The lonely runner with seven runners

J. Barajas and O. Serra*

Dept. Matemàtica Aplicada 4
Universitat Politècnica de Catalunya, Barcelona, Spain

{jbarajas,oserra}@ma4.upc.edu

Submitted: Nov 11, 2007; Accepted: Feb 28, 2008; Published: Mar 18, 2008
Mathematics Subject Classification: 11B75, 11J71, 05C15

Abstract

Suppose $k+1$ runners having nonzero constant pairwise distinct speeds run laps on a unit-length circular track starting at the same time and place. A runner is said to be lonely if she is at distance at least $1/(k+1)$ along the track to every other runner. The lonely runner conjecture states that every runner gets lonely. The conjecture has been proved up to six runners ($k \leq 5$). A formulation of the problem is related to the regular chromatic number of distance graphs. We use a new tool developed in this context to solve the first open case of the conjecture with seven runners.

1 Introduction

Consider $k+1$ runners on a unit length circular track. All the runners start at the same time and place and each runner has a constant speed. The speeds of the runners are pairwise distinct. A runner is said to be lonely at some time if she is at distance at least $1/(k+1)$ along the track from every other runner. The *Lonely Runner Conjecture* states that each runner gets lonely. The Lonely Runner Conjecture has been introduced by Wills [12] and independently by Cusick [7], and it has been given this pitturesque name by Goddyn [4]. For $k = 3$, there are four proofs in the context of diophantine approximations: Betke and Wills [3] and Cusick [7, 8, 9]. The case $k = 4$ was first proved by Cusick and Pomerance [10], with a proof requiring computer checking. Later, Bienia et al. [4] gave a simpler proof for the case $k = 4$. The case $k = 5$ was proved by Bohman, Holzman and Kleitman [5]. A simpler proof for this case was given later by Renault [11].

This problem appears in different contexts. Cusick [7] was motivated by an application in view obstruction problems in n–dimensional geometry, and Wills [12] considered the

*Research supported by the Spanish Ministry of Eduaction under grant MTM2005-08990-C02-01 and by the Catalan Research Council under grant 2005SGR00256

problem from the diophantine approximation point of view. Biennia et al. [4] observed that the solution of the lonely runner problem implies a theorem on nowhere zero flows in regular matroids. Zhu [13] used known results for the lonely runner problem to compute the chromatic number of distance graphs. In [2] a similar approach was used to study the chromatic number of circulant graphs.

A convenient and usual reformulation of the lonely runner conjecture can be obtained by assuming that all speeds are integers, not all divisible by the same prime, (see e.g. [5]) and that the runner to be lonely has zero speed. Let $\|x\|$ denote the distance of the real number x to its nearest integer. In this formulation the Lonely Runner Conjecture states that, for any set D of k positive integers, there is a real number t such that $\|td\| \geq 1/(k+1)$ for each $d \in D$. We shall consider a discrete version of the lonely runner problem.

Let N be a positive integer. For an integer $x \in \mathbb{Z}$ we denote by $|x|_N$ the residue class of x or $-x$ in the interval $[0, N/2]$. For a set $D \subset \mathbb{N}$ of positive integers we define the *regular chromatic number* $\chi_r(N, D)$ as

$$\chi_r(N, D) = \min\{k : \exists \lambda \in \mathbb{Z}_N \text{ such that } |\lambda d|_N \geq \frac{N}{k} \text{ for each } d \in D\},$$

if D contains no multiples of N and $\chi_r(N, D) = \infty$ otherwise. We define the *regular chromatic number* of D as

$$\chi_r(D) = \liminf_{N \to \infty} \chi_r(N, D).$$

The reason for calling chromatic numbers the parameters defined above stems from applications of the lonely runner problem to the study of the chromatic numbers of distance graphs and circulant graphs; see e.g [1, 2, 13]. In this terminology, the lonely runner conjecture can be equivalently formulated as follows.

Conjecture 1 *For every set $D \subset \mathbb{Z}$ of positive integers with* $\gcd(D) = 1$*,*

$$\chi_r(D) \leq |D| + 1.$$

In [1] the so–called *Prime Filtering Lemma* was introduced as a tool to obtain a characterization of sets D with $|D| = 4$ for which equality holds in Conjecture 1. The Prime Filtering Lemma provides a short proof of the conjecture for $|D| = 4$ (five runners) which we include in Section 3 just to illustrate the technique. In Section 2 we formulate a generalization of the lemma and we then use it in the rest of the paper to solve the first open case of the conjecture when $|D| = 6$. As it will become clear in the coming sections, the Prime Filtering Lemma essentially reduces the proof to a finite problem in $\mathbb{Z}_7$ which can be seen as a generalization of the Lonely Runner Problem in which the runners may have different starting points. Unfortunately the conjecture does not always hold in this new context and we have to proceed with a more detailed case analysis.

2 Notation and Preliminary results

For a positive integer x and a prime p, the p–adic valuation of x is

$$\nu_p(x) = \max\{k : \ x \equiv 0 \pmod{p^k}\}.$$

We also denote by $r_p(x)=(xp^{-\nu_p(x)})_p$ the congruence class modulo p of the least coefficient in the p-ary expansion of x.

We shall consider the discrete version of the lonely runner problem mostly in the integers modulo N with N a prime power. We denote by $(x)_N$ the residue class of x modulo N in $\{0,1,\ldots,N-1\}$ and we denote by $|x|_N$ the residue class of x or $-x$ modulo N in $\{0,1,\ldots,\lfloor N/2\rfloor\}$.

Let D be a set of positive integers, let $m=\max\nu_p(D)$ and set $N=p^{m+1}$. Note that, for each $x\in D$, $\nu_p(x)=\nu_p((x)_N)=\nu_p(|x|_N)$. By abuse of notation we still denote by D the set $\{(d)_N:\ d\in D\}$ as a subset of $\mathbb{Z}_N$ whenever the ambient group is clear from the context. The *p–levels* of D are

$$D_p(i)=\{d\in D:\ \nu_p(d)=i\}.$$

Let $q=q_{p,m}:\mathbb{Z}\to\mathbb{Z}_p$ be defined as

$$q(x)=\left(\left\lfloor\frac{x}{p^m}\right\rfloor\right)_p,$$

that is, $q(x)=k$ is equivalent to $(x)_N\in[k(\frac{N}{p}),(k+1)\frac{N}{p})$. We call the interval $[k(\frac{N}{p}),(k+1)\frac{N}{p})$ the k-th (N/p)*–arc*. Our goal is to find a multiplier λ for $D'=D\setminus D_p(m)$ such that

$$q(\lambda\cdot D')\cap\{0,p-1\}=\emptyset, \tag{1}$$

where $\lambda\cdot X=\{\lambda x; x\in X\}$. Indeed, if (1) holds, then $|\lambda d|_N\geq N/p$ for each $d\in D$ and $\chi_r(D)\leq\chi_r(N,D)\leq p$, giving Conjecture 1 whenever $|D|\geq p-1$.

We shall mostly use multipliers of the form $1+p^{m-j}k$. Let

$$\Lambda_{j,p}=\{1+p^{m-j}k,\ 0\leq k\leq p-1\},\quad j=0,1,\ldots,m-1,$$

and

$$\Lambda_{m,p}=\{1,2,\ldots,p-1\}.$$

Note that all elements in $U\mathbb{Z}_N$, the multiplicative group of invertible elements in $\mathbb{Z}_N$, can be obtained as a product of elements in $\Lambda_{0,p}\cup\Lambda_{1,p}\cup\cdots\Lambda_{m,p}$. In what follows, by a multiplier we shall always mean an invertible element in $\mathbb{Z}_N$ where N is a prime power for some specified prime p.

For each j and each $\lambda\in\Lambda_{j,p}$, we have

$$\nu_p(\lambda\cdot x)=\nu_p(x) \tag{2}$$

and, if $\lambda=1+kp^{m-j}$, then

$$q(\lambda\cdot x)=\begin{cases} q(x), & \text{if } \nu_p(x)>j,\\ q(x)+kr_p(x), & \text{if } \nu_p(x)=j.\end{cases} \tag{3}$$

In view of (3), when using a multiplier $\lambda\in\Lambda_{j,p}$, the values of q on the elements in the p–levels $D_p(i)$ of D with $i>j$ remain unchanged. The following result is based in this simple principle. It gives a sufficient condition for the existence of a multiplier λ such that multiplication by λ sends every element $d\in D$ outside a 'forbidden' set for d.

Lemma 2 (Prime Filtering) *Let p be a prime and let D be a set of positive integers. Set $m = \max\{\nu_p(d) : d \in D\}$ and $N = p^{m+1}$. For each $d \in D$ let $F_d \subset \mathbb{Z}_p$. Suppose that*

$$\begin{aligned}\sum_{d\in D_p(j)} |F_d| &\le p-1 \text{ for each } j = 0,1,\ldots,m-1, \text{ and}\\ \sum_{d\in D_p(m)} |F_d| &\le p-2,\end{aligned}$$

Then there is a multiplier λ such that, for each $d \in D$,

$$q(\lambda d) \notin F_d.$$

Proof. For each $d \in D(m)$ we have $q(\Lambda_{m,p} \cdot d) = \Lambda_{m,p}$. Hence there are at most $|F_d|$ elements λ in $\Lambda_{m,p}$ such that $q(\lambda d) \in F_d$. Since $\sum_{d\in D_p(m)} |F_d| \le p-2$, there is $\lambda \in \Lambda_{m,p}$ such that $q(\lambda d) \notin F_d$ for each $d \in D_p(m)$.

