Two Coefficients of the Dyson Product

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Abstract

In this paper, the closed-form expressions for the coefficients of $\frac{x_r^2}{x_s^2}$ and $\frac{x_r^2}{x_s x_t}$ in the Dyson product are found by applying an extension of Good's idea. As consequences, we find several interesting Dyson style constant term identities.

1 Introduction

For nonnegative integers $a_1, a_2, \dots, a_n$, define

$$D_n(\mathbf{x}, \mathbf{a}) := \prod_{1 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right)^{a_i}, \qquad \text{(Dyson product)}$$

where $\mathbf{x} := (x_1, \dots, x_n)$ and $\mathbf{a} := (a_1, \dots, a_n)$.

Dyson [2] conjectured the following constant term identity in 1962.

Theorem 1.1 (Dyson's Conjecture).

$$\operatorname*{CT}_{\mathbf{x}} D_n(\mathbf{x}, \mathbf{a}) = \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1!\, a_2! \cdots a_n!}.$$

where $\mathrm{CT}_{\mathbf{x}} f(\mathbf{x})$ *means to take the constant term in the* x*'s of the series* $f(\mathbf{x})$.

Dyson's conjecture was first proved independently by Gunson [5] and by Wilson [10]. Later an elegant recursive proof was published by Good [4], and a combinatorial proof was given by Zeilberger [11]. Andrews [1] conjectured the q-analog of the Dyson conjecture which was first proved, combinatorially, by Zeilberger and Bressoud [12] in 1985. Recently, Gessel and Xin [3] gave a very different proof by using properties of formal Laurent series and of polynomials.

Good's idea has been extended by several authors. The current interest is to evaluate the coefficients of monomials $M := \prod_{i=0}^{n} x_i^{b_i}$, where $\sum_{i=0}^{n} b_i = 0$, in the Dyson product. Kadell [6] outlined the use of Good's idea for M to be $\frac{x_1}{x_n}$, $\frac{x_1 x_2}{x_{n-1} x_n}$ and $\frac{x_1 x_2}{x_n^2}$. Along this line, Zeilberger and Sills [9] presented a case study in experimental yet rigorous mathematics by describing an algorithm that automatically conjectures and proves closed-form. Using this algorithm, Sills [8] guessed and proved closed-form expressions for M to be $\frac{x_s}{x_r}$, $\frac{x_s x_t}{x_r^2}$ and $\frac{x_t x_u}{x_r x_s}$. These results and their q-analogs were recently generalized for M with a square free numerator by Lv, Xin and Zhou [7] by extending Gessel-Xin's Laurent series method [3] for proving the q-Dyson Theorem.

The cases for M having a square in the numerator are much more complicated. By extending Good's idea, we obtain closed forms for the simplest cases $M = \frac{x_r^2}{x_s^2}$ and $M = \frac{x_r^2}{x_s x_t}$. In doing so, we guess these two formulas simultaneously, written as a sum instead of a single product. Our main results are stated as follows.

Theorem 1.2. *Let r and s be distinct integers with $1 \le r, s \le n$. Then*

$$\operatorname*{CT}_{\mathbf{x}} \frac{x_s^2}{x_r^2} D_n(\mathbf{x}, \mathbf{a}) = \frac{a_r}{(1+a^{(r)})(2+a^{(r)})} \left[(a_r - 1) - \sum_{\substack{i=1 \\ i \neq r,s}}^{n} \frac{a_i(1+a)}{(1+a^{(r)}-a_i)} \right] C_n(\mathbf{a}), \tag{1.1}$$

where $a := a_1 + a_2 + \cdots + a_n$, $a^{(j)} := a - a_j$ and $C_n(\mathbf{a}) := \frac{(a_1+a_2+\cdots+a_n)!}{a_1!\, a_2! \cdots a_n!}$.

Theorem 1.3. *Let r, s and t be distinct integers with $1 \le r, s, t \le n$. Then*

$$\operatorname*{CT}_{\mathbf{x}} \frac{x_s x_t}{x_r^2} D_n(\mathbf{x}, \mathbf{a}) = \frac{a_r}{(1+a^{(r)})(2+a^{(r)})} \left[(a + a_r) - \sum_{\substack{i=1 \\ i \neq r,s,t}}^{n} \frac{a_i(1+a)}{(1+a^{(r)}-a_i)} \right] C_n(\mathbf{a}), \tag{1.2}$$

where $a, a^{(r)}$ and $C_n(\mathbf{a})$ are defined as Theorem 1.2.

The proofs will be given in Section 2. In Section 3, we construct several interesting Dyson style constant term identities.

