The Minor Crossing Number of Graphs with an Excluded Minor

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Abstract

The *minor crossing number* of a graph G is the minimum crossing number of a graph that contains G as a minor. It is proved that for every graph H there is a constant c, such that every graph G with no H-minor has minor crossing number at most $c|V(G)|$.

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1 Introduction

The *crossing number* of a graph[1] G, denoted by $\mathsf{cr}(G)$, is the minimum number of crossings in a drawing[2] of G in the plane; see [13, 28, 29, 37, 48–50] for surveys. The crossing number is an important measure of the non-planarity of a graph [48], with applications in discrete and computational geometry [27, 47] and VLSI circuit design [3, 20, 21]. In information visualisation, one of the most important measures of the quality of a graph drawing is the number of crossings [34–36].

We now outline various aspects of the crossing number that have been studied. First note that computing the crossing number is $\mathcal{NP}$-hard [15], and remains so for simple cubic graphs [19, 31]. Moreover, the exact or even asymptotic crossing number is not known for specific graph families, such as complete graphs [40], complete bipartite graphs [23, 38, 40], and Cartesian products [1, 5, 6, 17, 39]. Given that the crossing number seems so difficult, it is natural to focus on asymptotic bounds rather than exact values. The 'crossing lemma', conjectured by Erdős and Guy [13] and first proved by Leighton [20] and Ajtai et al. [2], gives such a lower bound. It states that for some constant c, $\mathsf{cr}(G) \geq c\|G\|^3/|G|^2$ for every graph G with $\|G\| \geq 4|G|$. See [22, 25] for recent improvements. Other general lower bound techniques that arose out of the work of Leighton [20, 21] include the bisection/cutwidth method [11, 26, 45, 46] and the embedding method [44, 45]. Upper bounds on the crossing number of general families of graphs have been less studied. One example, by Pach and Tóth [30], says that graphs G of bounded genus and bounded degree have $\mathcal{O}(|G|)$ crossing number. See [9, 12] for extensions. The present paper also focuses on crossing number upper bounds.

Graph minors[3] are a widely used structural tool in graph theory. So it is inviting to explore the relationship between minors and the crossing number. One impediment is that the crossing number is not minor-monotone; that is, there are graphs G and H with H a minor of G, for which $\mathsf{cr}(H) > \mathsf{cr}(G)$. Nevertheless, following an initial paper by Robertson and Seymour [41], there have been a number of recent papers on the relationship between crossing number and graph minors [7, 8, 14, 16, 18, 19, 24, 31, 51]. For example, Wood and Telle [51] proved the following upper bound (generalising the

[1]We consider finite, undirected, simple graphs G with vertex set $V(G)$ and edge set $E(G)$. Let $|G| := |V(G)|$ and $\|G\| := |E(G)|$. Let $\Delta(G)$ be the maximum vertex degree of G.

[2]A *drawing* of a graph represents each vertex by a distinct point in the plane, and represents each edge by a simple closed curve between its endpoints, such that the only vertices an edge intersects are its own endpoints, and no three edges intersect at a common point (except at a common endpoint). A *crossing* is a point of intersection between two edges (other than a common endpoint). A graph is *planar* if it has a crossing-free drawing.

[3]Let vw be an edge of a graph G. Let G' be the graph obtained by identifying the vertices v and w, deleting loops, and replacing parallel edges by a single edge. Then G' is obtained from G by *contracting* vw. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. A family of graphs $\mathcal{F}$ is *minor-closed* if $G \in \mathcal{F}$ implies that every minor of G is in $\mathcal{F}$. $\mathcal{F}$ is *proper* if it is not the family of all graphs. A deep theorem of Robertson and Seymour [43] states that every proper minor-closed family can be characterised by a finite family of excluded minors. Every proper minor-closed family is a subset of the H-minor-free graphs for some graph H. We thus focus on minor-closed families with one excluded minor.

above-mentioned results in [9, 12, 30] for graphs of bounded genus).

Theorem 1 ([51]). *For every graph H there is a constant $c = c(H)$, such that every H-minor-free graph G has crossing number* $\mathsf{cr}(G) \leq c\,\Delta(G)^2|G|$.

