Regularisation and the Mullineux map

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Abstract

We classify the pairs of conjugate partitions whose regularisations are images of each other under the Mullineux map. This classification proves a conjecture of Lyle, answering a question of Bessenrodt, Olsson and Xu.

1 Introduction

Suppose $n \geqslant 0$ and $\mathbb{F}$ is a field of characteristic p; we adopt the convention that the characteristic of a field is the order of its prime subfield. It is well known that the representation theory of the symmetric group $\mathfrak{S}_n$ is closely related to the combinatorics of partitions. In particular, for each partition λ of n, there is an important $\mathbb{F}\mathfrak{S}_n$-module S^λ called the *Specht module*. If $p = \infty$, then the Specht modules are irreducible and afford all irreducible representations of $\mathbb{F}\mathfrak{S}_n$. If p is a prime, then for each p-regular partition λ the Specht module S^λ has an irreducible cosocle D^λ, and the modules D^λ afford all irreducible representations of $\mathbb{F}\mathfrak{S}_n$ as λ ranges over the set of p-regular partitions of n.

Given this set-up, it is natural to express representation-theoretic statements in terms of the combinatorics of partitions. An example of this which is of central interest in this paper is the *Mullineux map*. Let sgn denote the one-dimensional sign representation of $\mathbb{F}\mathfrak{S}_n$. Then there is an involutory functor $- \otimes \mathrm{sgn}$ from the category of $\mathbb{F}\mathfrak{S}_n$-modules to itself. This functor sends simple modules to simple modules, and therefore for each p-regular partition λ there is some p-regular partition $M(\lambda)$ such that $D^\lambda \otimes \mathrm{sgn} \cong D^{M(\lambda)}$.

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The map M thus defined is now called the Mullineux map, since it coincides with a map defined combinatorially by Mullineux [8]; this was proved by Ford and Kleshchev [3], using an alternative combinatorial description of M due to Kleshchev [5].

Another important aspect of the combinatorics of partitions from the point of view of representation theory is *p-regularisation*. This combinatorial procedure was defined by James in order to describe, for each partition λ, a p-regular partition (which is denoted $G\lambda$ in this paper) such that the simple module $D^{G\lambda}$ occurs exactly once as a composition factor of S^λ. In this paper we study the relationship between the Mullineux map and regularisation. Our motivation is the observation that if $p = 2$ or p is large relative to the size of λ, then $MG\lambda = GT\lambda$, where $T\lambda$ denotes the conjugate partition to λ. However, this is not true for arbitrary p, and it natural to ask for which pairs (p, λ) we have $MG\lambda = GT\lambda$. The purpose of this paper is to answer this question, which was first posed by Bessenrodt, Olsson and Xu; the answer confirms a conjecture of Lyle.

If we replace the group algebra $\mathbb{F}\mathfrak{S}_n$ with the Iwahori–Hecke algebra of the symmetric group at a primitive eth root of unity in $\mathbb{F}$ (for some $e \geqslant 2$), then all of the above background holds true, with the prime p replaced by the integer e (and with an appropriate analogue of the sign representation). Therefore, in this paper, we work with an arbitrary integer $e \geqslant 2$ rather than a prime p.

In the remainder of this section we give all the definitions we shall need concerning partitions, and state our main result. Section 2 is devoted to proving one half of the conjecture, and Section 3 to the other half. While the first half of the proof consists of elementary combinatorics, the latter half of the proof is algebraic, being an easy consequence of two theorems about v-decomposition numbers in the Fock space. We introduce the background material for this as we need it.

1.1 Partitions

A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \dots$ and the sum $|\lambda| = \lambda_1 + \lambda_2 + \dots$ is finite. We say that λ is a partition of $|\lambda|$. When writing partitions, we usually group together equal parts and omit zeroes. We write $\varnothing$ for the unique partition of 0.

λ is often identified with its *Young diagram*, which is the subset

$$[\lambda] = \{(i, j) \mid j \leqslant \lambda_i\}$$

of $\mathbb{N}^2$. We refer to elements of $\mathbb{N}^2$ as *nodes*, and to elements of $[\lambda]$ as nodes of λ. We draw the Young diagram as an array of boxes using the English convention, so that i increases down the page and j increases from left to right.

If $e \geqslant 2$ is an integer, we say that λ is *e-regular* if there is no $i \geqslant 1$ such that $\lambda_i = \lambda_{i+e-1} > 0$, and otherwise we say that λ is *e-singular*. We say that λ is *e-restricted* if $\lambda_i - \lambda_{i+1} < e$ for all $i \geqslant 1$.

1.2 Operators on partitions

Here we introduce a variety of operators on partitions. These include regularisation and the Mullineux map, as well as other more familiar operators which will be useful.

1.2.1 Conjugation

Suppose λ is a partition. The *conjugate partition* to λ is the partition $T\lambda$ obtained by reflecting the Young diagram along the main diagonal. That is,

$$(T\lambda)_i = \left|\left\{j \geqslant 1 \;\middle|\; \lambda_j \geqslant i\right\}\right|.$$

We remark that $T\lambda$ is conventionally denoted λ'; we choose our notation in this paper so that all operators on partitions are denoted with capital letters written on the left. The letter T is taken from [1], and stands for 'transpose'.

