On rainbow trees and cycles

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Abstract

We derive sufficient conditions for the existence of rainbow cycles of all lengths in edge colourings of complete graphs. We also consider rainbow colorings of a certain class of trees.

1 Introduction

Let the edges of the complete graph K_n be coloured so that no colour is used more than $\max\{b, 1\}$ times. We refer to this as a b-bounded colouring. We say that a subset S of the edges of K_n is *rainbow coloured* if each edge of S is of a different colour. Various authors have considered the question of how large can $b = b(n)$ be so that any b-bounded edge colouring contains a rainbow Hamilton cycle. It was shown by Albert, Frieze and Reed [1] (see Rue [7] for a correction in the claimed constant) that b can be as large as $n/64$. This confirmed a conjecture of Hahn and Thomassen [5]. Our first theorem discusses the existence of rainbow cycles of all sizes. We give a kind of a *pancyclic* rainbow result.

Theorem 1 *There exists an absolute constant $c > 0$ such that if an edge colouring of K_n is cn-bounded then there exist rainbow cycles of all sizes $3 \le k \le n$.*

Having dealt with cycles, we turn our attention to trees.

Theorem 2 *Given a real constant $\varepsilon > 0$ and a positive integer Δ, there exists a constant $c = c(\varepsilon, \Delta)$ such that if an edge colouring of K_n is cn-bounded, then it contains a rainbow copy of* every *tree T with at most $(1 - \varepsilon)n$ vertices and maximum degree Δ.*

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We conjecture that that there is a constant $c = c(\Delta)$ such that every cn-bounded edge colouring of K_n contains a rainbow copy of every *spanning* tree of K_n which has maximum degree at most Δ. We are far from proving this and give a small generalisation of the known case where the tree in question is a Hamilton path. Let T^* be an arbitrary *rooted* tree with ν_0 nodes. Assume that ν_0 divides n and let $\nu_1 = n/\nu_0$. We define $T(\nu_1)$ as follows: It has a *spine* which is a path $P = (x_0, x_1, \ldots, x_{\nu_1-1})$ of length $\nu_1 - 1$. We then have ν_1 vertex disjoint copies $T_0, T_1, \ldots, T_{\nu_1-1}$ of T^*, where T_i is rooted at x_i for $i = 0, 1, \ldots, \nu_1 - 1$. $T(\nu)$ has n vertices. The edges of $T(\nu_1)$ are of two types, *spine-edges* in P and *teeth-edges*.

We state our theorem as

Theorem 3 *If an edge colouring of K_n is k-bounded and $\binom{\nu_1-2}{2} > 16kn$ then there exists a rainbow copy of every possible $T(\nu_1)$.*

2 Proof of Theorem 1

We will not attempt to maximise c as we will be far from the optimum.

The following lemma is enough to prove the theorem:

Lemma 4

(a) *Let $c_0 = 2^{-7}$ and suppose that $n \geq 2^{21}$. Then every $2c_0n$-bounded edge colouring of K_n contains rainbow cycles of length k, $n/2 \leq k \leq n$.*

(b) *If $n \geq e^{1000}$ and $cn \geq n^{2/3}$ and an edge colouring of K_n is cn-bounded, then there exists a set $S \subseteq [n]$ such that $|S| = N = n/2$ and the induced colouring of the edges of S is $c'N$-bounded where $c' = c(1 + 1/(\ln n)^2)$.*

