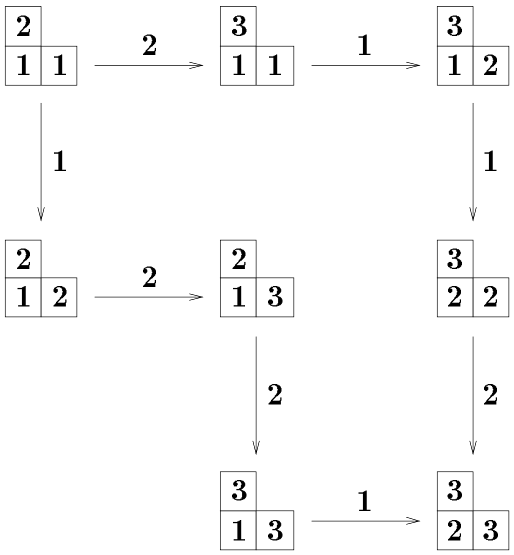


Figure 1: Crystal graph of the $U_q(\mathfrak{gl}_3)$-module $V_{(2,1)}$

Example 3.5 The crystal graph of the $U_q(\mathfrak{gl}_3)$-module $V_{(2,1)}$ is readily deduced from Examples 3.1, 3.2 and 3.3. It is shown in Figure 1.

It is clear from this definition that the crystal graph $\Gamma(M)$ of the direct sum $M = M_1 \oplus M_2$ of two $U_q(\mathfrak{gl}_n)$-modules is the disjoint union of $\Gamma(M_1)$ and $\Gamma(M_2)$. It follows from the complete reducibility of M that the connected components of $\Gamma(M)$ are the crystal graphs of the irreducible components of M. Note also that if one restricts the crystal graph $\Gamma(M)$ to its edges of colour i, one obtains a decomposition of this graph into *strings* of colour i corresponding to the U_i-decomposition of M. For a vertex v of $\Gamma(M)$, we shall denote by $\epsilon_i(v)$ (*resp.* $\phi_i(v)$) the distance of v to the origin (*resp.* end) of its string of colour i, that is,

$$\epsilon_i(v) = \max\{k \mid \tilde{e}_i^k v \neq 0\}, \quad \phi_i(v) = \max\{k \mid \tilde{f}_i^k v \neq 0\}.$$

The integers $\epsilon_i(v)$, $\phi_i(v)$ give information on the geometry of the graph $\Gamma(M)$ around the vertex v. They seem to be very significant from a representation theoretical point of view. Indeed, both the Littlewood-Richardson multiplicities $c^{\nu}_{\lambda\,\mu}$ and the q-weight multiplicities $K_{\lambda\,\mu}(q)$ can be computed in a simple way from the integers $\epsilon_i(v)$, $\phi_i(v)$ attached to the crystal graph $\Gamma(V_\lambda)$ [1, 18].

One of the nicest properties of crystal bases is that they behave well under the tensor product operation. We shall first consider an example, from which Kashiwara deduces by induction the general description of the crystal basis of a tensor product [13].

Example 3.6 We slightly modify the notations of Example 2.6 and write $\{u_k^{(l)}, k = 0, \ldots, l\}$ for the canonical basis of the $U_q(\mathfrak{sl}_2)$-module $V_{(l)}$. We recall that for convenience we set $u_{-1}^{(l)} = u_{l+1}^{(l)} = 0$. A first basis of the tensor product $V_{(1)} \otimes V_{(l)}$ is $\{u_j^{(1)} \otimes u_k^{(l)}, j =$

$0, 1,\ k = 0, \ldots, l\}$. We define

$$w_k = u_0^{(1)} \otimes u_k^{(l)} + q^{l-k+1} u_1^{(1)} \otimes u_{k-1}^{(l)}, \quad k = 0, \ldots, l+1, \tag{29}$$
$$z_k = q^{l-k-1}\,[l-k]\, u_1^{(1)} \otimes u_k^{(l)} - q^l\,[k+1]\, u_0^{(1)} \otimes u_{k+1}^{(l)}, \quad k = 0, \ldots, l-1. \tag{30}$$

It is straightforward to check that $e_1 w_0 = e_1 z_0 = 0,\ f_1 w_{l+1} = f_1 z_{l-1} = 0$, and

$$f_1 w_k = [k+1]\, w_{k+1}, \quad f_1 z_k = [k+1]\, z_{k+1}.$$

Hence $\{w_k\}$ and $\{z_k\}$ span submodules of $V_{(1)} \otimes V_{(l)}$ isomorphic to $V_{(l+1)}$ and $V_{(l-1)}$ respectively, and are the canonical bases of these irreducible representations of $U_q(\mathfrak{sl}_2)$. Therefore, the $\mathcal{A}$-lattice L spanned by $\{w_k\}$, $\{z_k\}$ is a crystal lattice and if we define $B = \{w_k \bmod qL\} \sqcup \{z_k \bmod qL\}$, then (B, L) is a crystal basis of $V_{(1)} \otimes V_{(l)}$. On the other hand, equations (29) (30) show that L coincides with the tensor product $L(1) \otimes L(l)$ of the crystal lattices of $V_{(1)}$ and $V_{(l)}$, and that

$$w_k \equiv u_0^{(1)} \otimes u_k^{(l)} \bmod qL,\ k = 0, \ldots, l, \quad w_{l+1} \equiv u_1^{(1)} \otimes u_l^{(l)} \bmod qL.$$

$$z_k \equiv u_1^{(1)} \otimes u_k^{(l)} \bmod qL, \quad k = 0, \ldots, l-1.$$

Thus, (L, B) coincides with $(L(1) \otimes L(l), B(1) \otimes B(l))$, where we have denoted by $B(1) \otimes B(l)$ the basis $\{u_j^{(1)} \otimes u_k^{(l)}, \bmod qL,\ j = 0, 1,\ k = 0, \ldots, l\}$. Finally, the action of $\tilde{f}_1$ on $B(1) \otimes B(l)$ is described by the crystal graph:

$$\begin{array}{ccccccccc}
u_0^{(1)} \otimes u_0^{(l)} & \to & u_0^{(1)} \otimes u_1^{(l)} & \to & \cdots & \to & u_0^{(1)} \otimes u_{l-1}^{(l)} & \to & u_0^{(1)} \otimes u_l^{(l)} \\
 & & & & & & & & \downarrow \\
u_1^{(1)} \otimes u_0^{(l)} & \to & u_1^{(1)} \otimes u_1^{(l)} & \to & \cdots & \to & u_1^{(1)} \otimes u_{l-1}^{(l)} & & u_1^{(1)} \otimes u_l^{(l)}
\end{array}$$

More generally, we have the following property [13].