Denote by $E(i) = \cup_{j\ge i} D(j)$. Let r be the smallest nonnegative integer i for which there is some $\lambda_i \in \prod_{j=i}^{m} \Lambda_{j,p}$ verifying $q(\lambda_i d) \notin F_d$ for every $d \in E(i)$. We have seen that $r \le m$.

Suppose that $r > 0$ and let $\lambda \in \Lambda_{r-1,p}$. It follows from (2) and (3) that, for each $d \in E(r)$, we have $(\lambda\lambda_r d)_N = (\lambda_r d)_N$. Note also that, for each $d \in D(r-1)$, we have $q(\lambda_r d \cdot \Lambda_{r-1,p}) = \mathbb{Z}_p$. Hence there are at most $|F_d|$ elements λ in $\Lambda_{r-1,p}$ for which $q(\lambda\lambda_r d) \in F_d$. Since $\sum_{d\in D_p(r-1)} |F_d| < p$ there is at least one $\lambda \in \Lambda_{r-1,p}$ for which $\lambda\lambda_r d \notin F_d$ for each $d \in E(r-1)$ contradicting the minimality of r. Thus $r = 0$ and we are done. □

We shall often use the following form of Lemma 2, in which all forbidden sets are the 0-th and $(p-1)$–th (N/p)–arcs.

Corollary 3 *With the notation of Lemma 2, suppose that $|d|_N \ge N/p$ for each $d \in D_p(i)$ and each $i \ge i_0$ for some positive integer $i_0 \le m$. If*

$$|D_p(j)| \le \frac{(p-1)}{2},\ j = 0,1,\ldots,i_0-1,$$

then

$$\chi_r(N,D) \le p.$$

Proof. Let $F_d = \{0, p-1\}$ for each $d \in D \setminus D_p(m)$. We can apply Lemma 2 to each element of $D' = D \setminus (\cup_{i\ge i_0} D_p(i))$ since $\sum_{d\in D_p(j)} |F_d| = 2|D_p(j)| \le (p-1)$ for each $j < i_0$. Thus there is $\lambda \in \prod_{j<i_0} \Lambda_{j,p}$ such that $q(\lambda d) \notin \{0, p-1\}$ for each $d \in D'$. With such λ we also have $|\lambda d|_N = |d|_N \ge N/p$ for each $d \in D \setminus D'$. Hence the inequality $|\lambda d|_N \ge N/p$ holds for each $d \in D$, which is equivalent to $\chi_r(N,D) \le p$. □

3 The case with three and five runners

Let us show that the cases with three ($|D| = 2$) and five ($|D| = 4$) runners can be easily handled. In other words, we prove in a simple way that $\chi_r(D) \le |D| + 1$ for those sets with $|D| = 2$ or $|D| = 4$.

For $|D| = 2$, either the two elements in D are relatively prime with 3 or they have different 3-adic valuations. In both cases Corollary 3 with $p = 3$ applies and we get $\chi_r(D) \le 3$.

Suppose now that $|D| = 4$. Let $m = \max \nu_5(D)$ and $N = 5^{m+1}$. Since we assume that $\gcd(D) = 1$, we always have $D_5(0) \neq \emptyset$. By definition we have $D_5(m) \neq \emptyset$ as well. If $|D_5(i)| \le 2$ for each $i < m$ then we are done by Corollary 3. Therefore we only have to consider the case $|D_5(0)| = 3$ and $|D_5(m)| = 1$.

Put $A = D_5(0) = \{d_1, d_2, d_3\}$ and $D_5(m) = \{d_4\}$. We shall show that, up to multiplication of elements in $\Lambda_{0,5} \cup \Lambda_{m,5}$, we have $q(A) \cap \{0, 4\} = \emptyset$. Since these multiplications preserve the inequality $|d_4|_N \ge N/5$ we will conclude that $\chi_r(D) \le 5$.

Let $d \in A$. For each $\lambda_k = 1 + k5^m \in \Lambda_{0,5}$ we have

$$q(\lambda_k d) = q(d) + kr_5(d), \tag{4}$$

and for each $j \in \{1, 2, 3\} \subset \Lambda_{m,5}$,

$$q((j+1)d) \subset q(jd) + q(d) + \{0, 1\} \subset (j+1)q(d) + \{0, 1, \dots, j\}. \tag{5}$$

Since we can replace each $d \in A$ by $-d$ we may assume that all elements in A belong to two nonzero congruence classes modulo 5, say $(A)_5 \subset \{1, 2\}$. Let $A_s = \{d \in A : (d)_5 = s\}$, $s \in \{1, 2\}$, denote the most popular congruence class.

Let us denote by $\ell(A)$ the cardinality of the smallest arithmetic progression of difference one in $\mathbb{Z}_5$ which contains $q(A)$. Let us show that

$$\ell(j\lambda_k \cdot A_s) \le |A_s| \tag{6}$$

for some $j \in \Lambda_{m,5}$ and some $\lambda_k \in \Lambda_{0,5}$.

Suppose that $|A_s| = 3$ and assume that (6) does not hold for $j = 1$. By (4) we may assume, up to multiplication by some λ_k, that $q(d_1) = 0$, $q(d_2) = 2$ and $q(d_3) = 3$.

By (5) we have $q(2d_1) \in \{0, 1\}$, $q(2d_2) \in \{0, 4\}$ and $q(2d_3) \in \{1, 2\}$. If (6) does not hold for $j = 2$ either, then $q(2d_2) = 4$ and $q(2d_3) = 2$. Again by (5) we have $q(3d_1) \subset \{0, 1, 2\}$, $q(3d_2) \subset \{1, 2\}$ and $q(3d_3) \subset \{0, 1\}$ and (6) holds for $j = 3$.

Hence (6) holds and, up to multiplication by some λ_k, we have $q(A) \cap \{0, 4\} = \emptyset$ as desired.

Suppose now that $A_s = \{d_1, d_2\}$ and $r_5(d_3) = \pm 2s$ and assume that (6) does not hold for $j = 1$. Without loss of generality we may assume that $q(d_1) = 0$ and $q(d_2) = 2$. By (5) we have $q(2d_1) \in \{0, 1\}$, $q(2d_2) \in \{0, 4\}$. If (6) does not hold for $j = 2$ then $q(2d_1) = 1$ and $q(2d_2) = 4$. Now, again by (5), $q(3d_1) \in \{1, 2\}$ and $q(3d_2) \in \{1, 2\}$ so that (6) holds for $j = 3$.

Hence we have $q(\lambda_k \cdot A_s) \cap \{0, 4\} = \emptyset$ at least for two values of k, and since $r_5(d_3) \neq \pm s$ at least for one of them we have $q(\lambda_k d_3) \neq 0, 4$ as well. This concludes the proof.

4 Overview of the proof for seven runners

In what follows, $m = \max \nu_7(D)$ and $N = 7^{m+1}$. We shall omit the subscript $p = 7$ and write $\nu(x) = \nu_7(x)$, $r(x) = r_7(x)$ and $\Lambda_j = \Lambda_{j,7}$.

Since we assume that $\gcd(D) = 1$, we always have $D_7(0) \neq \emptyset$. By definition we have $D_7(m) \neq \emptyset$ as well.

If $|D_7(i)| \leq 3$ for each $0 \leq i < m$ then we are done by Corollary 3. Therefore we may suppose that $|D_7(i)| \geq 4$ for some i. On the other hand, if $|D_7(i_0)| = 4$ for some $i_0 > 0$ then, again by Corollary 3, the problem can be reduced to the set $D' = \{d/p^{i_0} : d \in D \setminus D_7(0)\}$. Indeed, if we can find a multiplier λ' such that $|\lambda' d'|_{N'} \geq N'/p$ for each $d' \in D'$, where $N' = N/p^{i_0}$, then $|\lambda d|_N \geq N/p$ for each $d \in D_7(i)$, $i \geq i_0$ with $(\lambda)_N = (\lambda')_N$ and Corollary 3 applies. Therefore we only have to consider the cases $|D_7(0)| = 4$ and $|D_7(0)| = 5$. These two cases are dealt with by considering the congruence classes modulo seven of the elements in $A = D_7(0)$. Since we can replace every element $d \in A$ by $-d$, we may assume that all elements in A belong to three nonzero congruence classes modulo 7, say $(A)_7 \subset \{1, 2, 4\}$. Let $A_s = \{d \in A : \ (d)_7 = s\}$, $s \in \{1, 2, 4\}$. Recall that, for $\lambda_k = 1 + k7^m \in \Lambda_0$ we have

$$q(\lambda_k \cdot A_s) = q(A_s) + ks \tag{7}$$

The case $|A| = 4$ is simpler and is treated in Section 5. The case $|A| = 5$ is more involved and it is described in Section 6. In both cases the general strategy consists of *compressing* the sets A_s, $s \in \{1, 2, 4\}$ and then using (7). For this we often apply the Prime Filtering Lemma to subsets of $A - A$ or $2A - 2A$.

In what follows we shall denote by $\ell(X)$, where X is a set of integers, the length of the smallest arithmetic progression of difference one in $\mathbb{Z}_7$ which contains $q(X)$.