We will first show that the lemma implies the theorem. Assume first that $n \geq e^{1000}$. We let $N_i = 2^{-i}n$ for $0 \leq i \leq r = \lfloor \log_2(ne^{-1000}) \rfloor$ and note that $N_i \geq e^{1000} > 2^{21}$ for all $i \leq r$. Now define a sequence $c_0, c_1, c_2, \ldots, c_r$ by

$$c_{i+1} = c_i \left(1 + \frac{1}{(\ln N_i)^2}\right).$$

Then for $i \geq 1$ we have:

$$\begin{aligned} c_i &= c_0 \prod_{s=1}^{i} \left(1 + \frac{1}{(\ln n - s\ln 2)^2}\right) \\ &\leq c_0 \exp\left\{\frac{1}{(\ln n)^2} \sum_{s=1}^{i} \frac{1}{\left(1 - \frac{s}{\log_2 n}\right)^2}\right\} \end{aligned}$$

$$= \quad c_0 \exp\left\{ \left(\frac{\log_2 n}{\ln n}\right)^2 \sum_{s=1}^{i} \frac{1}{(\log_2 n - s)^2} \right\} .$$

Then for all $0 \le i \le r$ we have:

$$c_0 \le c_i \le c_0 \exp\left\{ \left(\frac{\log_2 n}{\ln n}\right)^2 \sum_{t=21}^{\infty} \frac{1}{t^2} \right\} \le c_0 \exp\left\{ 2.1 \int_{t=20}^{\infty} t^{-2} dt \right\} = c_0 \exp\left\{ \frac{2.1}{20} \right\} \le 2c_0.$$

Furthermore, for $0 \le i \le r$ we have $n/2^r > 2^{21}$ and so

$$c_i N_i^{1/3} \ge \frac{c_0 n^{1/3}}{2^{i/3}} \ge 1,$$

which implies that $c_i N_i \ge N_i^{2/3}$.

Assume now we are given a $c_0 n$-bounded coloring of K_n and that $n \ge e^{1000}$. Then by part (a) of the lemma we can find rainbow cycles of length k, $n/2 \le k \le n$. By part (b) there exists a subset S, $|S| = n/2 = N$, such that the induced coloring on S is $c_1 n$-bounded. Now we can apply part (a) of the lemma to the induced subgraph $G[S]$ to find rainbow cycles of length k, $n/4 \le k \le n/2$. We can continue this halving process for r steps, thus finding rainbow cycles of length k, $N_r \le k \le n$ where $e^{1000} \le N_r \le 2e^{1000}$.

To summarise: Assuming the truth of Lemma 4, if $n \ge e^{1000}$ and $c \le 2^{-7}$ then any cn-bounded coloring of K_n contains a rainbow cycle of length $2e^{1000} \le k \le n$.

Up to this point, the value of c is quite reasonable. We now choose a very small value of c in order to finish the proof without too much more effort.

Suppose now that $c \le e^{-3001}$, $n \ge e^{1000}$ and $3 \le k \le \min\{2e^{1000}, n\}$. Suppose that K_n is edge colored with q colors and that color i is used $m_i \le cn$ times. Choose a set S of k vertices. Let $\mathcal{E}$ be the event S contains two edges of the same color. at random. Then,

$$\begin{aligned} \mathbf{Pr}(\mathcal{E}) &\le \binom{k}{2}^2 \sum_{i=1}^{q} \left(\frac{m_i}{\binom{n}{2}}\right)^2 + \binom{k}{3} \sum_{i=1}^{q} \frac{\binom{m_i}{2}}{\binom{n}{3}} \qquad (1) \\ &\le \binom{k}{2}^2 \frac{\binom{n}{2}}{cn} \left(\frac{cn}{\binom{n}{2}}\right)^2 + \binom{k}{3} \frac{\binom{n}{2}}{cn} \frac{\binom{cn}{2}}{\binom{n}{3}} \\ &\le \frac{ck^2}{n-1} + \frac{ck^3}{4} \\ &< 1. \end{aligned}$$

The two sums in (1) correspond to having two disjoint edges with the same color and to two edges of the same color sharing a vertex, respectively.

All that is left is the case $n \le e^{1000}$ but now c is so small that $cn < 1$ and all edges have distinct colors.

2.1 Proof of Lemma 4

Part (a) follows immediately from [1] ($n \geq 2^{21}$ is easily large enough for the result there to hold). We can apply the main theorem of that paper to any subset of $[n]$ with at least $n/2$ vertices.