In this paper we write $l(\lambda)$ for $(T\lambda)_1$, i.e. the number of non-zero parts of λ.

1.2.2 Row and column removal

Suppose λ is a partition. Let $R\lambda$ denote the partition obtained by removing the first row of the Young diagram; that is, $(R\lambda)_i = \lambda_{i+1}$ for $i \geqslant 1$. Similarly, let $C\lambda$ denote the partition obtained by removing the first column from the Young diagram of λ, i.e. $(C\lambda)_i = \max\{\lambda_i - 1, 0\}$ for $i \geqslant 1$.

In this paper we shall use without comment the obvious relation $TR = CT$.

1.2.3 Regularisation

Now we introduce one of the most important concepts of this paper. Suppose λ is a partition and $e \geqslant 2$. The *e-regularisation* of λ is an e-regular partition associated to λ in a natural way. The notion of regularisation was introduced by James [4] in the case where e is a prime, where it plays a rôle in the computation of the e-modular decomposition matrices of the symmetric groups.

For $l \geqslant 1$, we define the lth *ladder* in $\mathbb{N}^2$ to be the set of nodes (i, j) such that $i + (e-1)(j-1) = l$. The regularisation of λ is defined by moving all the nodes of λ in each ladder as high as they will go within that ladder. It is a straightforward exercise to show that this procedure gives the Young diagram of a partition, and the e-regularisation of λ is defined to be this partition.

Example. Suppose $e = 3$ and $\lambda = (4, 3^3, 1^5)$. Then the e-regularisation of λ is $(5, 4, 3^2, 2, 1)$, as we can see from the following Young diagrams, in which we label each node with the number of the ladder in which it lies.

$$
\begin{array}{|c|c|c|c|}
\hline
1 & 3 & 5 & 7 \\ \hline
2 & 4 & 6 \\ \cline{1-3}
3 & 5 & 7 \\ \cline{1-3}
4 & 6 & 8 \\ \cline{1-3}
5 \\ \cline{1-1}
6 \\ \cline{1-1}
7 \\ \cline{1-1}
8 \\ \cline{1-1}
9 \\ \cline{1-1}
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|c|c|}
\hline
1 & 3 & 5 & 7 & 9 \\ \hline
2 & 4 & 6 & 8 \\ \cline{1-4}
3 & 5 & 7 \\ \cline{1-3}
4 & 6 & 8 \\ \cline{1-3}
5 & 7 \\ \cline{1-2}
6 \\ \cline{1-1}
\end{array}
$$

We write $G\lambda$ for the e-regularisation of λ. Clearly $G\lambda$ is e-regular, and equals λ if λ is e-regular. We record here three results we shall need later; the proofs of the first two are easy exercises.

Lemma 1.1. *Suppose λ is a partition. If $(G\lambda)_1 = \lambda_1$, then $RG\lambda = GR\lambda$.*

Lemma 1.2. *Suppose λ and μ are partitions. If $l(\lambda) = l(\mu)$ and $GC\lambda = C\mu$, then $G\lambda = G\mu$.*

Lemma 1.3. *Suppose ζ is an e-regular partition, and $x \geqslant l(\zeta) + e - 1$. Let ξ be the partition obtained by adding a column of length x to ζ, and let η be the partition obtained by adding a column of length $x - e + 1$ to $C\zeta$. Then $G\eta = CG\xi$.*

Proof. For any $n \geqslant 1$ and any partition λ, let $\mathrm{lad}_n(\lambda)$ denote the number of nodes of λ in ladder n. Since $G\eta$ and $CG\xi$ are both e-regular, it suffices to show that $\mathrm{lad}_n(G\eta) = \mathrm{lad}_n(CG\xi)$ for all n.

η is obtained from ζ by adding the nodes $(l(\zeta) + 1, 1), \dots, (x - e + 1, 1)$, so we have

$$\mathrm{lad}_n(G\eta) = \mathrm{lad}_n(\eta) = \begin{cases} \mathrm{lad}_n(\zeta) + 1 & (l(\zeta) < n < x + e) \\ \mathrm{lad}_n(\zeta) & (\text{otherwise}). \end{cases}$$

It is also easy to compute

$$\mathrm{lad}_n(\xi) = \begin{cases} 1 & (1 \leqslant n < e) \\ \mathrm{lad}_{n-e+1}(\zeta) + 1 & (e \leqslant n \leqslant x) \\ \mathrm{lad}_{n-e+1}(\zeta) & (x < n). \end{cases}$$

Claim. $l(G\xi) = l(\zeta) + e - 1$.

Proof. Since ζ is e-regular and $(l(\zeta), 1) \in [\zeta]$, every node of ladder $l(\zeta)$ is a node of ζ. Hence every node of ladder $l(\zeta) + e - 1$ is a node of ξ; so when ξ is regularised, none of these nodes moves, and we have $(l(\zeta)+e-1, 1) \in [G\xi]$, i.e. $l(G\xi) \geqslant l(\zeta)+e-1$.