5 The case $|A| = 4$

Let $A = \{d_1, d_2, d_3, d_4\} \subset D_7(0)$ and $d_5 \in D_7(i_0)$, $0 < i_0 \leq m$. Recall that for any $d \in D_7(m)$ and $\lambda \in U\mathbb{Z}_N$ we have $|\lambda d|_N \geq N/7$. Set $|d_5|_N = u7^{i_0}$ and let u' such that $uu' \equiv 1 \pmod{7^{m+1-i_0}}$. Let

$$\Lambda = \{ju'(1 + 7^{m-i_0}) : \ 1 \leq j \leq 5\}.$$

For each $\lambda \in \Lambda_0$ and $\lambda' \in \Lambda$ we clearly have

$$|\lambda\lambda' d_5|_N = j(7^m + 7^{i_0}) \geq N/7. \tag{8}$$

We shall show that there are $\lambda \in \Lambda_0$ and $\lambda' \in \Lambda$ such that $q(\lambda\lambda' \cdot A) \cap \{0, 6\} = \emptyset$, thus concluding the case $|A| = 4$.

Let $\lambda_k = 1 + k7^m$, $0 \leq k \leq 6$, denote the elements in Λ_0 and $\lambda'_j = ju'(1 + 7^{m-i_0})$, $1 \leq j \leq 5$, the ones in Λ. For $d \in A$ we have

$$q(\lambda'_{j+1} d) \in q(\lambda'_j d) + q(\lambda'_1 d) + \{0, 1\} \subset (j + 1)q(\lambda'_1 d) + \{0, 1, \ldots, j\}, \tag{9}$$

and

$$q(\lambda_k d) = q(d) + kr(d). \tag{10}$$

We consider three cases according to the cardinality $|A_s|$ of the most popular congruence class in A.

Case 1. $|A_s| = 4$.

If we show that $\ell(\lambda' \cdot A) \leq 5$ for some $\lambda' \in \Lambda$ then, in view of (10), we have $\{0,6\} \cap q(\lambda_k \lambda' \cdot A) = \emptyset$ for at least one value of k and we are done.

Suppose this is not the case. Without loss of generality we may then assume that $q(\lambda'_1 \cdot A) = \{0,2,4,6\}$, say $q(\lambda'_1 d_i) = 2(i-1)$, $1 \leq i \leq 4$. In view of (9), we have $q(\lambda'_3 d_i) \in 6(i-1) + \{0,1,2\}$. Since $\{2,3\} \cap q(\lambda'_3 \cdot A) \neq \emptyset$ we must have $q(\lambda'_3 d_1) = 2$. Similarly, $\{3,4\} \cap q(\lambda'_3 \cdot A) \neq \emptyset$ implies $q(\lambda'_3 d_4) = 4$. Now, again by (9),

$$q(\lambda'_4 d_1) \in \{2,3\}, \;\; q(\lambda'_4 d_2) \in \{1,2,3,4\}, \;\; q(\lambda'_4 d_3) \in \{2,3,4,5\} \text{ and } q(\lambda'_4 d_4) \in \{3,4\},$$

which yields $\{0,6\} \cap q(\lambda'_4 \cdot A) = \emptyset$, a contradiction.

Case 2. $|A_s| = 3$.

Let $A_s = \{d_1, d_2, d_3\}$, so that either $d_4 \in A_{2s}$ or $d_4 \in A_{4s}$.

Suppose that

$$\ell(\lambda' \cdot A_s) \leq 4 \tag{11}$$

for some $\lambda' \in \Lambda$. Then, in view of (10), we have $\{0,6\} \cap q(\lambda_k \lambda' \cdot A_s) = \emptyset$ for $k \in \{k_0, k_0 + s^{-1}\}$ and some k_0 (the values taken modulo seven). By (10) one of these two values sends $\lambda' d_4$ outside of $\{0,6\}$ and we are done.

Suppose that (11) does not hold. Then we may assume that either $q(\lambda'_1 \cdot A_s) = \{0,1,4\}$ or $q(\lambda'_1 \cdot A_s) = \{0,2,4\}$, say $q(\lambda'_1 d_1) = 0$, $q(\lambda'_1 d_2) = 1$ or 2 and $q(\lambda'_1 d_3) = 4$. If $q(\lambda'_1 d_2) = 1$, by (9),

$$q(\lambda'_2 \cdot A_s) \subset \{0,2,1\} + \{0,1\} = \{0,1,2,3\},$$

and (11) holds, a contradiction. If $q(\lambda'_1 d_2) = 2$, using (9) with λ'_3, we have

$$(q(\lambda'_3 d_1), q(\lambda'_3 d_2), q(\lambda'_3 d_3)) \subset \{0,1,2\} \times \{6,0,1\} \times \{5,6,0\}.$$

Since $\{2,3,4\} \cap q(\lambda'_3 \cdot A_s) \neq \emptyset$ we have $q(\lambda'_3 d_1) = 2$, and $\{3,4,5\} \cap q(\lambda'_3 \cdot A_s) \neq \emptyset$ implies $q(\lambda'_3 d_3) = 5$. But then $q(\lambda'_4 d_1) \subset 2 + \{0,1\}$ and $q(\lambda'_4 d_3) \subset 2 + \{0,1\}$, so that

$$q(\lambda'_4 \cdot A_s) \subset (2 + \{0,1\}) \cup \{1,2,3,4\},$$

and (11) holds, again a contradiction.

Case 3. $|A_s| = 2$.

We may assume that either $|A_{2s}| = 2$ or $|A_{2s}| = |A_{4s}| = 1$. Let $A_s = \{d_1, d_2\}$.

Suppose that

$$\ell(\lambda' \cdot A_s) \leq 2 \tag{12}$$

for some $\lambda' \in \Lambda$. Then we have $\{0,6\} \cap q(\lambda_k \lambda' \cdot A_s) = \emptyset$ for $k \in \{k_0, k_0 + s^{-1}, k_0 + 2s^{-1}, k_0 + 3s^{-1}\}$ and some k_0 (the values taken modulo seven). It is a routine checking that for at least one of these four values of k we have $\{0,6\} \cap q(\lambda_k \lambda' \cdot (A \setminus A_s)) = \emptyset$ as well and we are done.

Suppose that (12) does not hold. We may assume that either (i) $q(\lambda'_1 \cdot A_s) = \{0,2\}$ or (ii) $q(\lambda'_1 \cdot A_s) = \{0,3\}$.

Assume that (i) holds. Then $(q(\lambda'_3 d_1), q(\lambda'_3 d_2)) \subset \{0,1,2\} \times \{6,0,1\}$. Since (12) does not hold, $(q(\lambda'_3 d_1), q(\lambda'_3 d_2))$ is one of the pairs $(1,6),(2,0)$ or $(2,6)$. In the two former ones we have $q(\lambda'_4 \cdot A_s) \subset \{1,2\}$ or $q(\lambda'_4 \cdot A_s) \subset \{2,3\}$ respectively, a contradiction; in the last one, $(q(\lambda'_4 d_1), q(\lambda'_4 d_2)) \subset \{2,3\} \times \{1,2\}$, so that $q(\lambda'_4 d_1) = 3$ and $q(\lambda'_4 d_2) = 1$, which in turn implies $q(\lambda'_5 \cdot A_s) \subset \{3,4\}$, again a contradiction.

Assume now that (ii) holds. Repeated use of (9) and the fact that (12) does not hold gives

$$\begin{array}{lll} (q(\lambda'_2 d_1), q(\lambda'_2 d_2)) \subset \{0,1\} \times \{6,0\} & \text{implies} & q(\lambda'_2 d_1) = 1 \text{ and } q(\lambda'_2 d_2) = 6 \\ (q(\lambda'_3 d_1), q(\lambda'_3 d_2)) \subset \{1,2\} \times \{2,3\} & \text{implies} & q(\lambda'_3 d_1) = 1 \text{ and } q(\lambda'_3 d_2) = 3 \end{array}$$

Hence,

$$q(\lambda'_5 \cdot A_s) \subset q(\lambda'_2 \cdot A_s) + q(\lambda'_3 \cdot A_s) + \{0,1\} = \{2,3\},$$

giving (12). This completes the proof for the case $|A| = 4$.

6 The case $|A| = 5$ and $m > 1$

Recall that $N = 7^{m+1}$ where we now assume that $m = \max(\nu(D)) \geq 2$, and that all elements in A belong to three nonzero congruence classes modulo 7, say $(A)_7 \subset \{1,2,4\}$. In particular, given any two elements in A we have either $r(y) = r(x)$ or $r(y) = 2r(x)$ or $r(x) = 2r(y)$. We find convenient to introduce the following notation:

$$e(x,y) = \begin{cases} 2x - y, & \text{if } r(y) = 2r(x) \\ 2y - x, & \text{if } r(x) = 2r(y) \\ x - y, & \text{if } r(y) = r(x), \end{cases} \quad \text{and} \quad \tilde{e}(x,y) = \begin{cases} 2q(x) - q(y), & \text{if } r(y) = 2r(x) \\ 2q(y) - q(x), & \text{if } r(x) = 2r(y) \\ q(x) - q(y), & \text{if } r(y) = r(x). \end{cases} \tag{13}$$

The following properties can be easily checked.

Lemma 4 *Let x, y be integers with $\nu(x) = \nu(y) = j < m$.*

(i) *For each $\lambda \in \cup_{i \leq j} \Lambda_i$ we have $\tilde{e}(x,y) = \tilde{e}(\lambda x, \lambda y)$.*

(ii) *$|\tilde{e}(x,y) - q(e(x,y))|_7 \leq 1$. Moreover, if $r(x) = r(y)$ then $\tilde{e}(x,y) - q(e(x,y)) \in \{0,6\}$.*

Proof. Let $\lambda = (1 + k7^{m-i})$. If $i < j$ then $q(\lambda x) = q(x)$ and $q(\lambda y) = q(y)$ so there is nothing to prove. If $i = j$ and $r(y) = 2r(x)$ then $q(\lambda x) = q(x) + kr(x)$ and $q(\lambda y) = q(y) + kr(y) = q(y) + 2kr(x)$ so that $\tilde{e}(\lambda x, \lambda y) = 2q(\lambda x) - q(\lambda y) = 2q(x) - q(y) = \tilde{e}(x,y)$. The case $r(y) = r(x)$ can be similarly checked.