We now prove part (b). Let S be a random $n/2$-subset of $[n]$. Now for each colour i we orient the i-coloured edges of K_n so that for each $v \in [n]$,

$$|d_i^+(v) - d_i^-(v)| \leq 1$$

where $d_i^+(v)$ (resp. $d_i^-(v)$) is the out-degree (resp. in-degree) of v in the digraph $D_i = ([n], E_i)$ induced by the edges of colour i. Now fix a colour i and let

$$L_i = \left\{v :\ d_i^+(v) \geq (\ln n)^6\right\}.$$

Then with (v, w) denoting an edge oriented from v to w we let

$$\begin{aligned}
A_1 &= \{(v,w) \in E_i :\ v \in L_i\} \\
A_2 &= \left\{(v,w) \in E_i :\ v \notin L_i, w \in L_i \textit{ and } \exists \geq (\ln n)^6 \textit{ edges of colour } i \textit{ from } \bar{L}_i \textit{ to } w\right\} \\
A_3 &= E_i \setminus (A_1 \cup A_2).
\end{aligned}$$

Let $|A_j| = \alpha_j n$ where $\alpha_1 + \alpha_2 + \alpha_3 \leq c$.

Let $Z_j, j = 1, 2, 3$, be the number of edges of A_j which are entirely contained in S and let $Z = Z_1 + Z_2 + Z_3$. We write

$$Z_1 = \sum_{v \in L_i} 1_{v \in S} X_{1,v}$$

where $X_{1,v}$ is the number of neighbours of v in D_i that are included in S.

Now

$$\mathbf{Pr}(X_{1,v} \geq \frac{1}{2} d_i^+(v) + \frac{1}{4} d_i^+(v)^{1/2} \ln n) \leq e^{-(\ln n)^2/24}.$$

This follows from the Chernoff bounds (more precisely, using Hoeffding's lemma [6] about sampling without replacement).

Note that

$$\frac{1}{2} d_i^+(v) + \frac{1}{4} d_i^+(v)^{1/2} \ln n \leq \frac{1}{2} d_i^+(v) \left(1 + \frac{1}{2(\ln n)^2}\right).$$

So, on using $n \geq e^{1000}$, we see that with probability at least

$$1 - ne^{-(\ln n)^2/24} = 1 - n^{1-(\ln n)/24} \geq 9/10$$

we have

$$Z_1 \leq \frac{1}{2} \alpha_1 n \left(1 + \frac{1}{2(\ln n)^2}\right).$$

The edges of A_2 are dealt with in exactly the same manner and we have that with probability at least 9/10,

$$Z_2 \leq \frac{1}{2}\alpha_2 n \left(1 + \frac{1}{2(\ln n)^2}\right).$$

To deal with Z_3 we observe that if we delete a vertex v of S then Z_3 can change by at most $2(\ln n)^6$. This is because the digraph induced by A_3 has maximum in-degree and out-degree bounded by $(\ln n)^6$. Applying a version of Azuma's inequality that deals with sampling without replacement (see for example Lemma 11 of [4]) we see that for $t > 0$,

$$\mathbf{Pr}\left(Z_3 \geq \frac{1}{4}\alpha_3 n + t\right) \leq \exp\left\{-\frac{2t^2}{n(\ln n)^{12}}\right\}.$$

So, putting $t = n^{3/5}$ and using $n \geq e^{1000}$ and $cn \geq n^{2/3}$ we see that with probability at least 9/10,

$$Z \leq \frac{1}{2}(\alpha_1 + \alpha_2)n\left(1 + \frac{1}{2(\ln n)^2}\right) + \frac{1}{4}\alpha_3 n + n^{3/5} \leq \frac{1}{2}cn\left(1 + \frac{1}{(\ln n)^2}\right).$$

So, with probability at least 7/10 the colouring of the edges of S is $c(1 + 1/(\ln n)^2)n/2$-bounded and Lemma 4 is proved. □