Theorem 3.7 *Let (L_1, B_1) and (L_2, B_2) be crystal bases of integrable $U_q(\mathfrak{gl}_n)$-modules M_1 and M_2. Let $B_1 \otimes B_2$ denote the basis $\{u \otimes v,\ u \in B_1,\ v \in B_2\}$ of $(L_1/qL_1) \otimes (L_2/qL_2)$ (which is isomorphic to $(L_1 \otimes L_2)/q(L_1 \otimes L_2)$). Then, $(L_1 \otimes L_2,\ B_1 \otimes B_2)$ is a crystal basis of $M_1 \otimes M_2$, the action of $\tilde{e}_i$, $\tilde{f}_i$ on $B_1 \otimes B_2$ being given by*

$$\tilde{f}_i(u \otimes v) = \begin{cases} u \otimes \tilde{f}_i v & \text{if} \quad \epsilon_i(u) < \phi_i(v) \\ \tilde{f}_i u \otimes v & \text{if} \quad \epsilon_i(u) \geq \phi_i(v) \end{cases}, \tag{31}$$

$$\tilde{e}_i(u \otimes v) = \begin{cases} \tilde{e}_i u \otimes v & \text{if} \quad \epsilon_i(u) > \phi_i(v) \\ u \otimes \tilde{e}_i v & \text{if} \quad \epsilon_i(u) \leq \phi_i(v) \end{cases}. \tag{32}$$

Theorem 3.7 enables one to describe the crystal graph of $V_{(1)}^{\otimes m}$ for any m, and to deduce from that the description of the crystal graph Γ_λ of the simple $U_q(\mathfrak{gl}_n)$-module V_λ. For convenience, we shall identify the tensor algebra $T(V_{(1)})$ with the free associative algebra $K(q)\langle A\rangle$ over the alphabet $A = \{1, \ldots, n\}$ via the isomorphism $v_i \mapsto i$, where $\{v_i\}$ is the canonical basis of $V_{(1)}$ defined in Example 2.1. Accordingly, the crystal lattice L spanned by the monomials in the v_i is identified with $\mathcal{A}\langle A\rangle$ and the K-vector space L/qL with $K\langle A\rangle$.

We recall some terminology on words. A *subword* of a word $w = x_1 \cdots x_m$ is a word of the form $u = x_{i_1} \cdots x_{i_k}$ with $1 \le i_1 < \cdots < i_k \le m$. A *factor* of w is a subword consisting of consecutive letters of w.

Define linear operators $\hat{e}_i$, $\hat{f}_i$ on $K\langle A\rangle$ in the following way. Consider first the case of a two-letter alphabet $A = \{i, i{+}1\}$. Let $w = x_1 \cdots x_m$ be a word on A. Delete every factor $((i{+}1)\, i)$ of w. The remaining letters constitute a subword w_1 of w. Then, delete again every factor $((i{+}1)\, i)$ of w_1. There remains a subword w_2. Continue this procedure until it stops, leaving a word $w_k = x_{j_1} \cdots x_{j_{r+s}}$ of the form $w_k = i^r (i{+}1)^s$. The image of w_k under $\hat{e}_i$, $\hat{f}_i$ is the word $y_{j_1} \cdots y_{j_{r+s}}$ given by

$$\hat{e}_i(i^r(i{+}1)^s) = \begin{cases} i^{r+1}(i{+}1)^{s-1} & (s \ge 1) \\ 0 & (s = 0) \end{cases}, \qquad \hat{f}_i(i^r(i{+}1)^s) = \begin{cases} i^{r-1}(i{+}1)^{s+1} & (r \ge 1) \\ 0 & (r = 0) \end{cases}.$$

The image of the initial word w is then $w' = y_1 \cdots y_m$, where $y_k = x_k$ for $k \notin \{j_1, \ldots, j_{r+s}\}$. For instance, if

$$w = (2\ 1)\ 1\ 1\ 2\ (2\ 1)\ 1\ 1\ 1\ 2\,,$$

we shall have

$$w_1 = .\ .\ 1\ 1\ (2\ .\ .\ 1)\ 1\ 1\ 2\,,$$

$$w_2 = .\ .\ \ 1\ 1\ .\ .\ .\ .\ \ 1\ 1\ 2\,.$$

Thus,

$$\hat{e}_1(w) = \ 2\ 1\ \mathbf{1}\ \mathbf{1}\ 2\ 2\ 1\ 1\ \mathbf{1}\ \mathbf{1}\ \mathbf{1}\,,$$

$$\hat{f}_1(w) = 2\ 1\ \mathbf{1}\ \mathbf{1}\ 2\ 2\ 1\ 1\ \mathbf{1}\ \mathbf{2}\ \mathbf{2}\,,$$

where the letters printed in bold type are those of the image of the subword w_2. Finally, in the general case, the action of the operators $\hat{e}_i$, $\hat{f}_i$ on w is defined by the previous rules applied to the subword consisting of the letters i, $i{+}1$, the remaining letters being unchanged.

The operators $\hat{e}_i$, $\hat{f}_i$ have been considered in [21], where they were used as building blocks for defining noncommutative analogues of Demazure symmetrization operators. It follows from Theorem 3.7 that they coincide in the above identification with the endomorphisms $\tilde{e}_i$, $\tilde{f}_i$ on the K-space L/qL [16].

It is straightforward to deduce from the definition of $\hat{e}_i$, $\hat{f}_i$ the following compatibility properties with the Robinson-Schensted correspondence:

(a) for any word w on A such that $\hat{e}_i w \neq 0$, we have $Q(\hat{e}_i w) = Q(w)$,

(b) for any pair of words w, u such that $P(w) = P(u)$ and $\hat{e}_i w \neq 0$, we have $P(\hat{e}_i w) = P(\hat{e}_i u)$.

In other words, the operators $\hat{e}_i$, $\hat{f}_i$ do not change the insertion tableau and are compatible with the plactic equivalence. Moreover, the words y such that $\hat{e}_i y = 0$ for any i are characterized by the Yamanouchi property: each right factor of y contains at least as many letters i than $i + 1$, and this for any i. It is well-known that for any standard Young tableau τ there exists a unique Yamanouchi word y such that $Q(y) = \tau$. This yields the following crystallization of (16).