2 Proof of Theorem 1.2 and Theorem 1.3

Good's proof [4] of the Dyson conjecture uses the recurrence

$$D_n(\mathbf{x}, \mathbf{a}) = \sum_{k=1}^{n} D_n(\mathbf{x}, \mathbf{a} - \mathbf{e}_k),$$

where $\mathbf{e}_k := (0, \ldots, 0, 1, 0, \ldots, 0)$ is the kth unit coordinate n-vector. It follows that the following recurrence holds for any monomial M of degree 0.

$$\operatorname*{CT}_{\mathbf{x}} \frac{1}{M} D_n(\mathbf{x}, \mathbf{a}) = \sum_{k=1}^{n} \operatorname*{CT}_{\mathbf{x}} \frac{1}{M} D_n(\mathbf{x}, \mathbf{a} - \mathbf{e}_k).$$

Thus if we can guess a formula, then we can prove it by checking the initial condition, the recurrence and the boundary conditions. This is the so called Good-style proof.

Our basic tool for guessing is Zeilberger and Sills' Maple package *GoodDyson*. For the cases $M = x_r^2/x_s^2$ and $M = x_r^2/(x_s x_t)$, the package can guess the formulas for $n = 2, 3, 4$, but not for $n \geq 5$. However, the results seem chaotic. Surprisingly, the formulas become nice when converted into partial fractions (by Maple). This leads us to come up with Theorems 1.2 and 1.3.

To prove our theorems, we denote by $F_L(r, s, \mathbf{a})$ (resp. $G_L(r, s, t, \mathbf{a})$) the left-hand side of (1.1) (resp. (1.2)), and by $F_R(r, s, \mathbf{a})$ (resp. $G_R(r, s, t, \mathbf{a})$) the right-hand side of (1.1) (resp. (1.2)). Without loss of generality, we may assume $r = 1, s = 2$ and $t = 3$ in Theorems 1.2 and 1.3, i.e., we need to prove that

$$F_L(\mathbf{a}) = F_R(\mathbf{a}), \qquad G_L(\mathbf{a}) = G_R(\mathbf{a}),$$

where $F_L(\mathbf{a}) := F(1, 2, \mathbf{a})$ and we use similar notations for $F_R(\mathbf{a}), G_L(\mathbf{a})$ and $G_R(\mathbf{a})$.

2.1 Initial Condition

We can easily verify that

$$F_L(\mathbf{0}) = F_R(\mathbf{0}) = 0, \qquad G_L(\mathbf{0}) = G_R(\mathbf{0}) = 0.$$

2.2 Recurrence

We need to show that $F_R(\mathbf{a})$ and $G_R(\mathbf{a})$ satisfy the recurrences

$$F_R(\mathbf{a}) = \sum_{k=1}^{n} F_R(\mathbf{a} - \mathbf{e}_k), \tag{2.1}$$

$$G_R(\mathbf{a}) = \sum_{k=1}^{n} G_R(\mathbf{a} - \mathbf{e}_k). \tag{2.2}$$

In order to do so, we define

$$\begin{aligned}
H_1(\mathbf{a}) &:= \frac{a_1(a_1 - 1)}{(1 + a^{(1)})(2 + a^{(1)})} C_n(\mathbf{a}), \\
H_2(\mathbf{a}) &:= \frac{a_1(a + a_1)}{(1 + a^{(1)})(2 + a^{(1)})} C_n(\mathbf{a}), \\
H_i(\mathbf{a}) &:= \frac{a_1 a_i (1 + a)}{(1 + a^{(1)})(2 + a^{(1)})(1 + a^{(1)} - a_i)} C_n(\mathbf{a}), \quad i = 3, 4, \ldots, n.
\end{aligned}$$

Then $F_R(\mathbf{a}) = H_1(\mathbf{a}) + \sum_{i=3}^{n} H_i(\mathbf{a})$ and $G_R(\mathbf{a}) = H_2(\mathbf{a}) + \sum_{i=4}^{n} H_i(\mathbf{a})$. Therefore to prove (2.1) and (2.2), it suffices to show the following:

Lemma 2.1. *For each $i = 1, 2, \ldots, n$, we have the recurrence $H_i(\mathbf{a}) = \sum_{k=1}^{n} H_i(\mathbf{a} - \mathbf{e}_k)$.*

Proof. 1. For $H_1(\mathbf{a})$,

$$\begin{aligned}
\sum_{k=1}^{n} H_1(\mathbf{a}-\mathbf{e}_k) &= \frac{(a_1-1)(a_1-2)}{(1+a^{(1)})(2+a^{(1)})}C_n(\mathbf{a}-\mathbf{e}_1) + \sum_{k=2}^{n} \frac{a_1(a_1-1)}{a^{(1)}(1+a^{(1)})}C_n(\mathbf{a}-\mathbf{e}_k) \\
&= \left[\frac{a_1(a_1-1)(a_1-2)}{a(1+a^{(1)})(2+a^{(1)})} + \sum_{k=2}^{n} \frac{a_k a_1(a_1-1)}{aa^{(1)}(1+a^{(1)})}\right]C_n(\mathbf{a}) \\
&= \left[\frac{a_1(a_1-1)(a_1-2)}{a(1+a^{(1)})(2+a^{(1)})} + \frac{a_1(a_1-1)}{a(1+a^{(1)})}\right]C_n(\mathbf{a}) \\
&= \frac{a_1(a_1-1)}{(1+a^{(1)})(2+a^{(1)})}C_n(\mathbf{a}) = H_1(\mathbf{a}).
\end{aligned}$$