Part (ii) follows directly from the definition of $q(x) = \left(\lfloor \frac{x}{7^m} \rfloor\right)_7$. $\square$

Recall that, for a subset $X \subset \mathbb{Z}$, $\ell(X)$ stands for the length of the shorter arithmetic progression of difference 1 in $\mathbb{Z}_7$ which contains $q(X)$. Let A_1 denote a class with larger length and denote by s its congruence class modulo 7. Denote by A_2 and A_4 the subsets of elements in A congruent with $2s$ and $4s$ modulo 7 respectively. As in the case $|A| = 4$, the general strategy consists in 'compressing' the sets $q(A_1), q(A_2), q(A_4)$. We summarize in lemmas 5 and 6 below some sufficient conditions in terms of the values of lengths of these three sets which allows one to conclude that (1) holds.

Lemma 5 *Assume that*

$$\ell(A_1) + \ell(A_2) + \ell(A_4) \leq 5.$$

Then there is $\lambda \in \Lambda_0$ such that

$$q(\lambda \cdot A) \cap \{0, 6\} = \emptyset,$$

unless $(\ell(A_1), \ell(A_2), \ell(A_4)) = (3, 1, 1)$ and $\tilde{e}(d, d') \in \{2, 4\}$ for each $d \in A_2$ and $d' \in A_4$.

Proof. The elements of Λ_0 will be denoted by $\lambda_k = 1+7^m ks^{-1}$. Observe that, for $d \in A_j$, we have $q(\lambda_k d) = q(d) + jk$. By (7) we may assume that $q(A_1) \subset \{1, 2, \ldots, \ell(A_1)\}$, so that $q(\lambda_k \cdot A_1) \cap \{0, 6\} = \emptyset$ for $k = 0, 1, \ldots, 5 - \ell(A_1)$. We may assume that $\ell(A_1) + \ell(A_2) + \ell(A_4) = 5$.

If $\ell(A_1) = 5$ we are done. If $\ell(A_1) = 4$ then we clearly have $q(\lambda_k \cdot (A \setminus A_1)) \cap \{0, 6\} = \emptyset$ for at least one of $k = 0, 1$.

Suppose that $\ell(A_1) = 3$. If either $\ell(A_2) = 2$ or $\ell(A_4) = 2$ then for at least one of the values of $k = 0, 1, 2$ we have $q(\lambda_k \cdot (A \setminus A_1)) \cap \{0, 6\} = \emptyset$. Let us consider the case $\ell(A_2) = \ell(A_4) = 1$. Let $q(A_2) = \{i\}$ and $q(A_4) = \{j\}$. Suppose that $q(\lambda_k \cdot A) \cap \{0, 6\} \neq \emptyset$ for each $k = 0, 1, 2$. Since at most one element in $\{i, i+2, i+4\}$ belongs to $\{0, 6\}$, two of the elements in $\{j, j+4, j+1\}$ must be in $\{0, 6\}$. The only possibility is $\{j, j+1\} = \{0, 6\}$ and $\{i+2\} \in \{0, 6\}$. This implies $2i - j \in \{2, 4\}$. Hence $\tilde{e}(d, d') \in \{2, 4\}$ for each $d \in A_2, d' \in A_4$.

Suppose finally that $\ell(A_1) = 2$. We may assume that $\ell(A_2) = 2$ and $\ell(A_4) = 1$. Let $q(A_2) = \{i, i+1\}$ and $q(A_4) = \{j\}$. Two of the four sets $\{i, i+1\} + 2k$, $k = 0, 1, 2, 3$, intersect $\{0, 6\}$ for two consecutive values of k in cyclic order. At most two of the sets $\{j\} + 4k$, $k = 0, 1, 2, 3$, intersect $\{0, 6\}$ for two non consecutive values of k. Hence there is some value of k for which $(q(A_2) + 2k) \cup (q(A_4) + 4k)$ does not intersect $\{0, 6\}$. This completes the proof. $\square$

Lemma 6 *Suppose that $q(A_1) \subset \{1, \ldots, \ell(A_1)\}$ and let $d \in A_1$ with $q(d) = 1$. There is $\lambda \in \Lambda_0$ such that*

$$q(\lambda \cdot A) \cap \{0, 6\} = \emptyset,$$

if one of the following conditions hold:

(i) *Either $(\ell(A_1), \ell(A_2), \ell(A_4)) = (5, 0, 1)$ and $\tilde{e}(d, d') \notin \{4, 6\}$ for each $d' \in A_4$, or $(\ell(A_1), \ell(A_2), \ell(A_4)) = (5, 1, 0)$ and $\tilde{e}(d, d') \notin \{2, 3\}$ for each $d' \in A_2$.*

(ii) *Either* $(\ell(A_1), \ell(A_2), \ell(A_4)) = (4,0,2)$, *or* $(\ell(A_1), \ell(A_2), \ell(A_4)) = (4,2,0)$ *and* $\tilde{e}(d,d') \neq 4$, *where* $q(A_2) = \{i, i+1\}$ *and* $d' \in A_2 \cap q^{-1}(i)$.

(iii) $(\ell(A_1), \ell(A_2), \ell(A_4)) = (3,3,0)$ *(or* $(3,0,3)$*.)*

Proof. We may assume that the elements of A_1 are congruent to 1 modulo 7, so that $q(\lambda_k \cdot A_1) \cap \{0,6\} = \emptyset$ for $k = 0, 1, \ldots, 5 - \ell(A_1)$.

(i) Suppose that $\ell(A_4) = 1$. If $q(A_4) = i \in \{0,6\}$ then $\tilde{e}(d,d') = 2i - 1 \in \{6,4\}$. Similarly, if $\ell(A_2) = 1$, then $i = q(A_2) \in \{0,6\}$ implies $\tilde{e}(d,d') = 2 - i \in \{2,3\}$.

(ii) Suppose that $\ell(A_4) = 2$, say $q(A_4) = \{i, i+1\}$. One of the two sets $\{i, i+1\}, \{i+4, i+5\}$ does not intersect $\{0,6\}$ so that the result holds for at least one λ_k, $k = 0, 1$. If $\ell(A_2) = 2$ then both $q(A_2) = \{i, i+1\}$ and $q(\lambda_1 \cdot A_2) = \{i+2, i+3\}$ intersect $\{0,6\}$ only if $i = 5$ and $\tilde{e}(d,d') = 2 - i = 4$.

(iii) Let $q(A_2) = \{i, i+1, i+2\}$. Now $q(\lambda_k \cdot A_1) \cap \{0,6\} = \emptyset$ for $k = 0, 1, 2$, and one of the three sets $\{i, i+1, i+2\}, \{i+2, i+3, i+4\}, \{i+4, i+5, i+6\}$ does not intersect $\{0,6\}$. The case $(3,0,3)$ is obtained by renaming A_1 as A_2, A_2 as A_4 and A_4 as A_1. $\square$

As shown in the lemmas 5 and 6 above, compression alone is usually not enough to conclude that (1) holds. The next lemmas provide additional tools to complete the proof. Further results of the same nature will appear later on in dealing with specific cases.

Lemma 7 *Let* $X \subset \mathbb{Z}_7$ *and let* d, d' *be two integers with* $\nu(d) = \nu(d') = 0$ *and* $r(d') = 2r(d)$. *There is* $\lambda \in \Lambda_h$ *for some* $h < m$ *such that*

$$\tilde{e}(\lambda d, \lambda d') \notin X$$

whenever one of the two following conditions holds:

(i) $\nu(2d - d') < m$ *and* $\ell(X) \leq 4$, *or*

(ii) $\nu(2d - d') = m$ *and* $r(2d - d') \notin X \cap (X+1)$.

Proof.

(i) Since $\ell(X) \leq 4$ there is $x \in \mathbb{Z}_7 \setminus (X + \{0,1,2\})$. Choose $\lambda \in \Lambda_h$, where $h = \nu(2d - d') < m$, such that $q(\lambda(2d - d')) = x - 1$. Using Lemma 4 we have $\tilde{e}(\lambda d, \lambda d') \in q(e(\lambda d, \lambda d') + \{-1,0,1\} = x + \{-2,-1,0\} \notin X$.

(ii) Since $\nu(2d - d') = m$ we have $r(2d - d') = q(2d - d') = q(2d) - q(d')$. Suppose that $q(2d - d') \notin X$. Choose $\lambda \in \Lambda_1$ such that $q(\lambda(7d)) = 0$. Let us show that this λ verifies the conditions of the lemma (recall that $m > 1$). Note that $q(2\lambda d) \in 2q(\lambda d)) + \{0,1\}$ and $q(2\lambda d) = 2q(\lambda d)) + 1$ implies $q(\lambda(7d)) = q(\lambda(2d + 2d + 2d + d)) \in q(2\lambda d) + q(2\lambda d) + q(2\lambda d) + q(\lambda d) + \{0,1,2,3\} = 7q(\lambda d) + 3 + \{0,1,2,3\}$, contradicting $q(\lambda(7d)) = 0$. Hence $q(2\lambda d) = 2q(\lambda d))$. Thus $\tilde{e}(\lambda d, \lambda d') = 2q(\lambda d) - q(\lambda d') = q(\lambda(2d - d')) = q(2d - d') \notin X$ as claimed. A similar argument applies when $q(2d - d') \notin (X+1)$ by choosing $\lambda \in \Lambda_1$ such that $q(\lambda(7d)) = 6$ so that $q(\lambda(2d)) = 2q(\lambda d)) + 1$. $\square$

Note that the proof of Lemma 7 (ii) requires $m > 1$. We give a last lemma before proceeding with the case analysis. First we note the following remark.