Theorem 3.8 *The crystal graph of the $U_q(\mathfrak{gl}_n)$-module $V_{(1)}^{\otimes m}$ is the colored graph whose vertices are the words of length m over A, and whose edges are given by:*

$$w \stackrel{i}{\longrightarrow} u \quad \Longleftrightarrow \quad \hat{f}_i w = u \,.$$

The connected components Γ_τ of $\Gamma(V_{(1)}^{\otimes m})$ are parametrized by the set of Young tableaux τ of weight (1^m). The vertices of Γ_τ are those words w which satisfy $Q(w) = \tau$. Moreover, if λ denotes the shape of τ, then Γ_τ is isomorphic to the crystal graph $\Gamma(V_\lambda)$.

Replacing the vertices w of Γ_τ by their associated Young tableaux $P(w)$, we obtain a labelling of $\Gamma(V_\lambda)$ by the set of Young tableaux of shape λ, as shown in Example 3.5. It follows from property *(b)* above that this labelling does not depend on the particular choice of τ among the standard Young tableaux of shape λ. We end this section by showing a less trivial example of crystal graph, which is computed easily using the previous description of $\hat{f}_i$.

Example 3.9 The crystal graph of the $U_q(\mathfrak{gl}_4)$-module $V_{(2,2)}$ is shown in Figure 2.

4 $\mathcal{F}_q[GL_n/B]$

The subalgebra $\mathcal{F}_q[GL_n/B]$ of $\mathcal{F}_q[\mathrm{Mat}_n]$ generated by quantum column-shaped tableaux can also be defined by generators and relations [29, 19]. The generators are denoted by columns

$$\boxed{\begin{array}{c} i_k \\ \vdots \\ i_2 \\ i_1 \end{array}} \qquad 1 \le k \le n, \quad 1 \le i_1, \ldots, i_k \le n,$$

and their products are written by juxtaposition. The relations are

(R_1) if $i_r = i_s$ for some indices r, s, then

$$\boxed{\begin{array}{c} i_k \\ \vdots \\ i_2 \\ i_1 \end{array}} = 0\,,$$

(R_2) for $w \in S_k$ and $i_1 < i_2 < \cdots < i_k$, we have

$$\boxed{\begin{array}{c} i_{w_k} \\ \vdots \\ i_{w_2} \\ i_{w_1} \end{array}} = (-q)^{-\ell(w)} \boxed{\begin{array}{c} i_k \\ \vdots \\ i_2 \\ i_1 \end{array}}\,,$$

(R_3) for $k \le l$ and $j_1 < j_2 < \cdots < j_l$, we have

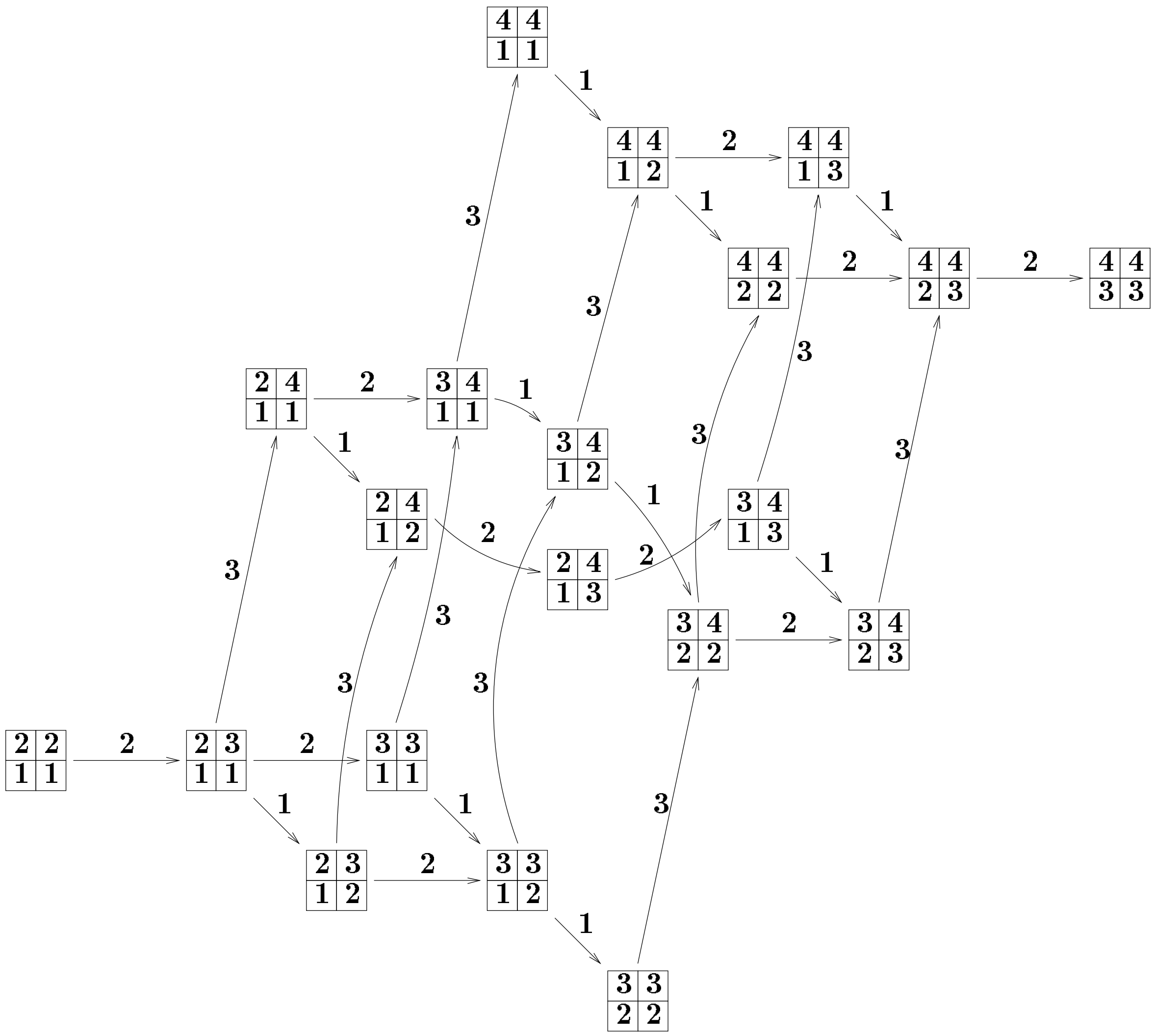


Figure 2: Crystal graph of the $U_q(\mathfrak{gl}_4)$-module $V_{(2,2)}$

$$\begin{array}{cc} & \begin{array}{|c|} \hline j_l \\ \vdots \\ j_{k+1} \end{array} \\ \begin{array}{|c|} \hline i_k \\ \vdots \\ i_1 \\ \hline \end{array} & \begin{array}{|c|} j_k \\ \vdots \\ j_1 \\ \hline \end{array} \end{array} = \sum_{w} (-q)^{\ell(w)} \begin{array}{cc} \begin{array}{|c|} \hline j_{w_l} \\ \vdots \\ j_{w_{k+1}} \end{array} & \\ \begin{array}{|c|} i_k \\ \vdots \\ i_1 \\ \hline \end{array} & \begin{array}{|c|} \hline j_{w_k} \\ \vdots \\ j_{w_1} \\ \hline \end{array} \end{array}$$