2. For $H_2(\mathbf{a})$,

$$\begin{aligned}
\sum_{k=1}^{n} H_2(\mathbf{a}-\mathbf{e}_k) &= \frac{(a_1-1)(a+a_1-2)}{(1+a^{(1)})(2+a^{(1)})}C_n(\mathbf{a}-\mathbf{e}_1) + \sum_{k=2}^{n} \frac{a_1(a+a_1-1)}{a^{(1)}(1+a^{(1)})}C_n(\mathbf{a}-\mathbf{e}_k) \\
&= \left[\frac{a_1(a_1-1)(a+a_1-2)}{a(1+a^{(1)})(2+a^{(1)})} + \sum_{k=2}^{n} \frac{a_k a_1(a+a_1-1)}{aa^{(1)}(1+a^{(1)})}\right]C_n(\mathbf{a}) \\
&= \left[\frac{a_1(a_1-1)(a+a_1-2)}{a(1+a^{(1)})(2+a^{(1)})} + \frac{a_1(a+a_1-1)}{a(1+a^{(1)})}\right]C_n(\mathbf{a}) \\
&= \frac{a_1(a+a_1)}{(1+a^{(1)})(2+a^{(1)})}C_n(\mathbf{a}) = H_2(\mathbf{a}).
\end{aligned}$$

3. For $H_i(\mathbf{a})$ with $i = 3, \ldots, n$, without loss of generality, we may assume $i = 3$.

$$\begin{aligned}
&\sum_{k=1}^{n} H_3(\mathbf{a}-\mathbf{e}_k) \\
=&\frac{aa_3(a_1-1)}{(1+a^{(1)})(2+a^{(1)})(1+a^{(1)}-a_3)}C_n(\mathbf{a}-\mathbf{e}_1) + \frac{aa_1a_3}{a^{(1)}(1+a^{(1)})(a^{(1)}-a_3)}C_n(\mathbf{a}-\mathbf{e}_2) \\
&+ \frac{aa_1(a_3-1)}{a^{(1)}(1+a^{(1)})(1+a^{(1)}-a_3)}C_n(\mathbf{a}-\mathbf{e}_3) + \sum_{k=4}^{n} \frac{aa_1a_3}{a^{(1)}(1+a^{(1)})(a^{(1)}-a_3)}C_n(\mathbf{a}-\mathbf{e}_k) \\
=&\frac{a_1a_3(a_1-1)}{(1+a^{(1)})(2+a^{(1)})(1+a^{(1)}-a_3)}C_n(\mathbf{a}) + \frac{a_1a_2a_3}{a^{(1)}(1+a^{(1)})(a^{(1)}-a_3)}C_n(\mathbf{a}) \\
&+ \frac{a_1a_3(a_3-1)}{a^{(1)}(1+a^{(1)})(1+a^{(1)}-a_3)}C_n(\mathbf{a}) + \frac{a_1a_3(a-a_1-a_2-a_3)}{a^{(1)}(1+a^{(1)})(a^{(1)}-a_3)}C_n(\mathbf{a}) \\
=&\frac{a_1a_3}{(1+a^{(1)})(1+a^{(1)}-a_3)}\left[\frac{a_1-1}{2+a^{(1)}} + \frac{a_3-1}{a^{(1)}} + \frac{(a-a_1-a_3)(1+a^{(1)}-a_3)}{a^{(1)}(a^{(1)}-a_3)}\right]C_n(\mathbf{a}) \\
=&\frac{a_1a_3(1+a)}{(1+a^{(1)})(2+a^{(1)})(1+a^{(1)}-a_3)}C_n(\mathbf{a}) = H_3(\mathbf{a}).
\end{aligned}$$

This completes the proof. ∎

2.3 Boundary Conditions

Now we consider the boundary conditions. For any k with $1 \le k \le n$,

$$D_n\big(\mathbf{x}, (a_1, \dots, a_{k-1}, 0, a_{k+1}, \dots, a_n)\big) = D_{n-1}(\mathbf{x}^{\langle k \rangle}, \mathbf{a}^{\langle k \rangle}) \times \prod_{\substack{i=1\\ i\neq k}}^{n} \left(1 - \frac{x_i}{x_k}\right)^{a_i},$$

where $\mathbf{x}^{\langle k \rangle} := (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$. Thus we have