Remark 8 *Let* $X \subset \mathbb{Z}$.

(i) *If* $q(X - X) \subset \{0,6\}$ *then* $\ell(k \cdot X) \leq k + 1$, $1 \leq k \leq 6$.

(ii) *If* $q(X - X) \subset \{0,1,5,6\}$ *then* $\ell(X) \leq 3$.

Lemma 9 *Let $B = \{b_1, b_2, b_3\} \subset \mathbb{Z}$ with $\nu(B) = \{0\}$ and $r(b_1) = r(b_2) = r(b_3)$. Set $x = b_1 - b_3$ and $y = b_2 - b_3$.*

(i) *If $\nu(x) \neq \nu(y)$ then there is a multiplier λ such that $\ell(\lambda \cdot B) \leq 2$.*

(ii) *If $\nu(x) = \nu(y) = h < m$ and $r(y) = jr(x)$, $j \in \{2, 3\}$ then there is a multiplier λ such that $q(\lambda y) \in \{0, 5, 6\}$ and $\ell(\lambda \cdot B) \leq j$. Moreover, if $j = 3$, then $\lambda \in \Lambda_h$.*

Proof. (i) Suppose $\nu(x) > \nu(y)$. By Lemma 2 there is a multiplier λ such that $q(\lambda x) = q(\lambda y) = 6$. Thus $q(\lambda x - \lambda y) = q(\lambda(b_1 - b_2)) \in \{0, 6\}$ and $q(\lambda \cdot B - \lambda \cdot B) \in \{0, 6\}$. By Remark 8 (i), we have $\ell(\lambda \cdot B) \leq 2$.

(ii) Suppose first that $r(y) = 2r(x)$ and set $e = e(x, y) = 2x - y$. We have $h' = \nu(e) > h$. Choose $\lambda \in \Lambda_{h'}$ such that either $q(\lambda e) = 0$ (if $\nu(e) \neq m$) or $\lambda e = N/7$ (if $\nu(e) = m$). By Lemma 4 we have $\tilde{e}(\lambda x, \lambda y) \in \{0, 1, 6\}$. Choose $\lambda' \in \Lambda_h$ such that $q(\lambda'\lambda x) = 0$ (if $\tilde{e}(\lambda x, \lambda y) \in \{0, 1\}$) or $q(\lambda'\lambda x) = 6$ (if $\tilde{e}(\lambda x, \lambda y) = 6$). Then $q(\lambda'\lambda y) \in \{0, 6\}$ and $q(\lambda'\lambda(y - x)) \in \{0, 6\}$. Thus $q(\lambda'\lambda \cdot (B - B)) \subset \{0, 6\}$. By Remark 8 (i), we have $\ell(\lambda'\lambda \cdot B) \leq 2$.

Suppose now that $r(y) = 3r(x)$. Choose $\lambda \in \Lambda_h$ such that

$$q(\lambda y) = \begin{cases} 0 & \text{if} \quad 3q(y) + 5q(x) \in \{0, 2\} \\ 6 & \text{if} \quad 3q(y) + 5q(x) \in \{1, 4, 6\} \\ 5 & \text{if} \quad 3q(y) + 5q(x) \in \{3, 5\} \end{cases}$$

Thus $q(\lambda y) \in \{0, 5, 6\}$, $q(\lambda x) \in \{0, 1, 5, 6\}$ and $q(\lambda y) - q(\lambda x) \in \{0, 1, 6\}$. By Lemma 4, $q(\lambda(y - x)) \in q(\lambda y) - q(\lambda x) + \{0, 6\} = \{0, 1, 5, 6\}$. Hence, by Remark 8 (ii), we have $\ell(\lambda \cdot B) \leq 3$. $\square$

6.1 Case 1: $|A_1| = 5$.

In what follows we shall use some appropriate numbering $\{d_1, d_2, d_3, d_4, d_5\}$ of the elements in A. We shall write $e_{ij} = e(d_i, d_j)$.

Lemma 10 *Suppose that $|A_1| \geq 4$ and let $E = (A_1 - A_1) \setminus \{0\}$. There is a numbering of the elements of A_1 such that one of the following holds:*

(i) $\nu(e_{21}) > \nu(e_{31})$, *or*

(ii) $\nu(e) = h$ *for each* $e \in E$ *and* $r(e_{31}) = 2r(e_{21})$ *and either*

(ii.1) $r(e_{41}) = 3r(e_{21})$, *or*

(ii.2) $r(e_{41}) = 4r(e_{21})$.

Proof. If $|\nu(E)| > 1$ then color the pair $\{d_i, d_j\}$ with $\nu(e_{ij})$. Two intersecting pairs must have different colors. We can rename d_1 their common element and d_2, d_3 the other two to get (i).

Suppose now that $|\nu(E)| = 1$. Consider the set $X = \{r(e_{il}), r(e_{jl}), r(e_{kl})\}$ where i, j, k, l are pairwise distinct subscripts. If $r(e_{il}) = r(e_{jl})$ then $\nu(e_{ij}) = \nu(e_{il} - e_{jl}) > \nu(e_{il})$ contradicting $|\nu(E)| = 1$. By symmetry the elements in X are pairwise distinct and two of them belong to one of the sets $\{1, 2, 4\}$ or $\{3, 5, 6\}$. We may thus assume that

$(r(e_{il}), r(e_{jl}), r(e_{kl})) = (x, 2x, y)$. If $y \in \{3x, 4x\}$ then (ii) holds with $l = 1, i = 2, j = 3$ and $k = 4$. If $y \in \{5x, 6x\}$ then $(r(e_{lj}), r(e_{ij}), r(e_{kj})) = (-2x, -x, y - 2x)$ and (ii) holds with $j = 1, i = 2, l = 3$ and $k = 4$. □

Let $A_1 = \{d_1, d_2, d_3, d_4, d_5\}$ where we use the labeling provided by Lemma 10. We shall show that, up to some multiplier, we have $\ell(A_1) \leq 5$. The result then follows by Lemma 5.

Let $B = \{d_1, d_2, d_3\}$ and $E = (A_1 - A_1) \setminus \{0\}$. Suppose that $\nu(E) \neq \{m\}$. Then, by Lemma 10 and Lemma 9, we may assume $\ell(B) \leq 2$. Thus, by (7) we may assume that $q(B) \subset \{0, 6\}$.

If $\{q(d_4), q(d_5)\} \neq \{2, 4\}$ then we have $\ell(A_1) \leq 5$ and we are done. Suppose that $\{q(d_4), q(d_5)\} = \{2, 4\}$. Then $q(3d_4), q(3d_5) \in \{0, 1, 5, 6\}$ and $q(3 \cdot B) \subset \{0, 1, 2, 4, 5, 6\}$. Furthermore, by Remark 8 (i), $\ell(3 \cdot B) \leq 4$. Thus we have either $q(3 \cdot A_1) \subset \{0, 1, 2, 5, 6\}$ or $q(3 \cdot A_1) \subset \{0, 1, 4, 5, 6\}$, so that $\ell(3 \cdot A_1) \leq 5$.

Suppose now that $\nu(E) = \{m\}$. Set $E_1 = \{e_{51}, e_{41}, e_{31}, e_{21}\} \subset \{kN/7,\ 1 \leq k \leq 6\}$. Denote by $iN/7$ and $jN/7$ the elements in the complement of E_1. Multiplying by $(i-j)^{-1} \in U\mathbb{Z}_7$ we may assume that the elements in E_1 are consecutive, so that $\ell(A_1) \leq 5$. This completes this case.

6.2 Case $|A_1| = 4$.

Let $A_1 = \{d_1, d_2, d_3, d_4\}$ and $r(d_5) \in \{2s, 4s\}$, where the elements in A_1 are labeled with the ordering of Lemma 10.

Let $B = \{d_1, d_2, d_3\}$ and $E = (A_1 - A_1) \setminus \{0\}$. Suppose that $\nu(E) \neq \{m\}$. Then, by Lemma 10 and Lemma 9 we may assume that $\ell(B) \leq 2$ and, by (7) we may assume that $q(B) \subset \{0, 6\}$. If $q(d_4) \neq 3$ then $\ell(A_1) \leq 4$, and if $q(d_4) = 3$ then $q(2 \cdot A_1) \subset \{0, 1, 5, 6\}$ and again $\ell(A_1) \leq 4$. The result follows by Lemma 5.