1.1 Minor Crossing Number

Bokal et al. [8] defined the *minor crossing number* of a graph G, denoted by $\mathsf{mcr}(G)$, to be the minimum crossing number of a graph that contains G as a minor. The main motivation for this definition is that for every constant c, the family of graphs G for which $\mathsf{mcr}(G) \leq c$ is closed under taking minors. Moreover, the minor crossing number corresponds to a natural style of graph drawing, in which each vertex is drawn as a tree. Bokal et al. [7] proved a number of lower bounds on the minor crossing number that parallel the lower bound techniques of Leighton. The main result of this paper is to prove the following upper bound, which is an analogue of Theorem 1 for the minor crossing number (without the dependence on the maximum degree).

Theorem 2. *For every graph H there is a constant $c = c(H)$, such that every H-minor-free graph G has minor crossing number* $\mathsf{mcr}(G) \leq c\,|G|$.

The restriction to graphs with an excluded minor in Theorem 2 is unavoidable in the sense that $\mathsf{mcr}(K_n) \in \Theta(n^2)$. The linear dependence in Theorem 2 is best possible since $\mathsf{mcr}(K_{3,n}) \in \Theta(n)$. Both these bounds were established by Bokal et al. [8]. An elegant feature of Theorem 2 and the minor crossing number is that there is no dependence on the maximum degree, unlike in Theorem 1, where some dependence on the maximum degree is unavoidable. In particular, the complete bipartite graph $K_{3,n}$ has no K_5-minor and has $\Theta(n^2)$ crossing number [23, 38].

2 Planar Decompositions

It is widely acknowledged that the theory of crossing numbers needs new ideas. Some tools that have been recently developed include 'meshes' [39], 'arrangements' [1], 'tile drawings' [4, 32, 32, 33], and the 'zip product' [4–6]. A feature of the proof of Theorem 1 by Wood and Telle [51] is the use of 'planar decompositions' as a new tool for studying the crossing number. Planar decompositions are also the key component in the proof of Theorem 2 in this paper.

Let G and D be graphs, such that each vertex of D is a set of vertices of G (called a *bag*). Note that we allow distinct vertices of D to be the same set of vertices in G; that is, $V(D)$ is a multiset. For each vertex v of G, let $D(v)$ be the subgraph of D induced by the bags that contain v. Then D is a *decomposition* of G if:

- $D(v)$ is connected and nonempty for each vertex v of G, and

- $D(v)$ and $D(w)$ touch[4] for each edge vw of G.

Decompositions, when D is a tree, were first studied in detail by Robertson and Seymour [42]. Diestel and Kühn [10][5] first generalised the definition for arbitrary graphs D.

We measure the 'complexity' of a graph decomposition D by the following parameters. The *width* of D is the maximum cardinality of a bag. The *order* of D is the number of bags. The *degree* of D is the maximum degree of the graph D. The decomposition D is *planar* if the graph D is planar.

Diestel and Kühn [10] observed that decompositions generalise minors in the following sense.

Lemma 1 ([10]). *A graph G is a minor of a graph D if and only if a graph isomorphic to D is a decomposition of G with width* 1*.*

Wood and Telle [51] describe a number of tools for manipulating decompositions, such as the following lemma for composing two decompositions.

Lemma 2 ([51]). *Suppose that D is a decomposition of a graph G with width k, and that J is a decomposition of D with width ℓ. Then G has a decomposition isomorphic to J with width $k\ell$.*

Lemma 2 has the following special case, which follows from Lemma 1.

Lemma 3. *If a graph G_1 is a minor of a graph G_2, and J is a decomposition of G_2 with width ℓ, then some graph isomorphic to J is a decomposition of G_1 with width ℓ.*

The next tool by Wood and Telle [51] reduces the order of a planar decomposition at the expense of increasing the width.

Lemma 4 ([51]). *Suppose that a graph G has a planar decomposition D of width k and order at most $c|G|$ for some $c \geq 1$. Then G has a planar decomposition of width $c'k$ and order $|G|$, for some c' depending only on c.*

Converse to Lemma 4, we now show that the width and degree of a planar decomposition can be reduced at the expense of increasing the order.