On the other hand, the node $(l(\zeta) + 1, 2)$ does not lie in $[\xi]$, so the node $(l(\zeta) + e, 1)$ cannot lie in $[G\xi]$, i.e. $l(G\xi) < l(\zeta) + e$.

From the claim we deduce that

$$\operatorname{lad}_n(CG\xi) = \begin{cases} \operatorname{lad}_{n+e-1}(\xi) - 1 & (n \leqslant l(\zeta)) \\ \operatorname{lad}_{n+e-1}(\xi) & (n > l(\zeta)), \end{cases}$$

and combining this with the statements above gives the result. □

1.2.4 The Mullineux map

Now we introduce the Mullineux map, which is the most important concept of this paper. We shall give two different recursive definitions of the Mullineux map: the original definition due to Mullineux [8], and an alternative version due to Xu [9].

Suppose λ is a partition, and define the *rim* of λ to be the subset of $[\lambda]$ consisting of all nodes (i, j) such that $(i + 1, j + 1) \notin \lambda$. Now fix $e \geqslant 2$, and suppose that λ is e-regular. Define the *e-rim* of λ to be the subset $\{(i_1, j_1), \dots, (i_r, j_r)\}$ of the rim of λ obtained by the following procedure.

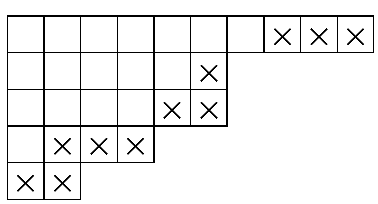
- If $\lambda = \varnothing$, then set $r = 0$, so that the e-rim of λ is empty. Otherwise, let (i_1, j_1) be the top-rightmost node of the rim, i.e. the node $(1, \lambda_1)$.
- For $k > 1$ with $e \nmid k - 1$, let (i_k, j_k) be the next node along the rim from (i_{k-1}, j_{k-1}), i.e. the node $(i_{k-1} + 1, j_{k-1})$ if $\lambda_{i_{k-1}} = \lambda_{i_{k-1}+1}$, or the node $(i_{k-1}, j_{k-1} - 1)$ otherwise.
- For $k > 1$ with $e \mid k - 1$, define (i_k, j_k) to be the node $(i_{k-1} + 1, \lambda_{i_{k-1}+1})$.
- Continue until a node (i_k, j_k) is reached in the bottom row of $[\lambda]$ (i.e. with $i_k = l(\lambda)$), and either $j_k = 1$ or $e \mid k$. Set $r = k$, and stop.

Less formally, we construct the e-rim of λ by working along the rim from top right to bottom left, and moving down one row every time the number of nodes we've seen is divisible by e.

The integer r defined in this way is called the *e-rim length* of λ. We define $I\lambda$ to be the partition obtained by removing the e-rim of λ from $[\lambda]$.

Examples.

1. Suppose $e = 3$, and $\lambda = (10, 6^2, 4, 2)$. Then the e-rim of λ consists of the marked nodes in the following diagram, and we see that $r = 11$ and $I\lambda = (7, 5, 4, 1)$.



2. Suppose $e = 2$, and λ is any 2-regular partition. The 2-rim of λ consists of the last two nodes in each row of $[\lambda]$ (or the last node, if there is only one). Hence when $e = 2$ the operator I is the same as C^2.

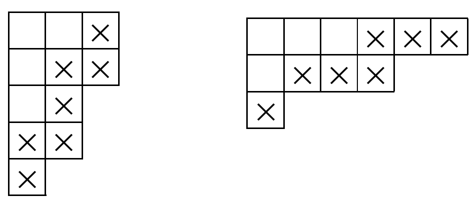
Now we can define the Mullineux map recursively. Suppose λ is an e-regular partition. If $\lambda = \varnothing$, then set $M\lambda = \varnothing$. Otherwise, compute the partition $I\lambda$ as above. Then $|I\lambda| < |\lambda|$, and $I\lambda$ is e-regular, so we may assume that $MI\lambda$ is defined. Let r be the e-rim length of λ, and define

$$m = \begin{cases} r - l(\lambda) & (e \mid r) \\ r - l(\lambda) + 1 & (e \nmid r). \end{cases}$$

It turns out that there is a unique e-regular partition μ which has e-rim length r and $l(\mu) = m$, and which satisfies $I\mu = MI\lambda$. We set $M\lambda = \mu$.

Examples.

1. Suppose $e = 3$, $\lambda = (3^2, 2^2, 1)$ and $\mu = (6, 4, 1)$. Then we have $I\lambda = (2, 1^2)$ and $I\mu = (3, 1)$, as we see from the following diagrams.



   Computing e-rims again, we find that $I^2\lambda = I^2\mu = \varnothing$. Now comparing the numbers of non-zero parts of these partitions with their e-rim lengths we find that $MI\lambda = I\mu$, and hence that $M\lambda = \mu$.