Suppose now that $\nu(E) = \{m\}$. If (ii.1) holds then multiplying by $(r(e_{21}))^{-1}$ (modulo 7) we may assume that $e_{41} = 3N/7$, $e_{31} = 2N/7$ and $e_{21} = N/7$ which yields $\ell(A_1) \leq 4$. The result follows by Lemma 5. Assume that (ii.2) holds. Up to multiplication by some $\lambda \in \Lambda_m$ we may assume that $q(\{e_{21}, e_{31}, e_{41}\}) \subset \{1, 2, 4\}$ so that $\ell(A_1) \leq 5$. By Lemma 7 we may also assume that $\tilde{e}(d_1, d_5) \notin \{4, 6\}$. Thus Lemma 6 (i) applies if $d_5 \in A_4$. Finally, if $d_5 \in A_2$, we may also assume that $\tilde{e}(d_1, d_5) \notin \{2, 3\}$ by using again Lemma 7 unless $e_{15} = 3N/7$. In this case we have $q(\{2e_{21}, 2e_{31}, 2e_{41}\}) \subset \{1, 4, 2\}$, so that $\ell(A_1) \leq 5$, while $2e_{15} = 6N/7$ and so $\tilde{e}(2d_1, 2d_5) \notin \{2, 3\}$, and the result also follows from Lemma 6 (i). This completes this case.

6.3 Case $|A_1| = 3$, $|A_2| = 1$ and $|A_4| = 1$.

First we consider a convenient labeling of the elements in A_1 to be used here and the two following subsections.

Lemma 11 *Let $s \in \mathbb{Z}_7^*$ be given. There is a labeling of the elements in A_1 such that one of the following holds:*

(i) $\nu(e_{13}) > \nu(e_{23}) = \nu(e_{12})$,

(ii) $\nu(e_{13}) = \nu(e_{23}) = h$ *and either*

(ii.1) $r(e_{13}) = 2r(e_{23})$, *or* (ii.2) $r(e_{13}) = 3r(e_{23}) = \pm s$,

Proof. Let $E = (A_1 - A_1) \setminus \{0\}$. If $|\nu(E)| > 1$ then we clearly can label the elements in D to get (i). Assume that $|\nu(E)| = 1$. Suppose that $|r(E)| < 6$. Note that we can not have $r(e_{ij}) = r(e_{ik})$ since otherwise $\nu(e_{kj}) = \nu(e_{ij} - e_{ik}) > \nu(e_{ij})$ contradicting $|\nu(E)| = 1$. Similarly, $r(e_{ij}) \neq r(e_{kj})$. Thus we may assume that the repeated values of r on E are $r(e_{ij}) = r(e_{jk})$ and we can label $i = 1$, $j = 2$ and $k = 3$. Suppose now that $|r(E)| = 6$. Thus we may assume that $r(e_{ik}) = s$. Observe that $r(e_{jk}) \in \{3s, 5s\}$. Indeed, if $r(e_{jk}) = 2s$ then $r(e_{ji}) = r(e_{jk}) - r(e_{ik}) = s$, if $r(e_{jk}) = 4s)$ then $r(e_{ij}) = r(e_{ik}) + r(e_{kj}) = 4s$ and if $r(e_{jk}) = 6s$ then $r(e_{kj}) = s$, contradicting in each case $|r(E)| = 6$. If $r(e_{jk}) = 3s$ then $r(e_{ji}) = -5s$, $r(e_{ki}) = -s$ and we can label $i = 3$, $j = 2$ and $k = 1$. In case $r(e_{jk}) = 5s$ we can label $i = 1$, $j = 2$ and $k = 3$. $\square$

Let $A_1 = \{d_1, d_2, d_3\}$, where we use the labeling of Lemma 11, $A_2 = \{d_4\}$ and $A_4 = \{d_5\}$. We divide the proof according to the cases of Lemma 11.

(i) and (ii.1) with $h < m$. By Lemma 9 applied to A_1 we may assume that $\ell(A_1) \leq 2$ and the result follows by Lemma 5.

(ii.2) with $h < m$. We consider two subcases:

(ii.2.a) $\nu(e_{45}) \neq \nu(e_{13})$. If $\nu(e_{45}) > \nu(e_{13})$, by Lemma 2 applied to e_{45} we may assume that $q(e_{45}) \in \{0, 6\}$ and thus $\tilde{e}(d_4, d_5) \in \{0, 1, 5, 6\}$. We can then apply Lemma 9 to $B = A_1$ so that we may assume that $\ell(A_1) \leq 3$, yielding the conditions of Lemma 5. A similar argument works when $\nu(e_{45}) < \nu(e_{13})$ by applying Lemma 9 first and then Lemma 2.

(ii.2.b) $h = \nu(e_{45}) = \nu(e_{13})$. By Lemma 11 we may assume $r(e_{45}) = \pm r(e_{13})$. Suppose first that $r(e_{45}) = r(e_{13})$. Set $f = e_{13} - e_{45}$, so that $\nu(f) > h$. By Lemma 2 we may assume $q(f) = 0$ (or $f = N/7$ with the same consequences) so that $\tilde{e}(e_{13}, e_{45}) = q(e_{13}) - q(e_{45}) \in \{0, 1\}$. Recall that this last value is invariant by multiplication of elements in Λ_j with $j \leq h$. Put $\tilde{u} = 3q(e_{13}) + 5q(e_{23})$. By Lemma 2, $q(e_{45})$ can be set to the value shown in the following table according to the values of $\tilde{e} = \tilde{e}(e_{13}, e_{45})$ and $\tilde{u}$:

<table>
<tr><td></td><td>$\tilde{u}$</td><td>0</td><td>1</td><td>2</td><td>3</td><td colspan="2">4</td><td>5</td><td>6</td></tr>
<tr><td rowspan="4">$\tilde{e} = 0$</td><td>$q(e_{45})$</td><td>0</td><td>6</td><td>0</td><td>3</td><td colspan="2">6</td><td>0</td><td>6</td></tr>
<tr><td>$q(e_{13})$</td><td>0</td><td>6</td><td>0</td><td>3</td><td colspan="2">6</td><td>0</td><td>6</td></tr>
<tr><td>$q(e_{23})$</td><td>0</td><td>5</td><td>6</td><td>3</td><td colspan="2">0</td><td>1</td><td>6</td></tr>
<tr><td>$q(e_{12})$</td><td>$\{0,6\}$</td><td>$\{0,1\}$</td><td>$\{0,1\}$</td><td>$\{0,6\}$</td><td colspan="2">$\{6,5\}$</td><td>$\{6,5\}$</td><td>$\{0,6\}$</td></tr>
<tr><td rowspan="4">$\tilde{e} = 1$</td><td>$q(e_{45})$</td><td>6</td><td>0</td><td>6</td><td>0</td><td>5</td><td>6</td><td>6</td><td>5</td></tr>
<tr><td>$q(e_{13})$</td><td>0</td><td>1</td><td>0</td><td>1</td><td>6</td><td>0</td><td>0</td><td>6</td></tr>
<tr><td>$q(e_{23})$</td><td>0</td><td>1</td><td>6</td><td>0</td><td>0</td><td>5</td><td>1</td><td>6</td></tr>
<tr><td>$q(e_{12})$</td><td>$\{0,6\}$</td><td>$\{0,6\}$</td><td>$\{0,1\}$</td><td>$\{0,1\}$</td><td>$\{6\}$</td><td>$\{1\}$</td><td>$\{5,6\}$</td><td>$\{6,6\}$</td></tr>
</table>

If $\tilde{u} = 4$ and $\tilde{e} = 1$ we set $q(e_{45}) = 5$ if $q(e_{12}) = q(e_{13}) - q(e_{23})$, and $q(e_{45}) = 6$ if $q(e_{12}) = q(e_{13}) - q(e_{23}) - 1$. In all cases except $\tilde{e} = 0$ and $\tilde{u} = 3$ we either have $\ell(A_1) \leq 3$, $\ell(A_2) = \ell(A_4) = 1$ and $\tilde{e}(d_4, d_5) \notin \{2, 4\}$ or $\ell(A_1) = 2$ and $\ell(A_2) = \ell(A_4) = 1$

so that Lemma 5 applies. If $\tilde{e} = 0$ and $\tilde{u} = 3$ we have $q(\{2e_{45}, 2e_{13}, 2e_{23}\}) \subset \{0, 6\}$ and $q(2e_{12}) \in \{0, 1, 5, 6\}$ reaching the same conditions.

A similar analysis applies if $r(e_{45}) = -r(e_{13})$ by exchanging $f = e_{13} - e_{45}$ by $f = e_{13} + e_{45}$ and $q(e_{45})$ by $q(-e_{45})$.

(ii.1) and $h = m$. Up to some multiplier in Λ_m we may assume that $e_{13} = 2N/7$ and $e_{23} = N/7$, which leads to $\ell(A_1) = 3$. By Lemma 7 we may also assume that $\tilde{e}(d_4, d_5) \notin \{2, 4\}$ and we are in the conditions of Lemma 5.

(ii.2) and $h = m$. Up to some multiplier in Λ_m we may assume that $e_{13} = 3N/7$ and $e_{23} = N/7$ which implies $\ell(A_1) = 4$. We may also assume that $q(A_1) \subset \{1, 2, 3, 4\}$ and $q(d_3) = 1$. Let $q(d_4) = i$ and $q(d_5) = j$. In this case (1) holds unless both $\{i, j\}$ and $\{i + 2, j + 4\}$ intersect $\{0, 6\}$, namely when $i \in \{0, 6\}$ and $j \in \{2, 3\}$ or $j \in \{0, 6\}$ and $i \in \{4, 5\}$. Thus $(\tilde{e}_{34}, \tilde{e}_{45})$ is one of the four pairs $\{(2, 4), (2, 5), (3, 2), (3, 3)\}$ in the first case and one of the four pairs $\{(4, 3), (4, 4), (5, 1), (5, 2)\}$ in the second one.