where the sum runs through all $w \in S_l$ such that $w_1 < \cdots < w_k$ and $w_{k+1} < \cdots < w_l$,

(R_4) for $k + s \leq l - 1$ and $j_1 < j_2 < \cdots < j_l$, we have

$$\sum_w (-q)^{\ell(w)} \begin{array}{|c|c|} \hline j_{w_l} & \\ \vdots & r_s \\ & \vdots \\ j_{w_{k+2}} & r_1 \\ j_{w_{k+1}} & j_{w_k} \\ i_{k-1} & j_{w_{k-1}} \\ \vdots & \vdots \\ i_1 & j_{w_1} \\ \hline \end{array} = 0$$

where the sum runs through all permutations $w \in S_l$ such that $w_1 < \cdots < w_k$ and $w_{k+1} < \cdots < w_l$.

Relations (R_1), (R_2) express the q-alternating property of quantum minors, while relations (R_3), (R_4) are the natural q-analogues of Sylvester's and Garnir's identities respectively (*cf.* [22]). Using (R_1), (R_2), (R_3) one can express every element of $\mathcal{F}_q[GL_n/B]$ as a linear combination of products of columns, the size of which weakly decreases from left to right. After that, following the same strategy as in the classical straightening algorithm, one can by means of (R_4) express all such products as linear combinations of quantum tableaux. Note that it follows from (R_3), (R_4) that all the quantum tableaux (τ) appearing in the straightening of a given tabloid (δ) have the same shape λ obtained by reordering the sizes of the columns of δ, which means that (δ) falls in the irreducible component V_λ of $\mathcal{F}_q[GL_n/B]$.

Example 4.1 We apply the q-straightening algorithm to

$$(\delta) = \begin{array}{|c|c|} \hline & 6 \\ 5 & 3 \\ 1 & 2 \\ \hline \end{array}$$

It follows from (R_1), (R_2), (R_3) that

$$\begin{array}{|c|c|} \hline & 6 \\ 5 & 3 \\ 1 & 2 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 6 & \\ 5 & 3 \\ 1 & 2 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 5 & \\ 3 & 6 \\ 1 & 2 \\ \hline \end{array} - q \begin{array}{|c|c|} \hline 5 & \\ 2 & 6 \\ 1 & 3 \\ \hline \end{array}$$

Using (R_4) for straightening the first term of the right-hand side, we obtain

$$\begin{array}{|c|c|} \hline & 6 \\ 5 & 3 \\ 1 & 2 \\ \hline \end{array} = (1-q^2) \begin{array}{|c|c|} \hline 5 & \\ 3 & 6 \\ 1 & 2 \\ \hline \end{array} + (q^3-q) \begin{array}{|c|c|} \hline 5 & \\ 2 & 6 \\ 1 & 3 \\ \hline \end{array} + q \begin{array}{|c|c|} \hline 6 & \\ 3 & 5 \\ 1 & 2 \\ \hline \end{array} - q^2 \begin{array}{|c|c|} \hline 6 & \\ 2 & 5 \\ 1 & 3 \\ \hline \end{array} - q^4 \begin{array}{|c|c|} \hline 3 & \\ 2 & 6 \\ 1 & 5 \\ \hline \end{array}$$

Letting q tend to 0 in this expansion, we get an illustration of the next Theorem, which can be regarded as a version of Theorem 1.1 for $\mathcal{F}_q[GL_n/B]$.

Theorem 4.2 *Let $\mathcal{B}$ denote the linear basis of $\mathcal{F}_q[GL_n/B]$ consisting of quantum tableaux, and let $\mathcal{L}$ be the $K[q]$-lattice generated by $\mathcal{B}$. Let λ be a partition, and (δ) a quantum tabloid in V_λ. Denote by w the word obtained by reading the columns of δ from top to bottom and left to right. Then,*

$$(\delta) \equiv \begin{cases} (P(w)) \mod q\mathcal{L} & \textit{if } P(w) \textit{ has shape } \lambda \\ \quad 0 \quad\ \mod q\mathcal{L} & \qquad \textit{otherwise} \end{cases} .$$

We shall not try, though this might be interesting, to derive Theorem 4.2 from a precise analysis of the q-straightening algorithm. The only consequence of this algorithm that we shall need is the following:

Proposition 4.3 *Any quantum tabloid lies in the $K[q,q^{-1}]$-linear span of the set of quantum tableaux.*

Our proof of Theorem 4.2 will be derived from the following result which describes several interesting bases of V_λ, and states that each of them gives rise to the same crystal basis at $q=0$. We first introduce some notations. Let λ be a partition, $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$ its conjugate, and $\sigma(\lambda') = (\lambda'_{\sigma_1}, \ldots, \lambda'_{\sigma_r})$ the composition obtained by permuting the parts of λ' using the permutation $\sigma \in S_r$. A tabloid δ is said to have *shape* $\sigma(\lambda')$ if it is a sequence of column-shaped Young tableaux $(c_1, \ldots, c_r)$, the size of the column c_i being equal to λ'_{σ_i}, $i = 1, \ldots, r$. The *column reading* of δ is the word u_δ obtained by reading the columns of δ from top to bottom and left to right.

Theorem 4.4 *For any $\sigma \in S_r$, the set of quantum tabloids*

$$B_\sigma = \{(\delta) \mid \delta \textit{ has shape } \sigma(\lambda') \textit{ and } P(u_\delta) \textit{ has shape } \lambda\}$$

is a linear basis of V_λ. The $\mathcal{A}$-lattice L spanned by B_σ is independent of σ as well as the projection B of B_σ in L/qL, and (L, B) is a crystal basis of V_λ. For $(\delta) \in B_\sigma$ and $(\gamma) \in B_\zeta$, we have

$$(\delta) \equiv (\gamma) \bmod qL \quad \Longleftrightarrow \quad P(u_\delta) = P(u_\gamma) .$$

Finally, if (δ) is such that $P(u_\delta)$ has not shape λ, then $(\delta) \equiv 0 \bmod qL$.