Lemma 5. *If a graph G has a planar decomposition D of width k, then G has:*
(a) *a planar decomposition D_1 of width k, order $|D_1| < 6|D|$ and degree $\Delta(D_1) \leq 3$,*
(b) *a planar decomposition D_2 of width* 2*, order $|D_2| < 3k(k+1)|D|$ and degree $\Delta(D_2) \leq 4$,*
(c) *a planar decomposition D_3 of width* 2*, order $|D_3| < 6k^2|D|$ and degree $\Delta(D_3) \leq 3$.*

[4] Let A and B be subgraphs of a graph G. Then A and B *intersect* if $V(A) \cap V(B) \neq \emptyset$, and A and B *touch* if they intersect or $v \in V(A)$ and $w \in V(B)$ for some edge vw of G.

[5] A decomposition was called a *connected decomposition* by Diestel and Kühn [10].

Proof. For the clarity of presentation, we assume that all the bags of D have width k, although this assumption is not used in the proof. We also assume that D has minimum degree at least 3; the reader can easily adapt the construction to vertices of degrees 1 and 2. (Alternatively, we can augment D to have minimum degree 3 by adding new edges, whenever D has at least 4 vertices.) Fix an embedding of D in the plane.

First we prove (a). Let D_1 be the graph with two vertices X_e and Y_e for every edge $e = XY \in E(D)$, where each bag X_e is a copy of X. We say that X_e *belongs* to X. Add the edge X_eY_e to D_1 for each edge $e = XY \in E(D)$. Add the edge X_eX_f to D_1 whenever the edges e and f are consecutive in the cyclic order of edges incident to a bag X in D.

As illustrated in Figure 1(b), each bag X is thus replaced by a cycle in D_1, each vertex of which has one more incident edge in D_1. Thus D_1 is a planar graph with maximum degree 3 and order $|D_1| = 2\|D\|$ (after adding edges to D, if necessary). Since D is planar, $\|D\| \leq 3|D| - 6$ and so $|D_1| \leq 6|D| - 12$. Since the set of bags of D_1 that belong to a specific bag of D induces a connected (cycle) subgraph of D_1, and $D(v)$ is a connected subgraph of D for each vertex v of G, $D_1(v)$ is a connected subgraph of D_1.

We now prove that $D_1(v)$ and $D_1(w)$ touch for each edge vw of G. If v and w are in a common bag X of D, then v and w are in every bag X_e of D_1. Otherwise, $v \in X$ and $w \in Y$ for some edge $e = XY$ of D, in which case $v \in X_e$, $w \in Y_e$, and X_eY_e is an edge of D_1. Thus $D_1(v)$ and $D_1(w)$ touch. Therefore D_1 is a planar decomposition of G. This completes the proof of (a).

Now we prove (b). Fix an arbitrary linear order $\preceq$ on $V(G)$, and arbitrarily orient the edges of D. For each arc $e = \overrightarrow{XY}$ of D, orient the edge X_eY_e of D_1 from X_e to Y_e.

Informally speaking, we now construct a planar decomposition D_2 from D_1 by replacing each bag X_e of D_1 by a set of $\binom{k+1}{2}$ bags, each of width 1 or 2, that form a wedge pattern, as illustrated in Figure 1(c). Depending on whether e is incoming or outgoing at X, the wedge is reflected appropriately to ensure the planarity of D_2.

Now we define D_2 formally. Consider a bag $X = \{v_1, v_2, \dots, v_k\}$ of D, where $v_1 \prec v_2 \prec \cdots \prec v_k$. For each pair of vertices v_i, v_j in X, and for each edge e incident to X, add a bag labelled $\{v_i, v_j\}_{X_e}$ to D_2, where $\{v_i, v_j\}_{X_e}$ is a copy of $\{v_i, v_j\}$. (The bag $\{v_i, v_i\}_{X_e}$ is a singleton $\{v_i\}$.) We say that $\{v_i, v_j\}_{X_e}$ *belongs* to X_e and to X. Thus there are $\binom{k+1}{2}$ bags that belong to each bag of D_1. Hence $|D_2| \leq \binom{k+1}{2}|D_1| < 3k(k+1)|D|$. Add an edge in D_2 between the bags $\{v_i, v_j\}_{X_e}$ and $\{v_i, v_{j+1}\}_{X_e}$ for $1 \leq i \leq k$ and $1 \leq j \leq k-1$. As illustrated in Figure 1(c), the subgraph of D_2 induced by the bags that belong to each bag X_e of D_1 form a planar grid-like graph.