If $\nu(e_{34}) < m$ then by Lemma 7 (i) applied to d_3 and d_4 we may assume that $\tilde{e}_{34} \notin \{2, 3, 4, 5\}$ and we are done.

If $\nu(e_{34}) = m$ and $\nu(e_{45}) < m$ then by applying Lemma 7 (i) to d_4 and d_5 we may assume that $\tilde{e}_{45} \notin \{2, 3, 4, 5\}$ if $\tilde{e}_{34} \in \{2, 3\}$ and $\tilde{e}_{45} \notin \{1, 2, 3, 4\}$ if $\tilde{e}_{34} \in \{4, 5\}$ thus avoiding the two bad cases.

Suppose that $\nu(e_{34}) = \nu(e_{45}) = m$. Observe that one of the four pairs $(q(e_{34}) - \epsilon_1, q(e_{45}) - \epsilon_2)$, $\epsilon_1, \epsilon_2 \in \{0, 1\}$ is not a bad pair. Observe also that $\tilde{e}(d_3, d_4) = 2(q(d_3)) - q(d_4) \in q(2d_3) - q(d_4) + \{0, 6\} = q(e_{34}) + \{0, 6\}$ and similarly $\tilde{e}(d_4, d_5) \in q(e_{45}) + \{0, 6\}$. We have $d_4 = 2d_3 + tN/7$, for some $t < 7$. By Lemma 2 we may assume that $q(7d_3) = 4\epsilon_1 + 2\epsilon_2$ for each choice of $\epsilon_1, \epsilon_2 \in \{0, 1\}$ (recall that $\nu(7d_3) = 1 < m$). By a routine checking we then conclude that $(\tilde{e}(d_3, d_4), \tilde{e}(d_4, d_5)) = (q(e_{34}) - \epsilon_1, q(e_{45}) - \epsilon_2)$. Thus each of the eight bad pairs can be avoided. This completes the proof of this case.

6.4 Case $|A_1| = 3$ and $|A_2| = 2$.

By using the labeling of Lemma 11 we have $A_1 = \{d_1, d_2, d_3\}$ and $A_2 = \{d_4, d_5\}$. We first prove the following Lemma.

Lemma 12 *Assume that $|A_1| = 3$ and $|A_2| = 2$. Let $d \in A_1$ such that $q(A_1) \subset \{q(d), \ldots, q(d) + \ell(A_1) - 1\}$ and $d' \in A_2$. If one of the following conditions hold then there is a multiplier λ such that*

$$q(A) \cap \{0, 6\} = \emptyset.$$

(i) $\ell(A_1) \leq 3$.
(ii) $\ell(A_1) = 4$ *and* $\tilde{e}(d, d') \in \{0, 1, 6\}$

Proof. (i) Since $\ell(A_1) \leq 3$ there are three good multipliers for A_1 in Λ_0. At most one of them is bad for each of the two elements in A_2.

(ii) We may assume $q(A_1) \subset \{1, 2, 3, 4\}$. Let $\lambda_k = 1 + 7^m k s^{-1} \in \Lambda_0$, so that λ_0, λ_1 are good multipliers for A_1. Since $\tilde{e}(d, d') \in \{0, 1, 6\}$, then $q(\lambda_0 d') \in \{1, 2, 3\}$, and $q(\lambda_1 d') \in \{3, 4, 5\}$. At most one of λ_0, λ_1 is bad for the second element in A_2. □

We divide the proof according to the cases in Lemma 11.

(i) or (ii) with $h < m$. By Lemma 9 applied to $B = A_1$ we may assume that $\ell(A_1) \le 3$ and the result follows from Lemma 12 (i).

(ii.1) with $h = m$. Up to a multiplier in Λ_m we may assume that $e_{13} = 2N/7$ and $e_{23} = N/7$ so that $\ell(A_1) = 3$. The result follows from Lemma 12 (i).

(ii.2) with $h = m$. We can apply Lemma 2 to set $e_{13} = 3N/7$ and $e_{23} = N/7$. Thus $\ell(A_1) = 4$. If $\nu(e_{34}) < m$ then by Lemma 7 (i) we can set $\tilde{e}_{34} \notin \{2,3,4,5\}$ and the result follows from Lemma 12 (ii). We can do the same if $\nu(e_{35}) < m$. Suppose that $\nu(e_{34}) = \nu(e_{35}) = m$. Then $\nu(e_{45}) = m$. Set $s = (3r(e_{45}))_7$. By renaming d_4 and d_5 if necessary we may assume $e_{45} = e_{23} = N/7$ so that $\ell(A_1) = 4$ and $\ell(A_2) = 2$. Moreover, by Lemma 7, we may also assume that $\tilde{e}_{35} \neq 4$. The result follows from Lemma 6 (ii).

6.5 Case $|A_1| = 3$ and $|A_4| = 2$.

By using the labeling of Lemma 11 we have $A_1 = \{d_1, d_2, d_3\}$ and $A_2 = \{d_4, d_5\}$. Suppose that, up to a multiplier,

$$\ell(A_1) \le 3 \quad \text{and} \quad \ell(A_4) \le 3 \qquad \text{or} \qquad \ell(A_1) \le 4 \quad \text{and} \quad \ell(A_4) \le 2. \tag{14}$$

Then either Lemma 5 or Lemma 6 (ii) or (iii) applies.

We divide the proof according to the cases of Lemma 11.

(i) or (ii.1) with $h < m$. By Lemma 9 we may assume that $\ell(A_1) \le 2$. If $\ell(A_4) \le 3$ we are in the conditions of Lemma 5. Otherwise we have $q(e_{45}) \in \{2,3,4\}$. If $|e_{45}|_N \ge 5N/14$ then $|2e_{45}|_N \le 2N/7$ which yields $\ell(2 \cdot A_1) \le 3, \ell(2 \cdot A_4) \le 3$ and (14) holds. If $|e_{45}|_N \in [2N/7, 5N/14]$ then $|3e_{45}|_N \le N/7$ which yields $\ell(3 \cdot A_1) \le 4, \ell(3 \cdot A_4) \le 2$ and (14) holds.

(ii.2) with $h < m$. If $\nu(e_{13}) = \nu(e_{45})$ we choose $s = r(e_{45})$. By renaming d_4 and d_5 if necessary, we may assume $r(e_{45}) = r(e_{13})$. Let $f = e_{45} - e_{13}$, so that $\nu(f) > \nu(e_{13})$. By Lemma 2 we may assume $q(f) = 0$ (if $\nu(f) \neq m$) or $f = N/7$ (if $\nu(f) = m$). By Lemma 9 (ii) we may also assume that $q(e_{13}) \in \{0,5,6\}$ and $\ell(A_1) \le 3$. Hence, $q(e_{45}) = q(e_{13} + f) \in q(e_{13}) + q(f) + \{0,1\}$ which implies $|q(e_{45})|_7 \le 2$ and $\ell(A_4) \le 3$. Therefore (14) holds.

Suppose now $\nu(e_{13}) \neq \nu(e_{45})$. If $\nu(e_{13}) < \nu(e_{45})$ we can apply Lemma 2 to e_{45} to set $q(e_{45}) = 0$ (if $\nu(e_{45}) \neq m$) or $e_{45} = N/7$ (if $\nu(e_{45}) = m$) and then Lemma 9 to A_1 to set $\ell(A_1) \le 3$ and $\ell(A_4) \le 2$ and (14) holds. A similar argument applies if $\nu(e_{13}) > \nu(e_{45})$ by applying Lemma 9 first and then Lemma 2.

(ii.1) with $h = m$. We may assume that $e_{13} = 2N/7$ and $e_{23} = N/7$ so that $\ell(A_1) = 3$. By renaming d_4 and d_5 if necessary we may assume $r(e_{45}) \in \{1,2,4\}$. We consider two cases.

(a) Either $\nu(e_{45}) < m$ or $e_{45} \in \{N/7, 2N/7\}$. In the first case, by Lemma 2, we may assume $q(e_{45}) \in \{0,1,5,6\}$. Thus, in both cases we have $\ell(A_4) \le 3$ yielding (14).

(b) $\nu(e_{45}) = m$ and $e_{45} \notin \{N/7, 2N/7\}$. We may then assume that $e_{45} = 4N/7$, $q(A_1) = \{1,2,3\}$ and $q(A_4) = \{i, i+4\}$. There are three available multipliers in Λ_0 for which $q(\lambda \cdot A_1) \cap \{0,6\} = \emptyset$. It can be easily checked that one of them verifies

$q(\lambda \cdot A_4) \cap \{0,6\}$ as well unless $i \in \{2,6\}$, and so, $\tilde{e}(d_3,d_4) \in \{4,5\}$. By Lemma 7 we may assume that $\tilde{e}(d_3,d_4)$ takes none of these two values unless $e_{34} = 5N/7$. If this is the case, we have $\ell(2 \cdot A_1) = 5$, $\ell(2 \cdot A_4) = 2$, $2e_{34} = 3N/7$ and, by Lemma 7(ii), we can avoid $\tilde{e}(2d_3, 2d_4) = 2$. By (7) we may assume $q(2 \cdot A_1) = \{1,3,5\}$ and, since $\tilde{e}(2d_3,2d_4) = 3$, we have $q(2 \cdot A_4) = \{1,2\}$.