2. Suppose $e = 2$, and λ is a 2-regular partition. From above, we see that the 2-rim length of λ is $2l(\lambda)$, if $\lambda_{l(\lambda)} \geqslant 2$, or $2l(\lambda) - 1$ if $\lambda_{l(\lambda)} = 1$. Either way, we get $m = l(\lambda)$, and this implies inductively that in the case $e = 2$ the Mullineux map is the identity.

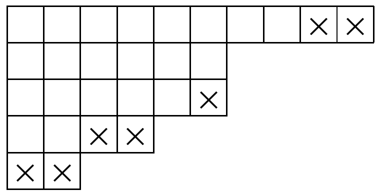
3. Suppose e is large relative to λ; in particular, suppose e is greater than the number of nodes in the rim of λ. Then the e-rim of λ coincides with the rim, so that the e-rim length is $\lambda_1 + l(\lambda) - 1$. Hence $m = \lambda_1$, and from this it is easy to prove by induction that $M\lambda = T\lambda$.

Now we give Xu's alternative definition of the Mullineux map. Suppose λ is a partition with e-rim length r, and define

$$l' = \begin{cases} l(\lambda) & (e \mid r) \\ l(\lambda) - 1 & (e \nmid r). \end{cases}$$

Define $J\lambda$ to be the partition obtained by removing the e-rim from λ, and then adding a column of length l'. Another way to think of this is to define the *truncated e-rim* of λ to be the set of nodes (i, j) in the e-rim of λ such that $(i, j-1)$ also lies in the e-rim, together with the node $(l(\lambda), 1)$ if $e \nmid r$, and to define $J\lambda$ to be the partition obtained by removing the truncated e-rim.

Example. Returning to an earlier example, take $e = 3$ and $\lambda = (10, 6^2, 4, 2)$. Then the truncated e-rim of λ consists of the marked nodes in the following diagram, and we see that $I\lambda = (8, 6, 5, 2)$.



If λ is e-regular, then it is a simple exercise to show that $J\lambda$ is e-regular and $|J\lambda| < |\lambda|$. So we assume that $MJ\lambda$ is defined recursively, and we define $M\lambda$ to be the partition obtained by adding a column of length $|\lambda| - |J\lambda|$ to $MJ\lambda$. Xu [9, Theorem 1] shows that this map coincides with Mullineux's map M. In other words, we have the following.

Proposition 1.4. *Suppose λ and μ are e-regular partitions, with $|\lambda| = |\mu|$. Then $M\lambda = \mu$ if and only if $MJ\lambda = C\mu$.*

1.3 Hooks

Now we set up some basic notation concerning hooks in Young diagrams. Suppose λ is a partition, and (i, j) is a node of λ. The *(i, j)-hook* of λ is defined to be the set $H_{ij}(\lambda)$ of nodes in $[\lambda]$ directly to the right of or directly below (i, j), including the node (i, j) itself. The *arm length* $a_{ij}(\lambda)$ is the number of nodes directly to the right of (i, j), i.e. $\lambda_i - j$, and the *leg length* $l_{ij}(\lambda)$ is the number of nodes directly below (i, j), i.e. $(T\lambda)_j - i$. The *(i, j)-hook length* $h_{ij}(\lambda)$ is the total number of nodes in $H_{ij}(\lambda)$, i.e. $a_{ij}(\lambda) + l_{ij}(\lambda) + 1$.

Now fix $e \geqslant 2$. The *e-weight* of λ is defined to be the number of nodes (i, j) of λ such that $e \mid h_{ij}(\lambda)$. If $(i, j) \in [\lambda]$ with $e \mid h_{ij}(\lambda)$, we say that $H_{ij}(\lambda)$ is

- *shallow* if $a_{ij}(\lambda) \geqslant (e-1)l_{ij}(\lambda)$, or
- *steep* if $l_{ij}(\lambda) \geqslant (e-1)a_{ij}(\lambda)$.

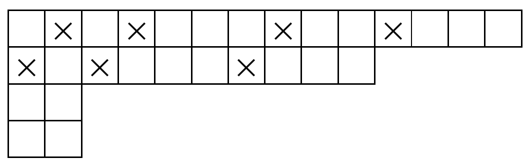
Example. Suppose $e = 3$ and $\lambda = (5, 2, 1^4)$. Then we have $(2, 1) \in [\lambda]$, with $a_{2,1}(\lambda) = 1$, $l_{2,1}(\lambda) = 4$, and hence $h_{2,1}(\lambda) = 6$. $H_{2,1}(\lambda)$ is steep if $e = 3$, but not if $e = 6$.

1.4 Lyle's Conjecture

Suppose $e \geqslant 2$ and λ is an e-regular partition. As noted above, if e is large relative to $|\lambda|$, then $M\lambda = T\lambda$. Of course, there is no hope that this is true in general, since $T\lambda$ will not in general be an e-regular partition. But e-regularisation provides a natural way to obtain an e-regular partition from an arbitrary partition, and it is therefore natural to ask: for which e-regular partitions λ do we have $M\lambda = GT\lambda$? When e is large relative to λ we have $G\lambda = \lambda$ and (from the example above) $M\lambda = T\lambda$, so certainly $M\lambda = GT\lambda$ in this case. We also have $M\lambda = GT\lambda$ for all partitions λ when $e = 2$: we have seen that for $e = 2$ the Mullineux map is the identity, and it is a simple exercise to show that λ and $T\lambda$ have the same 2-regularisation for any λ. But it is not generally true that $M\lambda = GT\lambda$ for an e-regular partition λ. Bessenrodt, Olsson and Xu [1] have given a classification of the partitions for which this does hold, as follows.