Proof: We fix $\sigma \in S_r$ and we write $\mu = (\mu_1, \ldots, \mu_r) := \sigma(\lambda')$. Recall from Example 2.2 that the q-analogue $V_{(1^k)}$ of the k th exterior power of the basic representation has a natural basis labelled by column-shaped Young tableaux. We introduce the $K(q)$-linear map φ_μ:

$$V_{(1^{\mu_1})} \otimes V_{(1^{\mu_2})} \otimes \cdots \otimes V_{(1^{\mu_r})} \quad \longrightarrow \quad V_\lambda$$

$$\boxed{\begin{matrix} b \\ \vdots \\ a \end{matrix}} \otimes \boxed{\begin{matrix} d \\ \vdots \\ \vdots \\ c \end{matrix}} \otimes \cdots \otimes \boxed{\begin{matrix} f \\ \vdots \\ e \end{matrix}} \quad \longrightarrow \quad \begin{array}{|c|c|} \hline & d \\ \cline{1-1} b & \vdots \\ \vdots & \vdots \\ a & c \\ \hline \end{array} \ \cdots \ \begin{array}{|c|} \hline f \\ \vdots \\ e \\ \hline \end{array}$$

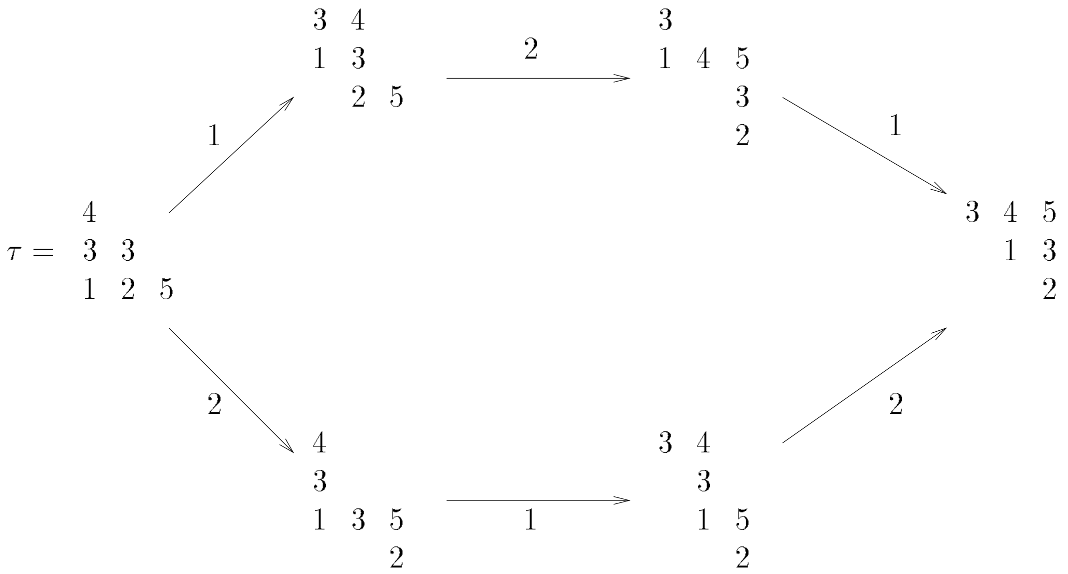
Comparing (12) (13) (14) to (18) (19) (20) we see immediately that φ_μ commutes with the action of $U_q(\mathfrak{gl}_n)$ and hence is in fact a morphism of $U_q(\mathfrak{gl}_n)$-modules. The image V_λ is irreducible and by (15) it appears with multiplicity one in $V_{(1^{\mu_1})} \otimes \cdots \otimes V_{(1^{\mu_r})}$. Therefore, the restriction of φ_μ to the irreducible component of $V_{(1^{\mu_1})} \otimes \cdots \otimes V_{(1^{\mu_r})}$ with highest weight vector the tensor product $y^\mu := (y_{(1^{\mu_1})}) \otimes \cdots \otimes (y_{(1^{\mu_r})})$ of all highest weight vectors in $V_{(1^{\mu_1})}, \ldots, V_{(1^{\mu_r})}$ is an isomorphism, and the kernel of φ_μ is the sum of all components $V_\nu,\ \nu \neq \lambda$ appearing in $V_{(1^{\mu_1})} \otimes \cdots \otimes V_{(1^{\mu_r})}$.

We now turn to the crystal bases of these various $U_q(\mathfrak{gl}_n)$-modules. It follows from Example 2.2 and Theorem 3.4 that, if we choose $u_{(1^k)} = y_{(1^k)}$, $L(1^k)$ is the $\mathcal{A}$-lattice spanned by the basis of column-shaped quantum tableaux, while $B(1^k)$ is the projection of this basis in $L(1^k)/qL(1^k)$. Using Theorem 3.7 we deduce that a crystal basis of $V_{(1^{\mu_1})} \otimes \cdots \otimes V_{(1^{\mu_r})}$ can be constructed by tensoring the crystal bases of the factors. Denote this basis by $(L^\mu,\, B^\mu)$. Then $\varphi_\mu(L^\mu)$ is a crystal lattice in V_λ. According to Theorem 3.4 such a lattice is unique up to an overall scalar multiple which can be determined by considering the highest weight space. Here we have $(L^\mu)_\lambda = \mathcal{A}\, y^\mu$ and $\varphi_\mu(y^\mu) = (y_\lambda)$. Therefore, $\varphi_\mu(L^\mu)$ is the crystal lattice L of V_λ specified by $L_\lambda = \mathcal{A}\,(y_\lambda)$, and it does not depend on σ. Now φ_μ induces an isomorphism from the K-subspace of L^μ/qL^μ spanned by the subset of B^μ consisting of those b in the connected component of the crystal graph with source y^μ mod qL^μ, to the K-space L/qL. It is easy to check from Theorem 3.7 and the explicit description of the operators $\hat{e}_i$, $\hat{f}_i$ which follows it that this connected component is labelled by the elements $b = c_1 \otimes \cdots \otimes c_r$ mod L^μ such that the word u_b obtained by reading c_1 from top to bottom, then c_2 from top to bottom, and so on, satisfies: $P(u_b)$ has shape λ. Hence its image is B_σ mod qL. This proves that B_σ is a basis of V_λ and that $L = \sum_{b \in B_\sigma} \mathcal{A}\, b$. Moreover, $(L,\, B_\sigma \bmod qL)$ is a crystal basis, which proves by unicity that B_σ mod qL does not depend on σ. On the other hand, the elements b of B^μ which belong to the other connected components are sent to 0 in L/qL. This means that the tabloids (δ) such that $P(u_\delta)$ has not shape λ belong to qL.