(ii.2) with $h = m$. Choose $s = r(e_{45})$. By exchanging d_4 with d_5 if necessary we may assume that $r(e_{13}) = r(e_{45})$, and by Lemma 2 we may assume that $e_{13} = N/7$ and $e_{23} = 5N/7$, so that $\ell(A_1) \leq 4$. If $\nu(e_{45}) = m$ we have $e_{45} = N/7$ and $\ell(A_4) = 2$. If $\nu(e_{45}) < m$ then, by Lemma 2 we may set $q(e_{45}) \in \{0,6\}$ and $\ell(A_4) = 2$ again. In both cases Lemma 6 (ii) applies.

6.6 Case $|A_1| = 2$.

We may assume that $A_1 = \{d_1, d_2\}$, $A_2 = \{d_3, d_4\}$ and $A_4 = \{d_5\}$.

Up to renaming the elements in A we may assume that $r(e_{12}), r(e_{34}) \in \{1,2,4\}$. Suppose that

$$\ell(A_1) \leq 2 \quad \text{and} \quad \ell(A_2) \leq 2. \tag{15}$$

Then the result follows from Lemma 5.

If $\nu(e_{12}) \neq \nu(e_{34})$ then, by Lemma 2 applied to e_{12} and e_{34} we may assume that $q(e_{12}), q(e_{34}) \in \{0,6\}$. Hence (15) holds.

Assume now that $\nu(e_{12}) = \nu(e_{34})$. Suppose first that $r(e_{12}) = r(e_{34})$. Let $f = e_{12} - e_{34}$. Note that $\nu(e_{12}) < \nu(f)$. If $\nu(f) \leq m$, by Lemma 2 applied to f and e_{34} we may assume that $q(f) = 6$ and $q(e_{34}) = 0$. On the other hand, if $\nu(f) > m$, so that $q(f) = 0$, we can apply Lemma 2 to e_{34} to have $q(e_{34}) = 6$. In both cases, Lemma 4 yields $q(e_{12}) = q(f + e_{34}) \in \{0,6\}$ and (15) holds.

Suppose now that $r(e_{12}) \neq r(e_{34})$. Then either $r(e_{12}) = 2r(e_{34})$ or $2r(e_{12}) = r(e_{34})$. If $\nu(e_{12}) = \nu(e_{34}) < m$ then, by Lemma 2, there is λ such that $q(\lambda e_{12}), q(\lambda e_{34}) \in \{0,1,5,6\}$. Since $\lambda e_{12} \neq 5N/7$ and $\lambda e_{34} \neq 5N/7$, (15) holds.

Assume $\nu(e_{12}) = \nu(e_{34}) = m$. We consider two cases:

(a) $r(e_{12}) = 2r(e_{34})$. Up to a multiplier in Λ_m we may assume that $e_{12} = 2N/7$ and $e_{34} = N/7$. Let $\lambda_k \in \Lambda_0$ be such that $q(\lambda_k d_2) = k$, $k = 1,2,3$. Since $e_{12} = 2N/7$ we have $q(\lambda_k \cdot A_1) = \{k, k+2\}$. On the other hand, since $q(d_4) = 2q(d_2) - \tilde{e}(d_2,d_4)$ and $e_{34} = N/7$, we have $q(\lambda_k \cdot A_2) = (2k - \tilde{e}(d_2,d_4), (2k-1) - \tilde{e}(d_2,d_4))$. Finally, $q(\lambda_k d_5) = 4\tilde{e}(d_2,d_5) + 4k$. Thus, if

$$(\tilde{e}(d_2,d_4), \tilde{e}(d_2,d_5)) \notin \{((4,2),(4,4),(5,4),(6,4),(6,6)\} \tag{16}$$

then $q(\lambda_k \cdot A) \cap \{0,6\} = \emptyset$ for some k. Now, if either $\nu(e_{24}) < m$, by using Lemma 7(i), or $e_{24} \in \{N/7, 2N/7, 3N/7\}$ we can assume that $\tilde{e}(d_2,d_4) \notin \{4,5,6\}$ and so (16) holds; if $\nu(e_{24}) = m$ and $\nu(e_{25}) < m$ we can avoid each of the pairs in (16) by setting $\tilde{e}(d_2,d_5) \notin \{2,4\}$ if $e_{24} \in \{4N/7, 5N/7\}$ and $\tilde{e}(d_2,d_5) \notin \{4,6\}$ if $e_{24} \in \{0, 6N/7\}$; finally, if $\nu(e_{24}) = \nu(e_{25}) = m$, we observe that one of the four pairs $(q(e_{24}) - \epsilon_1, q(e_{25}) - \epsilon_2)$, $\epsilon_1, \epsilon_2 \in \{0,1\}$ avoids each of the bad pairs in (16). By repeating the argument in the final paragraph of case 6.3 applied to $7d_5$, (we here can also assume that $q(7d_5) = 4\epsilon_2 + 2\epsilon_1$) we get (16).

(b) $r(e_{12}) = 4r(e_{34})$. Up to a multiplier in Λ_m we may assume that $e_{12} = N/7$ and $e_{34} = 2N/7$. Let $\lambda_k \in \Lambda_0$ be such that $q(\lambda_k d_4) = k$, $k = 1, 2, 3$, so that $q(\lambda_k \cdot A_2) = \{k, k+2\}$. On the other hand, $q(\lambda_k d_2) = 4\tilde{e}(d_2, d_4) + 4k$, so that $q(\lambda_k \cdot A_1) = \{4\tilde{e}(d_2, d_4) + 4k, 4\tilde{e}(d_2, d_4) + 4k + 1\}$. Finally $q(\lambda_k d_5) = 2q(\lambda_k d_4) - \tilde{e}(d_4, d_5) = 2k - \tilde{e}(d_4, d_5)$. By Lemma 7 we can assume that $\tilde{e}(d_2, d_4) \notin \{2, 4\}$. It is then routine to check that, for every value of $\tilde{e}(d_2, d_4)$ and $\tilde{e}(d_4, d_5)$ there is $k \in \{1, 2, 3\}$ such that $q(\lambda_k \cdot A) \cap \{0, 6\} = \emptyset$.

This completes the proof of the case $|A| = 5$ and $m > 1$.

7 The case $|A| = 5$ and $m = 1$

In Section 6 we have used the hypothesis $m > 1$ in Lemma 7 and in particular situations in cases 6.3 and 6.6. However, when $m = 1$ we are led to consider the problem with $N = 49$ and $d_6 = 7k$, $1 \leq k \leq 3$, and it is unfortunately no longer true that all sets admit a good multiplier. By computer search we found that there is always a multiplier for which each of these sets can be included in the interval $[7, 42]$ except in the three (up to dilation) following ones:

$$\{1, 3, 4, 5, 18\} \ \ \{1, 4, 6, 10, 11\} \text{ and } \{1, 4, 6, 10, 22\}.$$

We consider each of this sets in $\mathbb{Z}_{N'}$ with $N' = 2N = 98$. There are at most 32 nonequivalent subsets in $\mathbb{Z}_{98}$ which are congruent to one of the above exceptional sets, and each of them has to be combined with the six possible values of d_6, namely $7k$, $k = 1, 2, \ldots, 6$. By checking all these possibilities, we found that every set D of integers which coincides with one of the found exceptions when considered in $\mathbb{Z}_{49}$ and verifying $\gcd(D) = 1$ admits a multiplier in $\mathbb{Z}_{98}$. This computation concludes the proof[1].

8 Acknowledgements

The authors want to thank the anonymous referee for valuable comments and remarks and for pointing out several typos which were quite crucial to follow the proofs properly.

References

[1] J. Barajas, O. Serra, Distance graphs with maximum chromatic number, *Discrete Math.* to appear (2007).

[2] J. Barajas, O. Serra, On the chromatic number of circulant graphs, *Discrete Math.* to appear (2007).

[3] U. Betke, J.M. Wills, Untere Schranken für zwei dophantische Approximations-Funktionen, *Monatsch. Math.* **76** (1972) 214–217.

[1] The described verifications were made by a simple program which runs in few seconds. A documented code in `MAPLE` for the the two verifications in $\mathbb{Z}_{49}$ and in $\mathbb{Z}_{98}$ can be found in `www-ma4.upc.edu/oserra/lonelyseven` by the names `check49.mws` and `check98.mws`.

[4] W. Biennia, L. Goddyn, P. Gvozdjak, A. Sebő and M. Tarsi, Flows, View obstructions and the Lonely Runner, *Journal of Combin. Theory Ser. B* **72** (1998) 1–9.

[5] T. Bohman, R. Holzman, D. Kleitman, Six lonely runners, *Electron. J. Combin.* **8** (2001), no. 2, Research Paper 3, 49 pp.

[6] Y.G. Chen, On a conjecture about diophantine approximations III, *J. Number Theory* **39** (1991) 91–103.

[7] T.W. Cusick, View-obstruction problems, *Aequationes Math.* **9** (1973) 165–170.

[8] T.W. Cusick, View-obstruction problems in n-dimensional geometry, *J. Combin. Theory Ser. A* **16** (1974), 1–11.

[9] T. W. Cusick, View-obstruction problems II, *Proc. Amer. Math. Soc.* **84** (1982) 25–28.

[10] T.W. Cusick, C. Pomerance, View obstruction problems III, *Journal of Number Theory* **19** (1984) 131–139.

[11] J. Renault, View-obstruction: a shorter proof for 6 lonely runners. *Discrete Math.* **287** (2004) 93–101.

[12] J.M. Wills, Zwei Säzte über inhomogene diophantische Approximation von Irrationalzahlen, *Monatsch. Math.* **71** (1967) 263–269.

[13] X. Zhu, Circular chromatic number of distance graphs with distance sets of cardinality 3, *J. Graph Theory* **41** (2002) 195–207.