Consider two edges $e = XY$ and $f = XZ$ of D that are consecutive in the cyclic order of edges incident to a bag X of D (defined by the planar embedding). Without loss of generality, XZ is clockwise from XY. We now add edges to D_2 between certain bags that belong to X_e and X_f depending on the orientations of the edges XY and XZ. Since D has minimum degree at least 3, the bags corresponding to X form a cycle in D_2. (For D-vertices of degree less than 3, the construction is slightly different; we leave the details of $\deg_D(v) \leq 2$ to the reader.) For $1 \leq i \leq k$, let P_i be the bag $\{v_i, v_k\}_{X_e}$ if $\overrightarrow{XY}$ and $\{v_i, v_i\}_{X_e}$ if $\overrightarrow{YX}$, and let Q_i be the bag $\{v_i, v_i\}_{X_f}$ if $\overrightarrow{XZ}$ and $\{v_i, v_k\}_{X_f}$ if $\overrightarrow{ZX}$. Add an edge between P_i and Q_i for $1 \leq i \leq k$. As illustrated in Figure 1(c), the subgraph of D_2

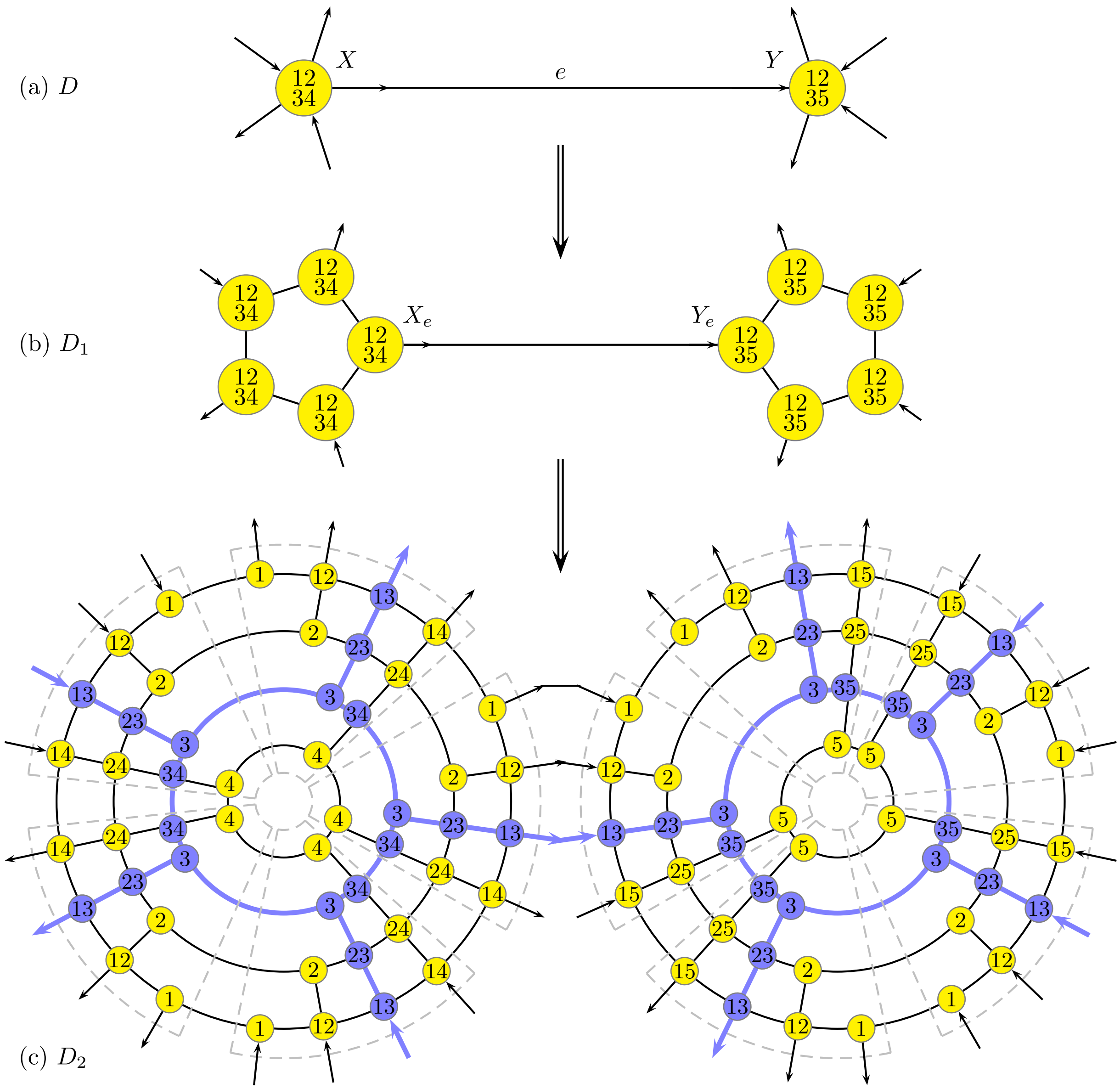

Figure 1: (a) The planar decomposition D. (b) The planar decomposition D_1 obtained from D by replacing each bag of degree d by d bags of degree 3. (c) The planar decomposition D_2 obtained from D_1 by replacing each bag of width k by $\binom{k+1}{2}$ bags of width 2. The subgraph $D_2(3)$ is highlighted.

induced by the bags that belong to each bag X of D is planar.

Now consider an edge $e = \overrightarrow{XY}$ of D, where $X = \{v_1, v_2, \dots, v_k\}$ with $v_1 \prec v_2 \prec \dots \prec v_k$, and $Y = \{w_1, w_2, \dots, w_k\}$ with $w_1 \prec w_2 \prec \dots \prec w_k$. Whenever $v_i = w_j$, add an edge between $\{v_1, v_i\}_{X_e}$ and $\{w_1, w_j\}_{Y_e}$ to D_2. This completes the construction of D_2. Observe that the bags $\{v_1, v_1\}_{X_e}, \{v_1, v_2\}_{X_e}, \dots, \{v_1, v_k\}_{X_e}$ are ordered clockwise on the outer