3 Proof of Theorem 2

We proceed as follows. We choose a large $d = d(\varepsilon, \Delta) > 0$ and a small $c \ll 1/d^{3/2}$ and consider a cn-bounded edge colouring of K_n. We then define $G_1 = G_{n,p}$, $p = d/n$. We remove any edge of G_1 which has the same colour as another edge of G_1. Call the remaining graph G_2. The edge set of G_2 is rainbow coloured. We then remove vertices of low and high degree to obtain a graph G_3. We then show that **whp** G_3 satisfies the conditions of a theorem of Alon, Krivelevich and Sudakov [2], implying that G_3 contains a copy of every tree with $\leq (1-\varepsilon)n$ vertices and maximum degree $\leq \Delta$. The theorem we need from [2] is the following:
Definition: Given two positive numbers a_1 and $a_2 < 1$, a graph $G = (V, E)$ is called an (a_1, a_2)-expander if every subset of vertices $X \subseteq V$ of size $|X| \leq a_1|V|$ satisfies $|N_G(X)| \geq a_2|X|$. Here $N_G(X)$ is the set of vertices in $V(G) \setminus X$ that are neighbours of vertices in X.

Theorem 5 *Let $\Delta \geq 2$, $0 < \varepsilon < 1/2$. Let H be a graph on N vertices of minimum degree δ_H and maximum degree Δ_H. Suppose that*

T1

$$N \geq \frac{480\Delta^3 \ln(2/\varepsilon)}{\varepsilon}.$$

T2
$$\Delta_H^2 \le \frac{1}{K} e^{\delta_H/(8K)-1} \ \textit{where}\ K = \frac{20\Delta^2 \ln(2/\varepsilon)}{\varepsilon}.$$

T3 *Every subgraph H_0 of H with minimum degree at least $\frac{\varepsilon\delta_H}{40\Delta^2\ln(2/\varepsilon)}$ is a $\left(\frac{1}{2\Delta+2}, \Delta+1\right)$-expander.*

Then H contains a copy of every tree with $\le (1-\varepsilon)N$ vertices and maximum degree $\le \Delta$.

We now get down to details. In the following we assume that $cd \ll 1 \ll d$. We will prove that **whp**,

P1 The number of edges using repeated colours is at most d^2cn.

P2 Every set $X \subseteq [n]$, $|X| \le n/d^{1/5}$ contains less than $\alpha d|X|$ edges of G_1 where, with $\Delta = 2d$,
$$\alpha = \frac{\varepsilon}{(100\Delta^2(\Delta+2)\ln(2/\varepsilon))}.$$

P3 G_1 contains at most $ne^{-d/10}$ vertices of degree outside $[d/2, 2d]$.

P4 Every pair of disjoint sets $S, T \subseteq [n]$ of size $n/d^{1/4}$ are joined by at least $d^{1/2}n/2$ edges in G_1.

Before proving that **P1**–**P4** hold **whp**, let us show that they are sufficient for our purposes.

Starting with $G_1 = G_{n,p}$ we remove all edges using repeated colours to obtain G_2. Then let X_0 denote the set of vertices of G_2 whose degree is not in $[d/3, 2d]$. It follows from **P1,P3** that
$$|X_0| \le n(e^{-d/10} + 12cd). \tag{2}$$
Note that $12cdn$ bounds the number of vertices that lose more than $d/6$ edges in going from G_1 to G_2.

Now consider a sequence of sets $X_0, X_1, \ldots,$ where $X_i = X_{i-1} \cup \{x_i\}$ and x_i has at least $2\alpha d$ neighbours in X_{i-1}. We continue this process as long as possible. Let G_3 be the resulting graph. We claim that the process stops before i reaches $|X_0|$. If not, we have a set with $2|X_0|$ vertices and at least $2\alpha d|X_0|$ edges. For this we need $2|X_0| \ge n/d^{1/5}$ (see **P2**) and this contradicts (2) if d is large and $c < 1/d^2$.