Finally, if ζ is another permutation and γ a tabloid of shape $\zeta(\lambda')$, the fact that $(\delta) \equiv (\gamma) \bmod qL$ means exactly that the words u_δ and u_γ label the same vertex in the two copies of $\Gamma(V_\lambda)$ to which they belong in $\Gamma(V_{(1)}^{\otimes k})$, that is, by Theorem 3.8, $P(u_\delta) = P(u_\gamma)$. □

Proof of Theorem 4.2: Let (δ) be a quantum tabloid in V_λ. Proposition 4.3 shows that (δ) is a $K[q, q^{-1}]$-linear combination of quantum tableaux. On the other hand, Theorem 4.4 shows that (δ) belongs to the $\mathcal{A}$-lattice L spanned by the set $\mathcal{B} = B_{\mathrm{id}}$ of quantum tableaux. Therefore, (δ) is in fact an element of the $K[q]$-lattice $\mathcal{L}$ spanned by $\mathcal{B}$. The other statements of Theorem 4.2 follow immediately from Theorem 4.4. □

We point out that the quantum tabloids $(\delta) \in B_\sigma$ are easily computed using Schützenberger's jeu de taquin [28]. For example, the following graph shows the δ's giving rise to quantum tabloids (δ) which are congruent to the quantum tableau (τ) modulo qL:



The edges labelled i connect tabloids δ, γ obtained one from the other by permuting the column lengths of the i th and $(i+1)$ th columns by means of jeu de taquin. The column readings u_δ of these tabloids are distinguished words of the plactic class of u_τ, called *frank words* [21]. The combinatorics of frank words occurs in several interesting problems (see [21, 24, 8]).

5 Proof of Theorem 1.1

The proof of Theorem 1.1 follows the same lines as that of Theorem 4.2, namely we first describe several realizations of the crystal basis of the bimodule $\mathcal{F}_q[\mathrm{Mat}_n]$ and then deduce Theorem 1.1 by comparing them, using the unicity property stated in Theorem 3.4.

We retain the notations of Section 4 (before Theorem 4.4). In particular, we fix a partition λ of k, and set $\mu = \sigma(\lambda')$ for $\sigma \in S_r$. We denote by W_λ the subspace of $\mathcal{F}_q[\mathrm{Mat}_n]$ whose decomposition as a $U_q(\mathfrak{gl}_n)$-bimodule is $W_\lambda \simeq \oplus_{\nu \le \lambda} V_\nu \otimes V_\nu$. Thus, $W_{(k)}$ is the homogeneous component of degree k of $\mathcal{F}_q[\mathrm{Mat}_n]$. For each $\nu \le \lambda$, we choose a standard Young tableau τ_ν of shape ν. We can now state

Theorem 5.1 *Let $B_{\lambda,\sigma}$ denote the following set of quantum bitabloids:*

$$B_{\lambda,\sigma} = \{(\delta|\delta') \mid \delta, \delta' \textit{ have shape } \mu \textit{ and } Q(u_\delta) = Q(u'_\delta) = \tau_\nu \textit{ for some } \nu \le \lambda\}\,.$$

Then $B_{\lambda,\sigma}$ is a basis of W_λ. The $\mathcal{A}$-lattice L_λ spanned by $B_{\lambda,\sigma}$ does not depend on σ or on the particular choice of Young tableaux $\{\tau_\nu\}$. The projection B_λ of $B_{\lambda,\sigma}$ in L_λ/qL_λ is also independent of σ, and (L_λ, B_λ) is a crystal basis of W_λ. Moreover for $(\delta|\delta') \in B_{\lambda,\sigma}$ and $(\gamma|\gamma') \in B_{\lambda,\zeta}$, we have

$$(\delta|\delta') \equiv (\gamma|\gamma') \bmod qL_\lambda \quad \Longleftrightarrow \quad P(u_\delta) = P(u_\gamma) \textit{ and } P(u_{\delta'}) = P(u_{\gamma'})\,.$$

Finally, the crystal graph attached to (L_λ, B_λ) is the coloured graph given by

$$(\delta|\delta') \xrightarrow{i_l^\dagger} (\gamma|\delta') \quad \Longleftrightarrow \quad f_i^\dagger(\delta|\delta') = (\gamma|\delta') \quad \Longleftrightarrow \quad \hat{f}_i u_\delta = u_\gamma\,,$$

$$(\delta|\delta') \xrightarrow{i_r} (\delta|\gamma') \iff f_i(\delta|\delta') = (\delta|\gamma') \iff \hat{f}_i u_{\delta'} = u_{\gamma'}\,.$$

Proof: We imitate the proof of Theorem 4.4 and introduce the $K(q)$-linear map Φ_μ:

$$\begin{array}{ccc} \left(V_{(1^{\mu_1})} \otimes \cdots \otimes V_{(1^{\mu_r})}\right) \otimes \left(V_{(1^{\mu_1})} \otimes \cdots \otimes V_{(1^{\mu_r})}\right) & \longrightarrow & W_\lambda \\ (c_1 \otimes \cdots \otimes c_r) \otimes (d_1 \otimes \cdots \otimes d_r) & \longrightarrow & (c_1 \cdots c_r | d_1 \cdots d_r) \end{array}$$

Here c_i, d_j denote elements of the canonical bases of $V_{(1^{\mu_i})}$, $V_{(1^{\mu_j})}$, *i.e.* columns of size μ_i, μ_j, and $(c_1 \cdots c_r|d_1 \cdots d_r)$ is the quantum bitabloid formed on these columns. The map Φ_μ is in fact a homomorphism of $U_q(\mathfrak{gl}_n)$-bimodules, and therefore sends the crystal lattice $\mathcal{L}_{\lambda,\sigma}$ spanned by the tensors $(c_1 \otimes \cdots \otimes c_r) \otimes (d_1 \otimes \cdots \otimes d_r)$ onto a crystal lattice $L_{\lambda,\sigma}$ in W_λ. To describe precisely $L_{\lambda,\sigma}$, it is enough to determine its submodule $L^+_{\lambda,\sigma}$ of highest weight vectors. We shall prove that $L^+_{\lambda,\sigma} = \oplus_{\nu \le \lambda} \mathcal{A}\,(y_\nu|y_\nu)$.