Theorem 1.5. [1, Theorem 4.8] *Suppose λ is an e-regular partition. Then $M\lambda = GT\lambda$ if and only if for every $(i, j) \in [\lambda]$ with $e \mid h_{ij}(\lambda)$, the hook $H_{ij}(\lambda)$ is shallow.*

Example. Suppose $e = 4$ and $\lambda = (14, 10, 2^2)$. The Young diagram is as follows; we have marked those nodes (i, j) for which $4 \mid h_{ij}(\lambda)$.



We see that all the hooks of length divisible by 4 are shallow, so λ satisfies the second hypothesis of Theorem 1.5. And it may be verified that $GT\lambda = M\lambda = (5^2, 4^2, 3^2, 2^2)$.

Bessenrodt, Olsson and Xu have also posed the following more general question [1, p. 454], which is essentially the same problem without the assumption that λ is e-regular.

For which partitions λ is it true that $MG\lambda = GT\lambda$?

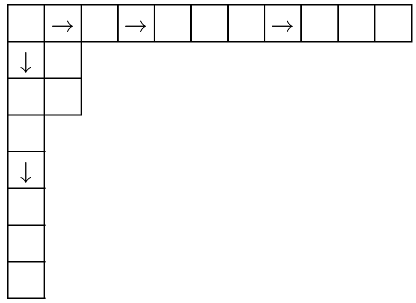
Motivated by the (now solved) problem of the classification of irreducible Specht modules for symmetric groups, Lyle conjectured the following solution in her thesis.

Conjecture 1.6. [7, Conjecture 5.1.18] *Suppose λ is a partition. Then $MG\lambda = GT\lambda$ if and only if for every $(i, j) \in [\lambda]$ with $e \mid h_{ij}(\lambda)$, the hook $H_{ij}(\lambda)$ is either shallow or steep.*

The purpose of this paper is to prove this conjecture. It is a simple exercise to show that a partition possessing a steep hook must be e-singular; so in the case where λ is e-regular, Conjecture 1.6 reduces to Theorem 1.5.

Let us define an *L-partition* to be a partition satisfying the second condition of Conjecture 1.6, i.e. a partition for which every $H_{ij}(\lambda)$ of length divisible by e is either shallow or steep.

Example. Suppose $e = 4$ and $\lambda = (11, 2^2, 1^5)$. The Young diagram of λ is as follows.



The nodes (i, j) with $4 \mid h_{ij}(\lambda)$ are marked; we see that those marked $\boxed{\rightarrow}$ correspond to shallow hooks, and those marked $\boxed{\downarrow}$ correspond to steep hooks. So λ is an L-partition when $e = 4$. We have $G\lambda = (11, 3, 2^2, 1^2)$, $GT\lambda = (8, 4, 3^2, 2)$, and it can be checked that $MG\lambda = GT\lambda$.

2 The 'if' part of Conjecture 1.6

In this section we prove the 'if' half of Conjecture 1.6, i.e. that $MG\lambda = GT\lambda$ whenever λ is an L-partition. We begin by noting some properties of L-partitions, and making some more definitions. Note that when $e = 2$, every partition is an L-partition; by the above remarks we have $MG\lambda = GT\lambda$ for every partition when $e = 2$, so Conjecture 1.6 holds when $e = 2$. Therefore, *we assume throughout this section that* $e \geqslant 3$. The following simple observations will be used without comment.

Lemma 2.1. *Suppose λ is a partition. Then λ is an L-partition if and only if $T\lambda$ is. If λ is an L-partition, then so are $R\lambda$ and $C\lambda$.*

Now we examine the structure of L-partitions in more detail. Suppose λ is an L-partition, and let $s(\lambda)$ be maximal such that $\lambda_{s(\lambda)} - \lambda_{s(\lambda)+1} \geqslant e$, setting $s(\lambda) = 0$ if λ is e-restricted. Similarly, set $t(\lambda) = 0$ if λ is e-regular, and otherwise let $t(\lambda)$ be maximal such that $(T\lambda)_{t(\lambda)} - (T\lambda)_{t(\lambda)+1} \geqslant e$. Clearly, we have $s(\lambda) = t(T\lambda)$.