$$\operatorname*{CT}_{\mathbf{x}} \ \frac{x_2^2}{x_1^2} D_n\big(\mathbf{x}, (a_1, \dots, a_{k-1}, 0, a_{k+1}, \dots, a_n)\big) = \operatorname*{CT}_{\mathbf{x}^{\langle k \rangle}} P_k \cdot D_{n-1}(\mathbf{x}^{\langle k \rangle}, \mathbf{a}^{\langle k \rangle}), \tag{2.3}$$

where P_k is given by

$$\begin{aligned} P_k : &= \operatorname*{CT}_{x_k} \frac{x_2^2}{x_1^2} \prod_{\substack{i=1\\ i\neq k}}^{n} \left(1 - \frac{x_i}{x_k}\right)^{a_i} \\ &= \begin{cases} 0, & k=1; \\ \binom{a_1}{2} + a_1 \sum_{i=3}^n a_i \frac{x_i}{x_1} + \sum_{i=3}^n \binom{a_i}{2} \frac{x_i^2}{x_1^2} + \sum\limits_{3\le i<j\le n} a_i a_j \frac{x_i x_j}{x_1^2}, & k=2; \\ \frac{x_2^2}{x_1^2}, & \text{otherwise.} \end{cases} \end{aligned}$$

Taking the constant term in the x's of (2.3), we obtain

$$\begin{aligned} &F_L(a_1, \dots, a_{k-1}, 0, a_{k+1}, \dots, a_n) \\ &\quad = \begin{cases} 0, & k=1; \\ \operatorname*{CT}_{\mathbf{x}^{\langle 2 \rangle}} \left(\binom{a_1}{2} + a_1 \sum\limits_{i=3}^n a_i \frac{x_i}{x_1} + \sum\limits_{i=3}^n \binom{a_i}{2} \frac{x_i^2}{x_1^2} + \sum\limits_{3\le i<j\le n} a_i a_j \frac{x_i x_j}{x_1^2} \right) D_{n-1}(\mathbf{x}^{\langle 2 \rangle}, \mathbf{a}^{\langle 2 \rangle}), & k=2; \\ \operatorname*{CT}_{\mathbf{x}^{\langle k \rangle}} \frac{x_2^2}{x_1^2} D_{n-1}(\mathbf{x}^{\langle k \rangle}, \mathbf{a}^{\langle k \rangle}), & \text{otherwise.} \end{cases} \end{aligned}$$

By Theorem 1.1 and [8, Theorem 1.1], we have

$$\begin{aligned} \operatorname*{CT}_{\mathbf{x}^{\langle 2 \rangle}} \binom{a_1}{2} D_{n-1}(\mathbf{x}^{\langle 2 \rangle}, \mathbf{a}^{\langle 2 \rangle}) &= \frac{a_1(a_1-1)}{2} C_{n-1}(\mathbf{a}^{\langle 2 \rangle}), \\ \operatorname*{CT}_{\mathbf{x}^{\langle 2 \rangle}} a_1 \sum_{i=3}^n a_i \frac{x_i}{x_1} D_{n-1}(\mathbf{x}^{\langle 2 \rangle}, \mathbf{a}^{\langle 2 \rangle}) &= -\frac{a_1^2\ (a^{(1)} - a_2)}{1 + a^{(1)} - a_2} C_{n-1}(\mathbf{a}^{\langle 2 \rangle}). \end{aligned}$$

So we obtain the following boundary conditions (also recurrences)

$$\begin{aligned} &F_L(a_1, \dots, a_{k-1}, 0, a_{k+1}, \dots, a_n) \\ &\quad = \begin{cases} 0, & k=1; \\ \left(\frac{a_1(a_1-1)}{2} - \frac{a_1^2\ a^{(1)}}{1+a^{(1)}} \right) C_{n-1}(\mathbf{a}^{\langle 2 \rangle}) & \\ \qquad + \sum\limits_{i=3}^n \binom{a_i}{2} F_L(1, i, \mathbf{a}^{\langle 2 \rangle}) + \sum\limits_{3\le i<j\le n} a_i a_j G_L(1, i, j, \mathbf{a}^{\langle 2 \rangle}), & k=2; \\ F_L(\mathbf{a}^{\langle k \rangle}), & \text{otherwise.} \end{cases} \end{aligned}$$

We need to show that $F_R(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)$ satisfies the same boundary conditions, i.e., the boundary conditions by replacing all F_L by F_R and all G_L by G_R:

$$F_R(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n) \\ = \begin{cases} 0, & k=1; \\ \left(\frac{a_1(a_1-1)}{2} - \frac{a_1^2 \ a^{(1)}}{1+a^{(1)}}\right) C_{n-1}(\mathbf{a}^{\langle 2 \rangle}) & \\ \qquad + \sum\limits_{i=3}^{n} \binom{a_i}{2} F_R(1, i, \mathbf{a}^{\langle 2 \rangle}) + \sum\limits_{3 \le i < j \le n} a_i a_j G_R(1, i, j, \mathbf{a}^{\langle 2 \rangle}), & k=2; \\ F_R(\mathbf{a}^{\langle k \rangle}), & \text{otherwise.} \end{cases} \tag{2.4}$$