face of the subgraph of D_2 induced by the bags belonging to X. Similarly, the bags $\{w_1, w_1\}_{Y_e}, \{w_1, w_2\}_{Y_e}, \dots, \{w_1, w_k\}_{Y_e}$ are ordered anticlockwise on the outer face of the subgraph of D_2 induced by the bags belonging to Y. Thus these edges do not introduce any crossings in D_2, as illustrated in Figure 1(c).

We now prove that each subgraph $D_2(v)$ is a nonempty connected subgraph of D_2 for each vertex v of G. Say v is in a bag $X = \{v_1, v_2, \dots, v_k\}$ of D, with $v_1 \prec v_2 \prec \dots \prec v_k$ and $v = v_i$. Observe that the set of bags $\{\{v_i, v_j\}_{X_e} : v_j \in X \in e \in E(D), i \leq j\}$ forms a cycle in D_2 (drawn as a circle in Figure 1(c)), and for each edge e incident to X, the bags $\{\{v_i, v_j\}_{X_e} : v_j \in X \in e \in E(D), j \leq i\}$ form a path between $\{v_1, v_i\}_{X_e}$ and $\{v_i, v_i\}_{X_e}$, where it attaches to this cycle. Thus the set of bags in D_2 that belong to X and contain v forms a connected subgraph of D_2. For each edge $e = XY$ of D with $v \in X \cap Y$, there is an edge in D_2 (between some bags $\{v_1, v_i\}_{X_e}$ and $\{w_1, w_j\}_{Y_e}$) that connects the set of bags that belong to X and contain v with the set of bags that belong to Y and contain v. Thus $D_2(v)$ is connected since $D(v)$ is connected.

We now prove that $D_2(v)$ and $D_2(w)$ touch for each edge vw of G. If v and w are in a common bag X of D, then v and w are in every bag $\{v, w\}_{X_e}$ of D_1. Otherwise, $v \in X$ and $w \in Y$ for some edge $e = XY$ of D, in which case v and w are in adjacent bags $\{v_1, v\}_{X_e}$ and $\{w_1, w\}_{Y_e}$, for appropriate vertices v_1 and w_1. Thus $D_2(v)$ and $D_2(w)$ touch. Therefore D_2 is a decomposition of G. Observe that $\Delta(D_2) \leq 4$. This completes the proof of (b).