Lemma 2.2. *If λ is an L-partition, then for $1 \leqslant i \leqslant s(\lambda)$ we have $\lambda_i - \lambda_{i+1} \geqslant e - 1$, while for $1 \leqslant j \leqslant t(\lambda)$ we have $(T\lambda)_j - (T\lambda)_{j+1} \geqslant e - 1$.*

Proof. We prove the first statement. Suppose this statement is false, and let $i < s(\lambda)$ be maximal such that $\lambda_i - \lambda_{i+1} < e - 1$. Put $j = \lambda_i - e + 2$. Then we have $(i, j) \in [\lambda]$, with $a_{ij}(\lambda) = e - 2$ and $l_{ij}(\lambda) = 1$, which (given our assumption that $e \geqslant 3$) contradicts the assumption that λ is an L-partition. □

Lemma 2.3. *Suppose λ is an L-partition and $(i, j) \in [\lambda]$ with $e \mid h_{ij}(\lambda)$.*

1. *If $i > s(\lambda)$, then $H_{ij}(\lambda)$ is steep.*

2. *If $j > t(\lambda)$, then $H_{ij}(\lambda)$ is shallow.*

Proof. We prove (1). Let $a = a_{ij}(\lambda)$ and $l = l_{ij}(\lambda)$. λ is an L-partition, so if $H_{ij}(\lambda)$ is not steep then it must be shallow, i.e. $a \geqslant (e-1)l$. In fact, since $e \mid h_{ij}(\lambda) = a + l + 1$, we find that $a \geqslant (e-1)l + e - 1$. The definition of l implies that $\lambda_{i+l+1} < j = \lambda_i - a$, so

$$\lambda_i - \lambda_{i+l+1} \;>\; a \;\geqslant\; (e-1)(l+1),$$

which implies that for some $k \in \{i, \dots, i+l\}$ we have $\lambda_k - \lambda_{k+1} \geqslant e$. But this contradicts the assumption that $i > s(\lambda)$. □

Now we define an operator S on L-partitions. Suppose λ is an L-partition, and let $s = s(\lambda)$. Define

$$S\lambda = (\lambda_1 - e + 1, \lambda_2 - e + 1, \dots, \lambda_s - e + 1, \lambda_{s+2}, \lambda_{s+3}, \dots).$$

Note that if λ is an e-restricted L-partition, then $S\lambda = R\lambda$. In general, we need to know that S maps L-partitions to L-partitions, in order to allow an inductive proof of Conjecture 1.6.

Lemma 2.4. *If λ is an L-partition, then so is $S\lambda$.*

Proof. Suppose λ is an L-partition, and that $(i, j) \in [S\lambda]$.

If $i > s(\lambda)$, then $(i+1, j) \in [\lambda]$, and we have

$$a_{ij}(S\lambda) = a_{(i+1)j}(\lambda), \qquad l_{ij}(S\lambda) = l_{(i+1)j}(\lambda).$$

So if $e \mid h_{ij}(S\lambda)$, then $e \mid h_{(i+1)j}(\lambda)$; so by Lemma 2.3(1) $H_{(i+1)j}(\lambda)$ is steep, and therefore $H_{ij}(S\lambda)$ is steep.

Next suppose $i \leqslant s(\lambda)$ and $j > \lambda_{s+1}$. Then $(i, j+e-1) \in [\lambda]$ and $a_{ij}(S\lambda) = a_{i(j+e-1)}(\lambda)$, $l_{ij}(S\lambda) = l_{i(j+e-1)}(\lambda)$. So if $e \mid h_{ij}(S\lambda)$, then $e \mid h_{i(j+e-1)}(\lambda)$, and so $H_{i(j+e-1)}(\lambda)$ is shallow, and hence $H_{ij}(S\lambda)$ is shallow.

Finally, suppose that $i \leqslant s(\lambda)$ and $j \leqslant \lambda_{s+1}$. Then $(i, j) \in [\lambda]$, and we have

$$a_{ij}(S\lambda) = a_{ij}(\lambda) - e + 1, \qquad l_{ij}(S\lambda) = l_{ij}(\lambda) - 1.$$

So if $e \mid h_{ij}(S\lambda)$, then $e \mid h_{ij}(\lambda)$, and hence $H_{ij}(\lambda)$ is either shallow or steep. If it is shallow, then we have

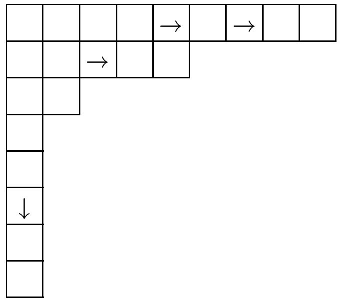
$$a_{ij}(S\lambda) \;=\; a_{ij}(\lambda) - e + 1 \;\geqslant\; (e-1)l_{ij}(\lambda) - e + 1 \;=\; (e-1)l_{ij}(S\lambda),$$

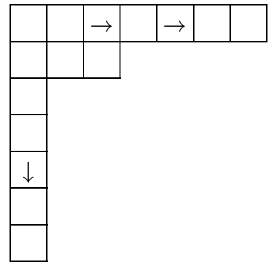
so that $H_{ij}(S\lambda)$ is shallow. On the other hand, if $H_{ij}(\lambda)$ is steep, then

$$l_{ij}(S\lambda) \;=\; l_{ij}(\lambda) - 1 \;\geqslant\; (e-1)a_{ij}(\lambda) - 1 \;>\; (e-1)a_{ij}(S\lambda)$$

so $H_{ij}(S\lambda)$ is steep. □

Example. Suppose $e = 3$, and let $\lambda = (9, 5, 2, 1^5)$. Then we have $s(\lambda) = 2$, so that $S\lambda = (7, 3, 1^5)$. We see that both λ and $S\lambda$ are L-partitions from the following diagrams.