For $G_L(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)$, similar computation for $\frac{1}{M} = \frac{x_2 x_3}{x_1^2}$ yields the boundary conditions:

$$G_L(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n) \\ = \begin{cases} 0, & k=1; \\ \frac{a_1^2}{1+a^{(1)}} C_{n-1}(\mathbf{a}^{\langle 2 \rangle}) - a_3 F_L(1, 3, \mathbf{a}^{\langle 2 \rangle}) - \sum_{i=4}^{n} a_i G_L(1, 3, i, \mathbf{a}^{\langle 2 \rangle}), & k=2; \\ \frac{a_1^2}{1+a^{(1)}} C_{n-1}(\mathbf{a}^{\langle 3 \rangle}) - a_2 F_L(1, 2, \mathbf{a}^{\langle 3 \rangle}) - \sum_{i=4}^{n} a_i G_L(1, 2, i, \mathbf{a}^{\langle 3 \rangle}), & k=3; \\ G_L(\mathbf{a}^{\langle k \rangle}), & \text{otherwise,} \end{cases}$$

so we need to prove that $G_R(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)$ satisfies the following boundary conditions:

$$G_R(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n) \\ = \begin{cases} 0, & k=1; \\ \frac{a_1^2}{1+a^{(1)}} C_{n-1}(\mathbf{a}^{\langle 2 \rangle}) - a_3 F_R(1, 3, \mathbf{a}^{\langle 2 \rangle}) - \sum_{i=4}^{n} a_i G_R(1, 3, i, \mathbf{a}^{\langle 2 \rangle}), & k=2; \\ \frac{a_1^2}{1+a^{(1)}} C_{n-1}(\mathbf{a}^{\langle 3 \rangle}) - a_2 F_R(1, 2, \mathbf{a}^{\langle 3 \rangle}) - \sum_{i=4}^{n} a_i G_R(1, 2, i, \mathbf{a}^{\langle 3 \rangle}), & k=3; \\ G_R(\mathbf{a}^{\langle k \rangle}), & \text{otherwise.} \end{cases} \tag{2.5}$$

These are summarized by the following lemma.

Lemma 2.2. *If $a_k = 0$ with $k = 1, 2, \ldots, n$, then $F_R(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)$ satisfies the boundary conditions* (2.4) *and $G_R(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)$ satisfies the boundary conditions* (2.5).

Proof. We only prove the first part for brevity and similarity.

Since the cases $k = 1, 3, \ldots, n$ are straightforward, we only prove the case $k = 2$. Note that during the proof of this lemma, we have $a^{(1)} = a_2 + a_3 + \cdots + a_n = a_3 + \cdots + a_n$ because $a_2 = 0$.

Since

$$\begin{aligned}
\sum_{i=3}^{n}\binom{a_i}{2}\sum_{\substack{j=3\\ j\neq i}}^{n}\frac{a_j}{1+a^{(1)}-a_j} &= \sum_{i=3}^{n}\frac{a_i(a_i-1)}{2}\sum_{j=3}^{n}\Big(\frac{a_j}{1+a^{(1)}-a_j}-\frac{a_i}{1+a^{(1)}-a_i}\Big)\\
&= \frac{1}{2}\Big(\sum_{i=3}^{n}(a_i^2-a_i)\sum_{j=3}^{n}\frac{a_j}{1+a^{(1)}-a_j}-\sum_{i=3}^{n}\frac{a_i^3-a_i^2}{1+a^{(1)}-a_i}\Big)\\
&= \frac{1}{2}\Big(\sum_{j=3}^{n}\frac{a_j}{1+a^{(1)}-a_j}\sum_{i=3}^{n}a_i^2-a^{(1)}\sum_{j=3}^{n}\frac{a_j}{1+a^{(1)}-a_j}-\sum_{i=3}^{n}\frac{a_i^3-a_i^2}{1+a^{(1)}-a_i}\Big), \qquad (2.6)
\end{aligned}$$

