THE INDEPENDENCE NUMBER OF DENSE GRAPHS WITH LARGE ODD GIRTH

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Abstract. Let G be a graph with n vertices and odd girth $2k+3$. Let the degree of a vertex v of G be $d_1(v)$. Let $\alpha(G)$ be the independence number of G. Then we show $\alpha(G) \geq 2^{-\left(\frac{k-1}{k}\right)} \left[\sum_{v \in G} d_1(v)^{\frac{1}{k-1}}\right]^{(k-1)/k}$. This improves and simplifies results proven by Denley [1].

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Let G be a graph with n vertices and odd girth $2k+3$. Let $d_i(v)$ be the number of points of degree i from a vertex v. Let $\alpha(G)$ be the independence number of G. We will prove lower bounds for $\alpha(G)$ which improve and simplify the results proven by Denley [1].

We will consider first the case $k=1$. We need the following lemma.

Lemma 1: Let G be a triangle-free graph. Then

$$\alpha(G) \geq \sum_{v \in G} d_1(v)/[1 + d_1(v) + d_2(v)].$$

Proof. Randomly label the vertices of G with a permutation of the integers from 1 to n. Let A be the set of vertices v such that the minimum label on vertices at distance $0, 1$ or 2 from v is on a vertex at distance 1. Clearly the probability that A contains a vertex v is $d_1(v)/[1+d_1(v)+d_2(v)]$. Hence the expected size of A is $\sum_{v \in G} d_1(v)/[1+d_1(v)+d_2(v)]$. Furthermore, A must be an independent set since if A contains an edge it is easy to see that it must lie in a triangle of G a contradiction. The result follows at once.

We can now prove the following theorem.

Theorem 1. Suppose G contains no 3 or 5 cycles. Let $\bar{d}$ be the average degree of vertices of G. Then

$$\alpha(G) \geq \sqrt{n\bar{d}/2}.$$

Proof. Since G contains no 3 or 5 cycles, we have $\alpha(G) \geq d_1(v)$ (consider the neighbors of v) and $\alpha(G) \geq 1 + d_2(v)$ (consider v and the points at distance 2 from v) for any vertex v of G. Hence $\alpha(G) \geq \sum_{v \in G} d_1(v)/[1+d_1(v)+d_2(v)] \geq \sum_{v \in G} d_1(v)/2\alpha(G)$ (by lemma 1 and the preceding remark). Therefore $\alpha(G)^2 \geq n\bar{d}/2$ or $\alpha(G) \geq \sqrt{n\bar{d}2}$ as claimed.

This improves Denley's Theorems 1 and 2. It is sharp for the regular complete bipartite graphs K_{aa}.

The above results are readily extended to graphs of larger odd girth.

Lemma 2: Let G have odd girth $2k+1$ or greater ($k \geq 2$). Then

$$\alpha(G) \geq \sum_{v \in G} \frac{\frac{1}{2}(1 + d_1(v) + \cdots + d_{k-1}(v))}{1 + d_1(v) + \cdots + d_k(v)}.$$

Proof. Randomly label the vertices of G with a permutation of the integers from 1 to n. Let A (respectively B) be the set of vertices v of G such that the minimum label on vertices at distance k or less from v is at even (respectively odd) distance $k-1$ or less. It is easy to see that A and B are independent sets and that the expected size of $A \cup B$ is $\sum_{v \in G} \frac{(1 + d_1(v) + \cdots + d_{k-1}(v))}{1 + d_1(v) + \cdots + d_k(v)}$. The lemma follows at once.

Theorem 2: Let G have odd girth $2k+3$ or greater ($k \geq 2$). Then

$$\alpha(G) \geq 2^{-\left(\frac{k-1}{k}\right)} \left[\sum_{v \in G} d_1(v)^{\frac{1}{k-1}}\right]^{\frac{k-1}{k}}.$$

Proof. By Lemmas 1, 2

$$\alpha(G) \geq \sum_{v \in G} \left[\left[\frac{d_1(v)}{1 + d_1(v) + d_2(v)}\right] + \frac{1}{2}\left[\frac{1 + d_1(v) + d_2(v)}{1 + d_1(v) + d_2(v) + d_3(v)}\right] + \cdots + \frac{1}{2}\left[\frac{1 + d_1(v) + \cdots + d_{k-1}(v)}{1 + d_1(v) + \cdots + d_k(v)}\right]\right] /(k-1).$$

Since the arithmetic mean is greater than the geometric mean, we can conclude that $\alpha(G) \geq \sum_{v \in G} \left[\frac{d_1(v)2^{-(k-2)}}{1 + d_1(v) + \cdots + d_k(v)}\right]^{1/k-1}$. Since the points at even (odd) distance less than or equal k from any vertex v in G form independent sets we have $2\alpha(G) \geq 1+$

$d_1(v)+\cdots+d_k(v)$. Hence $\alpha(G) \geq \sum_{v\in G}\left[\frac{d_1(v)}{2^{k-1}\alpha(G)}\right]^{\frac{1}{k-1}}$ or $\alpha(G)^{\frac{k}{k-1}} \geq \frac{1}{2}\left[\sum_{v\in G} d_1(v)^{\frac{1}{k-1}}\right]$ or $\alpha(G) \geq 2^{-(\frac{k-1}{k})}\left[\sum_{v\in G} d_1(v)^{\frac{1}{k-1}}\right]^{\frac{k-1}{k}}$ as claimed.

Corollary 1: Let G be regular degree d and odd girth $2k+3$ or greater ($k \geq 2$). Then

$$\alpha(G) \geq 2^{-\left(\frac{k-1}{k}\right)} n^{\frac{k-1}{k}} d^{\frac{1}{k}}.$$

Proof. Immediate from Theorem 3.

This improves Denley's Theorem 4.

References

1. Denley, T., *The Independence number of graphs with large odd girth*, The Electronic Journal of Combinatorics **1** (1994) #R9.