Now we examine the relationship between the operator S and e-regularisation.

Lemma 2.5. *Suppose λ is an L-partition. Then*

$$GTS\lambda = CGT\lambda.$$

Proof. We use induction on $s(\lambda)$. In the case $s(\lambda) = 0$ both λ and $S\lambda = R\lambda$ are e-restricted, i.e. $T\lambda$ and $TS\lambda$ are e-regular, and so $GTS\lambda = TS\lambda = TR\lambda = CT\lambda = CGT\lambda$.

Now suppose $s(\lambda) > 0$. Then $s(R\lambda) = s(\lambda) - 1$, so we may assume that the result holds with λ replaced by $R\lambda$. Put $\zeta = GCT\lambda$; then by the inductive hypothesis $GTSR\lambda = CGTR\lambda = C\zeta$. Let ξ and η be as defined in Lemma 1.3, with $x = \lambda_1$. Note that

$$x = \lambda_1 \geqslant \lambda_2 + e - 1 = l(CT\lambda) + e - 1 \geqslant l(GCT\lambda) + e - 1 = l(\zeta) + e - 1,$$

as required by Lemma 1.3.

Claim. $GT\lambda = G\xi$.

Proof. We have $l(T\lambda) = \lambda_1 = l(\xi)$ and $GCT\lambda = \zeta = C\xi$, and Lemma 1.2 gives the result.

Claim. $GTS\lambda = G\eta$.

Proof. Since $s(\lambda) > 0$, $S\lambda$ may be obtained from $SR\lambda$ by adding a row of length $\lambda_1 - e + 1$; hence $TS\lambda$ may be obtained from $TSR\lambda$ by adding a column of length $\lambda_1 - e + 1$. So we have $l(TS\lambda) = \lambda_1 - e + 1 = l(\eta)$, and

$$GCTS\lambda = GTSR\lambda = C\zeta = C\eta,$$

and again we may appeal to Lemma 1.2.

Now Lemma 1.3 combined with these two claims gives the result. □

Next we prove a simple lemma which gives an equivalent statement to the condition $MG\lambda = GT\lambda$ in the presence of a suitable inductive hypothesis.

Lemma 2.6. *Suppose λ is an L-partition, and that $MG\mu = GT\mu$ for all L-partitions μ with $|\mu| < |\lambda|$. Then $MG\lambda = GT\lambda$ if and only if $GS\lambda = JG\lambda$.*

Proof. Since $|G\lambda| = |GT\lambda|$, we have

$$\begin{aligned} MG\lambda = GT\lambda \quad &\iff \quad MJG\lambda = CGT\lambda && \text{by Proposition 1.4} \\ &\iff \quad MJG\lambda = GTS\lambda && \text{by Lemma 2.5} \\ &\iff \quad MJG\lambda = MGS\lambda && \text{by the inductive hypothesis and Lemma 2.4} \\ &\iff \quad JG\lambda = GS\lambda. && \square \end{aligned}$$

We now require one more lemma concerning the regularisations of L-partitions.

Lemma 2.7. *Suppose λ is an L-partition with $s(\lambda) > 0$ and $\lambda_1 \geqslant l(\lambda)$. Then:*

1. $(G\lambda)_1 = \lambda_1$;

2. $(G\lambda)_1 - (G\lambda)_2 \geqslant e - 1$;

3. $(GS\lambda)_1 = (S\lambda)_1$.

Proof.

1. Obviously $(G\lambda)_1 \geqslant \lambda_1$, so it suffices to show that $[\lambda]$ does not contain a node in ladder $(e-1)\lambda_1 + 1$. If it does, let (i, j) be the rightmost such node. Since $(i, j) \neq (1, \lambda_1 + 1)$, we have $i \geqslant e$ and we know that the node $(i - e + 1, j + 1)$ does not lie in λ; in other words, $(T\lambda)_j - (T\lambda)_{j+1} \geqslant e$. This means that $j \leqslant t(\lambda)$, and so by Lemma 2.2 we have $i \leqslant l(\lambda) - (e-1)(j-1)$, so that

$$l(\lambda) \geqslant i + (e-1)(j-1) = (e-1)\lambda_1 + 1 > \lambda_1,$$

contrary to hypothesis.

2. By part (1), we must show that $(G\lambda)_2 \leqslant \lambda_1 - e + 1$, i.e. that $[\lambda]$ does not contain a node in ladder $2 + (e-1)(\lambda_1 - e + 1)$. Supposing otherwise, we let (i, j) be the rightmost such node. Arguing as above, we find that

$$\lambda_1 \geqslant l(\lambda) \geqslant i + (e-1)(j-1) = 2 + (e-1)(\lambda_1 - e + 1),$$

and this rearranges to yield $\lambda_1 < e$, which is absurd given that $s(\lambda) > 0$.