we have

$$\begin{aligned}
&\sum_{i=3}^{n}\binom{a_i}{2}F_R(1,i,\mathbf{a}^{\langle 2\rangle})\\
&\quad= \frac{a_1}{(1+a^{(1)})(2+a^{(1)})}\sum_{i=3}^{n}\binom{a_i}{2}\Big[(a_1-1)-\sum_{\substack{j=3\\ j\neq i}}^{n}\frac{a_j(1+a)}{1+a^{(1)}-a_j}\Big]C_{n-1}(\mathbf{a}^{\langle 2\rangle})\\
&\quad= -\frac{a_1(1+a)}{2(1+a^{(1)})(2+a^{(1)})}\Big[\sum_{j=3}^{n}\frac{a_j}{1+a^{(1)}-a_j}\sum_{i=3}^{n}a_i^2-a^{(1)}\sum_{j=3}^{n}\frac{a_j}{1+a^{(1)}-a_j}\\
&\qquad -\sum_{i=3}^{n}\frac{a_i^3-a_i^2}{1+a^{(1)}-a_i}\Big]C_{n-1}(\mathbf{a}^{\langle 2\rangle})+\frac{a_1(a_1-1)}{2(1+a^{(1)})(2+a^{(1)})}\Big(\sum_{i=3}^{n}a_i^2-a^{(1)}\Big)C_{n-1}(\mathbf{a}^{\langle 2\rangle}) \quad \text{by (2.6)}\\
&\quad= -\frac{a_1a^{(1)}(a_1-1)}{2(1+a^{(1)})(2+a^{(1)})}C_{n-1}(\mathbf{a}^{\langle 2\rangle})-\lambda\Big[(1+a)\sum_{j=3}^{n}\frac{a_j}{1+a^{(1)}-a_j}\sum_{i=3}^{n}a_i^2\\
&\qquad\qquad -(1+a)\sum_{j=3}^{n}\frac{a_j^3-a_j^2+a_ja^{(1)}}{1+a^{(1)}-a_j}-(a_1-1)\sum_{i=3}^{n}a_i^2\Big], \qquad (2.7)
\end{aligned}$$

where $\lambda := \frac{a_1}{2(1+a^{(1)})(2+a^{(1)})}C_{n-1}(\mathbf{a}^{\langle 2\rangle})$.

Observe that

$$\sum_{3\le i<j\le n}a_ia_j=\frac{1}{2}\Big[(a^{(1)})^2-\sum_{k=3}^{n}a_k^2\Big] \qquad (2.8)$$

and

$$\begin{aligned}
\sum_{3\le i<j\le n}a_ia_j\sum_{\substack{k=3\\ k\neq i,j}}^{n}\frac{a_k}{1+a^{(1)}-a_k} &= \sum_{3\le i<j\le n}a_ia_j\sum_{k=3}^{n}\Big(\frac{a_k}{1+a^{(1)}-a_k}-\frac{a_i}{1+a^{(1)}-a_i}-\frac{a_j}{1+a^{(1)}-a_j}\Big)\\
&= \sum_{3\le i<j\le n}a_ia_j\sum_{k=3}^{n}\frac{a_k}{1+a^{(1)}-a_k}-\sum_{3\le i<j\le n}\frac{a_i^2a_j}{1+a^{(1)}-a_i}-\sum_{3\le i<j\le n}\frac{a_ia_j^2}{1+a^{(1)}-a_j}\\
&= \frac{1}{2}\sum_{k=3}^{n}\frac{a_k}{1+a^{(1)}-a_k}\Big[\big(a^{(1)}\big)^2-\sum_{i=3}^{n}a_i^2\Big]-\sum_{i=3}^{n}\sum_{\substack{j=3\\ j\neq i}}^{n}\frac{a_i^2a_j}{1+a^{(1)}-a_i} \qquad \text{by (2.8)}\\
&= \frac{1}{2}\sum_{k=3}^{n}\frac{a_k}{1+a^{(1)}-a_k}\Big[\big(a^{(1)}\big)^2-\sum_{i=3}^{n}a_i^2\Big]-\sum_{i=3}^{n}\frac{a_i^2(a^{(1)}-a_i)}{1+a^{(1)}-a_i}. \qquad (2.9)
\end{aligned}$$