To do this we have to introduce some notation. We write for short $V^\mu := V_{(1^{\mu_1})} \otimes \cdots \otimes V_{(1^{\mu_r})}$, and we consider a source vertex

$$y = \boxed{\begin{matrix} y_1 \\ \vdots \\ y_{\mu_1} \end{matrix}} \otimes \cdots \otimes \boxed{\begin{matrix} y_{k-\mu_r+1} \\ \vdots \\ y_k \end{matrix}}$$

of the crystal graph of V^μ. This means that $y_1 \cdots y_k$ is a Yamanouchi word on $\{1, \ldots, n\}$ such that the columns of y are increasing from bottom to top. By definition of a crystal basis, there exists a highest weight vector T_y in V^μ such that $T_y \equiv y \bmod q\mathcal{L}_{\lambda,\sigma}$. Now, let

$$w = \boxed{\begin{matrix} w_1 \\ \vdots \\ w_{\mu_1} \end{matrix}} \otimes \cdots \otimes \boxed{\begin{matrix} w_{k-\mu_r+1} \\ \vdots \\ w_k \end{matrix}}$$

be any monomial tensor of the same weight ν as y. Then $\Phi_\mu(T_y \otimes w) = \alpha_{y,w}(q)\,(y_\nu|y_\nu)$ for some scalar $\alpha_{y,w}(q) \in K(q)$. Indeed,

$$e_i^\dagger \Phi_\mu(T_y \otimes w) = \Phi_\mu(e_i^\dagger T_y \otimes w) = \Phi_\mu((e_i T_y) \otimes w) = 0\,,$$

and on the other hand $\Phi_\mu(T_y \otimes w)$ has weight (ν, ν). Therefore $\Phi_\mu(T_y \otimes w)$ is a scalar multiple of the highest weight vector $(y_\nu|y_\nu)$. But from the definition of Φ_μ, we have

$$\Phi_\mu(T_y \otimes w) = \sum_{\mathbf{u}} \kappa_{\mathbf{u}}(q)\,(\mathbf{u}|\mathbf{w})\,. \tag{33}$$

Here $\mathbf{w}$ is the tabloid made of the columns of w, $\mathbf{u}$ runs through the tabloids of the same shape and weight as $\mathbf{w}$, and $\kappa_{\mathbf{u}}(q) \in \mathcal{A}$. Moreover, $\kappa_{\mathbf{y}}(0) = 1$ and $\kappa_{\mathbf{u}}(0) = 0$ for $\mathbf{u} \neq \mathbf{y}$. Expanding (33) on the basis

$$\beta_w = \{t_{u_1 w_1} \cdots t_{u_k w_k} \mid u_1 \cdots u_k \text{ has weight } \nu \text{ and } w_i = w_j \Longrightarrow u_i \le u_j\}$$

and comparing to the similar expansion of $(y_\nu|y_\nu)$, we obtain by checking the coefficient of $t_{w_1 w_1} \cdots t_{w_k w_k}$ that $\alpha_{y,w}(q) = \kappa_{\mathbf{w}}(q)$. Therefore, $\Phi_\mu(T_y \otimes w) \equiv \delta_{wy}\,(y_\nu|y_\nu) \bmod q(y_\nu|y_\nu)$, and if y' is another source vertex of $\Gamma(V^\mu)$, we have

$$\Phi_\mu(T_y \otimes T_{y'}) \equiv \delta_{y\,y'}\,(y_\nu|y_\nu) \bmod q(y_\nu|y_\nu)\,. \tag{34}$$

This proves that $L^+_{\lambda,\sigma} = \oplus_{\nu\leq\lambda}\mathcal{A}\,(y_\nu|y_\nu)$, as stated.

It follows that $L_{\lambda,\sigma}$ does not in fact depend on σ or on $\{\tau_\nu\}$, and we may write $L_{\lambda,\sigma} = L_\lambda$. We can now consider the map induced by Φ_μ from $\mathcal{L}_{\lambda,\sigma}/q\mathcal{L}_{\lambda,\sigma}$ to L_λ/qL_λ (we still denote it by Φ_μ). By (34), it maps to zero all the connected components of the crystal graph of $V^\mu \otimes V^\mu$ with source vertex $y \otimes y'$ such that $y \neq y'$. This means precisely that $(\delta|\delta') \equiv 0 \bmod qL_\lambda$ whenever $Q(u_\delta) \neq Q(u_{\delta'})$. On the other hand, to each τ_ν corresponds a unique source vertex $y = y(\tau_\nu)$ such that $Q(u_y) = \tau_\nu$. The vertices $w \otimes w'$ of the connected components of $\Gamma(V^\mu \otimes V^\mu)$ with origin $y(\tau_\nu) \otimes y(\tau_\nu)$, $\nu \leq \lambda$, span a subspace of $\mathcal{L}_{\lambda,\sigma}/q\mathcal{L}_{\lambda,\sigma}$ isomorphic under Φ_μ to L_λ/qL_λ. Therefore $(L_\lambda,\, B_{\lambda,\sigma} \bmod qL_\lambda)$ is a crystal basis of W_λ, and this implies that $B_{\lambda,\sigma} \bmod qL_\lambda$ is independent of σ. The other statements follow now easily from Theorem 3.8. □