Now we prove (c). Construct a planar decomposition D_3 from D_2 by the following operation applied to each bag W of D_2 with degree 4. Say the neighbours of W are Z_1, Z_2, Z_3, Z_4 in clockwise order in the embedding of D_2. Replace W by two bags W_1 and W_2, both copies of W, where W_1 is adjacent to W_2, Z_1, Z_2, and W_2 is adjacent to W_1, Z_3, Z_4. Clearly D_3 is a planar decomposition of G with maximum degree 3. For each bag X_e of D_1, there are k bags of degree 3 in D_2 and $\frac{1}{2}k(k-1)$ bags of degree 4 that belong to X_e. Since each bag of degree 4 in D_2 is replaced by two bags in D_3, there are $k + 2(\frac{1}{2}k(k-1)) = k^2$ bags in D_3 that belong to X_e. Thus $|D_3| \leq 2k^2\|D\| < 6k^2|D|$. This completes the proof of (c). $\square$

Note that the upper bound of $|D_1| \leq 6|D|$ in Lemma 5(a) can be improved to $|D_1| \leq 4|D|$ by replacing each bag of degree d by $d-2$ bags of degree 3, as illustrated in Figure 2. We omit the details.

3 Planar Decompositions and Crossing Number

In this section we review some of the results by Wood and Telle [51] that link planar decompositions and crossing number.

Lemma 6 ([51]). *If D is a planar decomposition of a graph G with width k, then G has crossing number*

$$\mathsf{cr}(G) \;\leq\; k(k+1)\,\Delta(G)^2\,|D| \;.$$

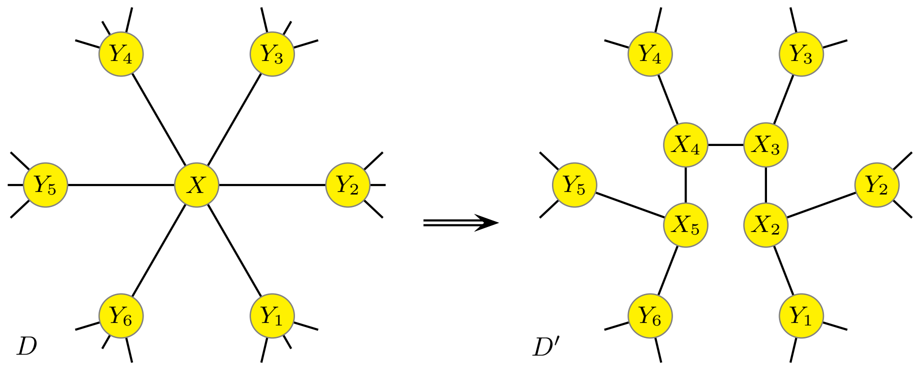

Figure 2: Replacing a bag of degree 6 by four bags of degree 3.

Lemma 7 ([51]). *For every graph H there is an integer $k = k(H)$, such that every H-minor-free graph G has a planar decomposition of width k and order $|G|$.*

Observe that Lemmas 6 and 7 imply Theorem 1. The next lemma is converse to Lemma 6.

Lemma 8 ([51]). *Every graph G has a planar decomposition of width 2 and order $|G| + \mathsf{cr}(G)$.*

We have the following characterisation of graphs with linear crossing number.

Theorem 3 ([51]). *The following are equivalent for a graph G of bounded degree:*

1. $\mathsf{cr}(G) \leq c_1|G|$ *for some constant* c_1,
2. G *has a planar decomposition with width* c_2 *and order* $|G|$ *for some constant* c_2,
3. G *has a planar decomposition with width* 2 *and order* $c_3|G|$ *for some constant* c_3.

Proof. Lemma 8 implies that (1) $\Rightarrow$ (3). Lemma 4 implies that (3) $\Rightarrow$ (2). Lemma 6 implies that (2) $\Rightarrow$ (1). $\square$

Note that Lemma 5(c) provides a more direct proof that (2) $\Rightarrow$ (3) in Theorem 3 (without the dependence on degree).

4 Planar Decompositions and Minor Crossing Number

Lemma 6 can be extended to give the following upper bound on the minor crossing number. Basically we replace the dependence on $\Delta(G)$ in Lemma 6 by $\Delta(D)$.