Thus we obtain that

$$
\begin{aligned}
&\sum_{3\le i<j\le n} a_i a_j G_R(1,i,j,\mathbf{a}^{\langle 2\rangle})\\
&\quad= \frac{a_1}{(1+a^{(1)})(2+a^{(1)})}\sum_{3\le i<j\le n} a_i a_j\Bigg[(a+a_1)-\sum_{\substack{k=3\\ k\ne i,j}}^{n}\frac{a_k(1+a)}{1+a^{(1)}-a_k}\Bigg]C_{n-1}(\mathbf{a}^{\langle 2\rangle})\\
&\quad= \frac{a_1(a+a_1)}{(1+a^{(1)})(2+a^{(1)})}\sum_{3\le i<j\le n} a_i a_j C_{n-1}(\mathbf{a}^{\langle 2\rangle})\\
&\qquad\quad -\frac{a_1(1+a)}{(1+a^{(1)})(2+a^{(1)})}\sum_{3\le i<j\le n} a_i a_j\sum_{\substack{k=3\\ k\ne i,j}}^{n}\frac{a_k}{1+a^{(1)}-a_k}C_{n-1}(\mathbf{a}^{\langle 2\rangle})\\
&\quad= \frac{a_1(a+a_1)}{2(1+a^{(1)})(2+a^{(1)})}\Bigg[(a^{(1)})^2-\sum_{k=3}^{n}a_k^2\Bigg]C_{n-1}(\mathbf{a}^{\langle 2\rangle})+\frac{a_1(1+a)}{2(1+a^{(1)})(2+a^{(1)})}\\
&\qquad \times\Bigg[\sum_{k=3}^{n}\frac{a_k}{1+a^{(1)}-a_k}\Big(\sum_{i=3}^{n}a_i^2-\big(a^{(1)}\big)^2\Big)+2\sum_{i=3}^{n}\frac{a_i^2(a^{(1)}-a_i)}{1+a^{(1)}-a_i}\Bigg]C_{n-1}(\mathbf{a}^{\langle 2\rangle}) \qquad \text{by (2.9)}\\
&\quad= \frac{a_1(a+a_1)(a^{(1)})^2}{2(1+a^{(1)})(2+a^{(1)})}C_{n-1}(\mathbf{a}^{\langle 2\rangle})+\lambda\Bigg[(1+a)\sum_{k=3}^{n}\frac{a_k}{1+a^{(1)}-a_k}\sum_{i=3}^{n}a_i^2\\
&\qquad\qquad -(a+a_1)\sum_{k=3}^{n}a_k^2-(1+a)\sum_{k=3}^{n}\frac{a_k(a^{(1)})^2-2a_k^2a^{(1)}+2a_k^3}{1+a^{(1)}-a_k}\Bigg]. \qquad (2.10)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&(1+a)\sum_{j=3}^{n}\frac{a_j^3-a_j^2+a_ja^{(1)}}{1+a^{(1)}-a_j}+(a_1-1)\sum_{i=3}^{n}a_i^2\\
&\qquad\qquad -(a+a_1)\sum_{k=3}^{n}a_k^2-(1+a)\sum_{k=3}^{n}\frac{a_k(a^{(1)})^2-2a_k^2a^{(1)}+2a_k^3}{1+a^{(1)}-a_k}\\
=&(1+a)\sum_{i=3}^{n}\frac{-a_i^3-a_i^2+a_ia^{(1)}-a_i(a^{(1)})^2+2a_i^2a^{(1)}}{1+a^{(1)}-a_i}-(1+a)\sum_{i=3}^{n}a_i^2\\
=&(1+a)\sum_{i=3}^{n}\frac{(1+a^{(1)}-a_i)(a_i^2+2a_i-a_ia^{(1)})-2a_i}{1+a^{(1)}-a_i}-(1+a)\sum_{i=3}^{n}a_i^2\\
=&(1+a)\sum_{i=3}^{n}(2a_i-a_ia^{(1)})-(1+a)\sum_{i=3}^{n}\frac{2a_i}{1+a^{(1)}-a_i}\\
=&a^{(1)}(1+a)(2-a^{(1)})-(1+a)\sum_{i=3}^{n}\frac{2a_i}{1+a^{(1)}-a_i} \qquad (2.11)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{a_1a^{(1)}(1+a)(2-a^{(1)})}{2(1+a^{(1)})(2+a^{(1)})}-\frac{a_1a^{(1)}(a_1-1)}{2(1+a^{(1)})(2+a^{(1)})}+\frac{a_1(a+a_1)(a^{(1)})^2}{2(1+a^{(1)})(2+a^{(1)})}+\frac{a_1(a_1-1)}{2}-\frac{a_1^2a^{(1)}}{1+a^{(1)}}\\
&=\frac{a_1(a_1-1)}{(1+a^{(1)})(2+a^{(1)})}. \qquad (2.12)
\end{aligned}
$$

Therefore by (2.7), (2.10), (2.11) and (2.12), we have

$$\left[\frac{a_1(a_1-1)}{2} - \frac{a_1^2\ a^{(1)}}{1+a^{(1)}}\right] C_{n-1}(\mathbf{a}^{\langle 2\rangle}) + \sum_{i=3}^{n} \binom{a_i}{2} F_R(1,i,\mathbf{a}^{\langle 2\rangle}) + \sum_{3\le i<j\le n} a_i a_j G_R(1,i,j,\mathbf{a}^{\langle 2\rangle})$$
$$= F_R(a_1, 0, a_3, \dots, a_n).$$

That is to say $F_R(a_1, 0, a_3, \dots, a_n)$ satisfies boundary conditions (2.4). □

2.4 The Proof

Now we can prove Theorems 1.2 and 1.3. Without loss of generality, we may assume $r = 1, s = 2$ and $t = 3$ in Theorems 1.2 and 1.3.