Proof of Theorem 1.1: Denote by L the crystal lattice of $\mathcal{F}_q[\mathrm{Mat}_n]$ whose submodule of highest weight vectors is equal to $\oplus_\nu\,\mathcal{A}\,(y_\nu|y_\nu)$. It follows from the proof of Theorem 5.1 that the crystal lattice L_λ of W_λ is nothing but $L \cap W_\lambda$. We also see that L is spanned over $\mathcal{A}$ by the set of bitableaux $(\tau\,|\,\tau')$. Indeed, if we choose τ_λ to be the standard Young tableau whose first column contains $1, 2, \ldots, \lambda'_1$, whose second column contains $\lambda'_1+1, \ldots, \lambda'_1+\lambda'_2$, and so on, we have

$$BT_\lambda := \{(\tau\,|\,\tau') \mid \tau,\, \tau' \text{ are Young tableaux of shape } \lambda\} \subset B_{\lambda,\mathrm{id}}$$

and $BT_\lambda \bmod L_\lambda$ is the part of B_λ which labels the connected component of $\Gamma(W_\lambda)$ corresponding to $V_\lambda \otimes V_\lambda$. Therefore, $(L, \sqcup_\lambda BT_\lambda \bmod L)$ is a crystal basis of $\mathcal{F}_q[\mathrm{Mat}_n]$ and $L = \oplus_\lambda\,\mathcal{A}\,BT_\lambda$. On the other hand, taking $\lambda = (k)$ in Theorem 5.1, we find that the set of monomials

$$B_k = \{t_{i_1j_1}\cdots t_{i_kj_k} \mid Q(i_1\cdots i_k) = Q(j_1\cdots j_k) = \tau_\nu \text{ for some } \nu\}$$

gives rise to the same crystal basis under the guise $(L, \sqcup_k B_k \bmod L)$. Hence, applying again Theorem 3.4, it follows from the description of the crystal graph that

$$t_{i_1j_1}\cdots t_{i_kj_k} \equiv \begin{cases} (P(i_1\cdots i_k)|P(j_1\cdots j_k)) & \bmod qL \quad \text{if } Q(i_1\cdots i_k) = Q(j_1\cdots j_k) \\ 0 & \bmod qL \quad \text{otherwise} \end{cases} \tag{35}$$

Finally, Proposition 4.3 extends to the case of bitableaux [10], that is, the coefficients of the straightening of a bitabloid belong to $K[q, q^{-1}]$. Since we have just proved that they also belong to $\mathcal{A}$, we can replace in (35) the crystal lattice L by the $K[q]$-lattice $\mathcal{L}$ spanned by the quantum bitableaux. □

References

[1] A. D. Berenstein, A. V. Zelevinsky, *Canonical bases for the quantum group of type A_r and piecewise-linear combinatorics*, Preprint Institut Gaspard Monge, 1994.

[2] C. Burdík, R. C. King, T. A. Welsh, *The explicit construction of irreducible representations of the quantum algebras $U_q(sl_n)$*, preprint 1993.

[3] V. Chari, A. Pressley, *A guide to quantum groups*, Cambridge Universty Press, 1994.

[4] E. Date, M. Jimbo, T. Miwa, Representations of $U_q(gl_n)$ at $q = 0$ and the Robinson-Schensted correspondence, in *Physics and mathematics of strings, Memorial volume of V. Knizhnik*, L. Brink, D. Friedan, A.M. Polyakov eds., World Scientific, 1990.

[5] J. Désarménien, *An algorithm for the Rota straightening formula*, Discrete Math. **30** (1980), 51-68.

[6] J. Désarménien, J.P.S. Kung, G.C. Rota, *Invariant theory, Young bitableaux and combinatorics*, Adv. Math. **27** (1978), 63-92.

[7] V. G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Soviet Math. Dokl. **32** (1985), 254-258.

[8] W. Fulton, A. Lascoux, *A Pieri formula in the Grothendieck ring of a flag bundle*, Duke Math. J., **76** (1994), 711-729.

[9] J.A. Green, Classical invariants and the general linear group, in *Representation theory of finite groups and finite dimensional algebras*, G.O. Michler, C.M. Ringel eds., Progress in Mathematics 95, Birkhäuser, 1991.

[10] R.Q. Huang, J.J. Zhang, *Standard basis theorem for quantum linear groups*, Adv. Math. **102** (1993), 202-229.

[11] M. Jimbo, *A q-difference analogue of $U(g)$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), 63-69.

[12] M. Jimbo, *A q-analogue of $U(gl(N+1))$, Hecke algebra, and the Yang-Baxter equation*, Lett. Math. Phys. **11** (1986), 247-252.

[13] M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Commun. Math. Phys. **133** (1990), 249-260.

[14] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465-516.

[15] M. Kashiwara, *Crystallization of quantized enveloping algebras*, Sugaku Expositiones, **7** (1994), 99-115.

[16] M. Kashiwara, T. Nakashima, *Crystal graphs for representations of the q-analogue of classical Lie algebras*, J. Algebra, **165** (1994), 295-345.

[17] D.E. Knuth, *Permutations, matrices and generalized Young tableaux*, Pacific. J. Math. **34** (1970), 709-727.

[18] A. Lascoux, B. Leclerc, J.Y. Thibon, *Crystal graphs and q-analogues of weight multiplicities for root systems of type A_n*, Lett. Math. Phys. *to appear.*

[19] V. Lakshmibai, N. Yu. Reshetikhin, Quantum deformations of SL_n/B and its Schubert varieties, in *Special Functions*, ICM-90 Satellite Conference Proceedings, M. Kashiwara, T. Miwa eds., Springer.

[20] A. Lascoux, M. P. Schützenberger, Le monoïde plaxique, in *Noncommutative structures in algebra and geometric combinatorics* A. de Luca Ed., Quaderni della Ricerca Scientifica del C. N. R., Roma, 1981

[21] A. Lascoux, M.P. Schützenberger, Keys and standard bases, in *Invariant theory and tableaux*, D. Stanton ed., Springer, 1990.

[22] B. Leclerc, *On identities satisfied by minors of a matrix*, Adv. in Math., **100** (1993), 101-132.

[23] M. Noumi, H. Yamada, K. Mimachi, *Finite dimensional representations of the quantum group $GL_q(n;C)$ and the zonal spherical functions on $U_q(n-1)\backslash U_q(n)$*, Japan. J. Math., **19** (1993), 31-80.

[24] V. Reiner, M. Shimozono, *Key polynomials and a flagged Littlewood-Richardson rule*, J. Comb. Theory A, **70** (1995), 107-143.

[25] N.Y. Reshetikhin, L.A. Takhtadzhyan, L.D. Faddeev, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J.,**1** (1990), 193–225.

[26] G. de B. Robinson, *On the representation theory of the symmetric group*, Amer. J. Math. **60** (1938), 745-760.

[27] C. Schensted, *Longest increasing and decreasing subsequences*, Canad. J. Math. **13** (1961), 179-191.

[28] M. P. Schützenberger, La correspondance de Robinson, in *Combinatoire et représentation du groupe symétrique*, Strasbourg 1976, D. Foata ed., Springer Lect. Notes in Math. 579.

[29] E. Taft, J. Towber, *Quantum deformation of flag schemes and Grassman schemes I – A q-deformation of the shape-algebra for $GL(n)$*, J. of Algebra, **142** (1991), 1–36.