Lemma 9. *If D is a planar decomposition of a graph G with width k, then G has minor crossing number*

$$\mathsf{mcr}(G) \ < \ k^3(k+1)(\Delta(D)+1)^2\,|D| \ .$$

Proof. Let G' be the graph with one vertex for each occurrence of a vertex of G in a bag of D. Consider a vertex x of G' in bag X and a distinct vertex y of G' in bag Y. Connect x and y by an edge in G' if and only if $X = Y$ or XY is an edge of D. (G' is a subgraph of the lexicographic product $D[K_k]$.) For each vertex v of G, the copies of v form a connected subgraph of G', since $D(v)$ is a connected subgraph of D. Since $D(v)$ and $D(w)$ touch for each edge vw of G, some copy of v is adjacent to some copy of w. Thus G is a minor of G', and $\mathsf{mcr}(G) \leq \mathsf{cr}(G')$. Moreover, D defines a planar decomposition of G' with width k. By Lemma 6 applied to G',

$$\mathsf{mcr}(G) \ \leq \ \mathsf{cr}(G') \ \leq \ k(k+1)\,\Delta(G')^2\,|D| \ .$$

A neighbour of a vertex x of G' is in the same bag as x or is in a neighbouring bag. Thus $\Delta(G') \leq (\Delta(D)+1)k - 1$. Thus

$$\mathsf{mcr}(G) \ < \ k(k+1)\,((\Delta(D)+1)k)^2\,|D| \ = \ k^3(k+1)\,(\Delta(D)+1)^2\,|D| \ .$$

□

Lemmas 9 and 5(a) imply that if D is a planar decomposition of a graph G with width k, then G has minor crossing number in $\mathcal{O}(k^4|D|)$. This bound can be improved by further transforming the decomposition into a decomposition with width 2. In particular, Lemmas 9 and 5(b) imply:

Lemma 10. *If D is a planar decomposition of a graph G with width k, then G has minor crossing number*

$$\mathsf{mcr}(G) \ < \ 2^3(2+1)(4+1)^2\,3k(k+1)|D| \ = \ 1800\,k(k+1)|D| \ .$$

Proof of Theorem 2. It follows immediately from Lemmas 7 and 10. □

We now set out to prove a converse result to Theorem 2.

Lemma 11. *For every graph G, there is a graph G' containing G as a minor, such that $\mathsf{mcr}(G) = \mathsf{cr}(G')$ and $|G'| \leq |G| + \mathsf{mcr}(G)$.*

Proof. By definition, there is a graph G' containing G as a minor, such that $\mathsf{mcr}(G) = \mathsf{cr}(G')$. Choose such a graph G' with the minimum number of vertices. There is a set $\{T_v : v \in V(G)\}$ of disjoint subtrees in G', such that for every edge vw of G, some vertex of T_v is adjacent to some vertex of T_w. Every vertex of G' is in some T_v, as otherwise we could delete the vertex from G'. Hence

$$|G'| \ = \sum_{v\in V(G)} |T_v| \ = \ |G| + \sum_{v\in V(G)} (|T_v|-1) \ = \ |G| + \sum_{v\in V(G)} \|T_v\| \ .$$

We can assume that every edge of every subtree T_v is in some crossing, as otherwise we could contract the edge. Thus $|G'| \ \leq \ |G| + \mathsf{cr}(G') \ = \ |G| + \mathsf{mcr}(G)$. □

The next lemma is an analogue of Lemma 8.

Lemma 12. *Every graph G has a planar decomposition with width 2 and order $|G| + 2\,\mathsf{mcr}(G)$.*

Proof. By Lemma 11, there is some graph G' containing G as a minor, such that $\mathsf{cr}(G') = \mathsf{mcr}(G)$ and $|G'| \leq |G| + \mathsf{mcr}(G)$. By Lemma 8, G' has a planar decomposition of width 2 and order $|G'| + \mathsf{cr}(G') = |G'| + \mathsf{mcr}(G) \leq |G| + 2\,\mathsf{mcr}(G)$. By Lemma 3, G has a planar decomposition with the same properties. ☐

We have the following characterisation of graphs with linear minor crossing number, which is analogous to Theorem 3 for crossing number (without the dependence on degree).

Theorem 4. *The following are equivalent for a graph G:*

1. $\mathsf{mcr}(G) \leq c_1|G|$ *for some constant* c_1,
2. G *has a planar decomposition with width* c_2 *and order* $|G|$ *for some constant* c_2,
3. G *has a planar decomposition with width* 2 *and order* $c_3|G|$ *for some constant* c_3.

Proof. Lemma 12 implies (1) $\Rightarrow$ (3). Lemma 4 implies that (3) $\Rightarrow$ (2). Lemma 10 implies that (2) $\Rightarrow$ (1). ☐

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