Proof of Theorems 1.2 *and* 1.3. We prove by induction on n for the two theorems simultaneously. Clearly, (1.1) and (1.2) hold when $n = 2, 3$. Assume they hold if n is replaced by $n - 1$. Then for $k = 1, 2, \dots, n$, (1.1) and (1.2) give

$$F_L(r, s, \mathbf{a}^{\langle k\rangle}) = F_R(r, s, \mathbf{a}^{\langle k\rangle}),$$
$$G_L(r, s, t, \mathbf{a}^{\langle k\rangle}) = G_R(r, s, t, \mathbf{a}^{\langle k\rangle}).$$

That is to say $F_L(a_1, \dots, a_{k-1}, 0, a_{k+1}, \dots, a_n)$ and $F_R(a_1, \dots, a_{k-1}, 0, a_{k+1}, \dots, a_n)$ (resp. $G_L(a_1, \dots, a_{k-1}, 0, a_{k+1}, \dots, a_n)$ and $G_R(a_1, \dots, a_{k-1}, 0, a_{k+1}, \dots, a_n)$) satisfy the same boundary conditions. Additionally $F_L(\mathbf{a})$ and $F_R(\mathbf{a})$ (resp. $G_L(\mathbf{a})$ and $G_R(\mathbf{a})$) have the same initial condition and recurrence. It follows that $F_L(\mathbf{a}) = F_R(\mathbf{a})$ (resp. $G_L(\mathbf{a}) = G_R(\mathbf{a})$). □

3 Several Dyson Style Constant Term Identities

By linearly combining Theorems 1.2 and 1.3, we obtain simple formulas.

Proposition 3.1. *Let $r, s, t, u,$ and v be distinct integers in $\{1, 2, \dots, n\}$. Then*

$$\operatorname*{CT}_{\mathbf{x}} \frac{(x_s - x_t)(x_u - x_v)}{x_r^2} D_n(\mathbf{x}, \mathbf{a}) = 0, \tag{3.1}$$

$$\operatorname*{CT}_{\mathbf{x}} \frac{(x_s - x_u)(x_s - x_v)}{x_r^2} D_n(\mathbf{x}, \mathbf{a}) = -\frac{a_r(1+a)}{(2+a^{(r)})(1+a^{(r)}-a_s)} C_n(\mathbf{a}), \tag{3.2}$$

$$\operatorname*{CT}_{\mathbf{x}} \frac{(x_s - x_t)^2}{x_r^2} D_n(\mathbf{x}, \mathbf{a}) = -\frac{a_r(1+a)}{2+a^{(r)}} \sum_{i=s,t} \frac{1}{1+a^{(r)}-a_i} C_n(\mathbf{a}). \tag{3.3}$$

It is worth mentioning that (3.3) follows from (3.1) and (3.2), since

$$(x_s - x_u)(x_s - x_v) + (x_t - x_u)(x_t - x_v) = (x_s - x_t)^2 + (x_s - x_u)(x_t - x_v) + (x_s - x_v)(x_t - x_u).$$

A consequence of Proposition 3.1 is the following:

Corollary 3.2. *Let $I := \{i_1, i_2, \ldots, i_{2m}\}$ be a $2m$-element subset of $\{1, 2, \ldots, n\}$ and let $r \le n$ be a positive integer with $r \notin I$. Then we have*

$$\operatorname*{CT}_{\mathbf{x}} \frac{\left(\sum_{j=1}^{2m}(-1)^j x_{i_j}\right)^2}{x_r^2} D_n(\mathbf{x}, \mathbf{a}) = -\frac{a_r(1+a)}{2+a^{(r)}} \sum_{j\in I} \frac{1}{1+a^{(r)}-a_j} C_n(\mathbf{a}).$$

Proof. Observe that

$$\begin{aligned}\left(\sum_{j=1}^{2m}(-1)^j x_{i_j}\right)^2 &= \Big[(x_{i_2}-x_{i_1}) + (x_{i_4}-x_{i_3}) + \cdots + (x_{i_{2m}}-x_{i_{2m-1}})\Big]^2 \\ &= (x_{i_2}-x_{i_1})^2 + \cdots + (x_{i_{2m}}-x_{i_{2m-1}})^2 + \sum_{k=1}^{m}\sum_{\substack{l=1\\ l\neq k}}^{m}(x_{i_{2k}}-x_{i_{2k-1}})(x_{i_{2l}}-x_{i_{2l-1}}).\end{aligned}$$

The corollary then follows by (3.1) and (3.3). □

Discussions: As we have seen in the proof, we need to guess the formulas of F_R and G_R simultaneously. This is unlike the coefficients for $M = x_s x_t / x_u^2$ and $M = x_s x_t/(x_u x_v)$, which have reasonable product formulas and are equal!

The *cubic cases* are M with $x_r^2 x_s$ or x_r^3 in the numerator. In both cases, we have three sub-cases for the denominator, and need to guess three coefficients simultaneously. The current difficulty is that we can not obtain enough data: the *GoodDyson* package is no longer effective for $n \ge 5$.

Our next project, suggested by the referee, will be to find Zeilberger-style combinatorial proofs as in [11], at least of formula (3.1), and hope such proofs may lead the way for the cubic cases.

The study of the q-analogs of these formulas will follow a completely different route and will not be discussed in this paper.

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