Thus $H = G_3$ has at least $n(1 - 2(e^{-d/10} + 12cd))$ vertices and this implies that **T1** holds. Also,
$$d(1/3 - 2\alpha) \le \delta_H \le \Delta_H \le 2d.$$
So if $d \gg K^2$, **T2** will also hold.

Now consider a subgraph Γ of H which has minimum degree at least βd where $\beta = 2(\Delta+2)\alpha$. Let $\nu = |V(\Gamma)|$. Choose $S \subseteq V(\Gamma)$ where $|S| \le \frac{\nu}{2\Delta+2}$ and let $T = N_\Gamma(S)$. Suppose also that $|T| < (|\Delta|+1)|S|$.

Suppose first that $|S| \geq n/d^{1/4}$. Then $|S \cup T| \leq \nu(\Delta+2)/(2\Delta+2)$ and so $Y = V(\Gamma) \setminus (S \cup T)$ satisfies $|Y| \geq |S| \geq n/d^{1/4}$. The fact that there are no $S : Y$ edges contradicts **P1, P4**.

Now assume that $1 \leq |S| \leq n/d^{1/4}$. Then $|S \cup T| \leq (\Delta+2)n/d^{1/4} \leq n/d^{1/5}$ and $S \cup T$ contains at least $\beta d|S|/2 \geq \alpha d|S \cup T|$ edges, contradicting **P2**.

Thus, Γ is $\left(\frac{1}{2\Delta+2}, \Delta+1\right)$-expander and the minimum degree requirement is βd which is weaker than required by **T3**.

It only remains to verify **P1**–**P4**:

P1: Let Z denote the number of edges using repeated colours. Let there be $m_i \leq cn$ edges with colour i for $i = 1, 2, \ldots, \ell$. Then

$$\mathbf{E}(Z) \leq \sum_{i=1}^{\ell} \binom{m_i}{2} p^2 \leq \frac{\binom{n}{2}}{cn} \binom{cn}{2} \frac{d^2}{n^2} \leq \frac{cd^2}{4} n.$$

Now **whp** G_1 has at most dn edges and changing one edge can only change Z by at most 2. So, by Azuma's inequality, we have

$$\mathbf{Pr}(Z \geq \mathbf{E}(Z) + t) \leq \exp\left\{-\frac{2t^2}{4dn}\right\},$$

and we get (something stronger than) **P1** by taking $t = n^{3/4}$.

P2: The probability **P2** fails is at most

$$\sum_{k=2\alpha d}^{n/d^{1/5}} \binom{n}{k} \binom{\binom{k}{2}}{\alpha dk} p^{\alpha dk} \leq \sum_{k=2\alpha d}^{n/d^{1/5}} \left(\left(\frac{k}{2n}\right)^{\alpha d-1} \left(\frac{e}{\alpha}\right)^{\alpha d} e \right)^k = o(1).$$

P3: If now Z is the number of vertices with degrees outside $[d/2, 2d]$ then the Chernoff bounds imply that

$$\mathbf{E}(Z) \leq n(e^{-d/8} + e^{-d/3}),$$

and Azuma's inequality will complete the proof.

P4: The probability **P4** fails is at most

$$\binom{n}{n/d^{1/4}}^2 \sum_{k=0}^{d^{1/2}n/2} \binom{n^2/d^{1/2}}{k} p^k (1-p)^{n^2/d^{1/2}-k} \leq 4^n e^{-d^{1/2}n/8} = o(1).$$

4 Proof of Theorem 3

We will use the lop-sided Lovász local lemma as in Erdős and Spencer [3] and in Albert, Frieze and Reed [1]. We state the lemma as

Lemma 6 *Let $A_1, A_2, \ldots, A_N$ denote events in some probability space. Suppose that for each i there is a partition of $[N] \setminus \{i\}$ into X_i and Y_i. Let $m = \max\{|Y_i| : i \in [N]\}$ and $\beta = \max\{\mathbf{Pr}(A_i \mid \bigcap_{j \in S} \bar{A}_j) : i \in [N], S \subseteq X_i\}$. If $4m\beta < 1$ then $\mathbf{Pr}(\bigcap_{i=1}^{n} \bar{A}_i) > 0$.*

Suppose now that we have a k-bounded colouring of K_n and that H is chosen uniformly from the set of all copies of $T(\nu)$ in K_n where T is an arbitrary rooted tree with ν vertices. We show that the probability that H is a rainbow copy is strictly positive.

Let $\{e_i, f_i\}, i = 1, 2, \ldots, N$, be an enumeration of all pairs of edges of K_n where e_i, f_i have the same colour (thus $N = \sum_\ell \binom{n_\ell}{2}$ where n_ℓ is the number of edges of colour ℓ). Let A_i be the event $H \supset \{e_i, f_i\}$ for $i = 1, 2, \ldots, N$. We apply Lemma 6 with the definition

$$Y_i = \{j \neq i : (e_j \cup f_j) \cap (e_i \cup f_i) \neq \emptyset\}.$$

With this definition

$$m \leq 4kn.$$

We estimate β as follows: Fix $i, S \subseteq X_i$. We show that for each $T \in \mathcal{T}_1 = A_i \cap \bigcap_{j \in S} \bar{A}_j$ (this means that T is a copy of $T(\nu_0, \nu_1)$ containing both e_i, f_i and at most one edge from each pair e_j, f_j for $j \in S$) there exists a set $S(T) \subseteq \mathcal{T}_2 = \bar{A}_i \cap \bigcap_{j \in S} \bar{A}_j$ such that (i) $|S(T)| > 4kn$ and (ii) $S(T) \cap S(T') = \emptyset$ for $T \neq T' \in \mathcal{T}_1$. This shows that

$$\mathbf{Pr}(A_i \mid \bigcap_{j \in S} \bar{A}_j) \leq \frac{1}{4m+1}$$

and proves the theorem.

Fix $H \in \mathcal{T}_1$. If $e = (x_i, x_{i+1})$ and $f = (x_j, x_{j+1})$ are both spine-edges where $j - i \geq 2$, we define the tree $F_{spine}(H; e, f)$, which is also a copy of $T(\nu)$, as follows: We delete e, f from H and replace them by (x_i, x_j) and (x_{i+1}, x_{j+1}). Suppose now that $e = (a, b) \in T_i \setminus x_i$ and $f = (c, d) \in T_j \setminus x_j$ are both teeth-edges and that $\phi(e) = f$ in some isomorphism from T_i to T_j. Then we define $F_{teeth}(H; e, f)$ as follows: We delete e, f from H and replace them by (a, d) and (b, c) to get another copy of $T(\nu)$.

Observe that if $f \neq f_i$ then $H' = F_\sigma(H; e_i, f) \in \mathcal{T}_2$ for $\sigma \in \{spine, teeth\}$. This is because e_i is not an edge of H' and the edges that we added are all incident with e_i. We cannot therefore have caused the occurrence of A_j for any $j \in X_i$. Similarly, $F_\sigma(H'; f_i, g) \in \mathcal{T}_2$ for $g \neq e_i$.

We use F_{spine}, F_{teeth} to construct $S(H)$ as follows: We choose an edge $f \neq f_i$ of the same type as e_i and construct $H' = F_\sigma(H; e_i, f)$ for the relevant σ. We then choose $g \neq e_i$ of the same type as f_i and construct $H'' = F_{\sigma'}(H'; f_i, g)$. In this way we construct $S(H) \subseteq \mathcal{T}_2$ containing at least $\binom{\nu_1 - 2}{2}$ distinct copies of $T(\nu_1)$.

Notice that knowing e_i, f_i allows us to construct H' from H'' and then H from H'. This shows that $S(H) \cap S(H') = \emptyset$. After this, all we have to do is choose k, ν_1 so that $\binom{\nu_1 - 2}{2} > 16kn$ in order to finish the proof of Theorem 3.

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