3. Obviously $(GS\lambda)_1 \geqslant (S\lambda)_1 = \lambda_1 - e + 1$, so it suffices to show that $[S\lambda]$ does not contain a node in ladder $1 + (e-1)(\lambda_1 - e + 1)$. Arguing as above, such a node would have to be of the form (i, j) with $j \leqslant t(S\lambda) \leqslant t(\lambda)$. But then $(TS\lambda)_j = (T\lambda)_j - 1$, so $[\lambda]$ contains the node $(i + 1, j)$, which lies in ladder $2 + (e-1)(\lambda_1 - e + 1)$. But it was shown in (2) that this is not possible.

$\square$

Proof of Conjecture 1.6 ('if' part). We proceed by induction on $|\lambda|$. It is clear that λ is an L-partition if and only if $T\lambda$ is, so Conjecture 1.6 holds for λ if and only if it holds for $T\lambda$. If either λ or $T\lambda$ is e-regular, then the result follows from Theorem 1.5, so we assume that λ is neither e-regular nor e-restricted; in particular, $s(\lambda) > 0$. By replacing λ with $T\lambda$ if necessary, we assume also that $\lambda_1 \geqslant l(\lambda)$.

> **Claim.** $(JG\lambda)_1 = \lambda_1 - e + 1$, and $RJG\lambda = JGR\lambda$.
>
> **Proof.** This follows from Lemma 2.7(1–2), given the definition of the operator J.
>
> **Claim.** $(GS\lambda)_1 = \lambda_1 - e + 1$, and $RGS\lambda = GRS\lambda$.
>
> **Proof.** We have $(S\lambda)_1 = \lambda_1 - e + 1$ by definition, and $(GS\lambda)_1 = (S\lambda)_1$ by Lemma 2.7(3). The second statement follows from Lemma 1.1.

By induction (replacing λ with $R\lambda$) we have $MGR\lambda = GTR\lambda$, and by Lemma 2.6 (and the inductive hypothesis) this gives $JGR\lambda = GSR\lambda$. Since obviously $GSR\lambda = GRS\lambda$, the two claims yield $JG\lambda = GS\lambda$. Now applying Lemma 2.6 again gives the result. □

3 The Fock space and v-decomposition numbers

In this section, we complete the proof of Conjecture 1.6 using v-decomposition numbers. We give only a very brief sketch of the background material needed, since this is discussed at length elsewhere; in particular, the article of Lascoux, Leclerc and Thibon [6] is an invaluable source.

Fix $e \geqslant 2$, let v be an indeterminate over $\mathbb{Q}$, and let $\mathcal{U}$ be the quantum algebra $U_v(\widehat{\mathfrak{sl}}_e)$ over $\mathbb{Q}(v)$. There is a module $\mathcal{F}$ for this algebra called the *Fock space*, which has a *standard basis* indexed by (and often identified with) the set of all partitions. The submodule generated by the empty partition is isomorphic to the *basic representation* of $\mathcal{U}$. This submodule has a *canonical* $\mathbb{Q}(v)$-basis

$$\left\{G(\mu) \;\middle|\; \mu \text{ an } e\text{-regular partition}\right\}.$$

The *v-decomposition numbers* are the coefficients obtained when the elements of the canonical basis are expanded in terms of the standard basis, i.e. the coefficients $d_{\lambda\mu}(v)$ in the expression

$$G(\mu) = \sum_{\lambda} d_{\lambda\mu}(v)\lambda.$$

We shall need to quote two results concerning v-decomposition numbers; one concerning the Mullineux map, and the other concerning e-regularisation. The first of these involves the e-weight of a partition, defined in §1.3.

Theorem 3.1. [6, Theorem 7.2] *Suppose λ and μ are partitions with e-weight w, and that μ is e-regular. Then*

$$d_{(T\lambda)(M\mu)}(v) = v^w d_{\lambda\mu}(v^{-1}).$$

The second result we need requires a definition. Given a partition λ, let $z(\lambda)$ be the number of nodes $(i, j) \in [\lambda]$ such that $e \mid h_{ij}(\lambda)$ and $H_{ij}(\lambda)$ is steep. Now we have the following result.

Theorem 3.2. [2, Theorem 2.2] *For any partition λ,*

$$d_{\lambda(G\lambda)}(v) = v^{z(\lambda)}.$$

Remark. Note that in [2] an alternative convention for the Fock space is used: our $d_{\lambda\mu}(v)$ is written in [2] as $d_{(T\lambda)(T\mu)}(v)$. Accordingly, the statement of [2, Theorem 2.2] involves shallow hooks rather than steep hooks. We hope that no confusion will result.

Now we combine these theorems. First we note the following obvious result about e-weight and the function z.

Lemma 3.3. *Suppose λ is a partition with e-weight w. Then $T\lambda$ also has e-weight w, and $z(T\lambda)$ equals the number of nodes $(i, j) \in [\lambda]$ such that $e \mid h_{ij}(\lambda)$ and $H_{ij}(\lambda)$ is shallow. Hence λ is an L-partition if and only if $w = z(\lambda) + z(T\lambda)$.*

Now we can complete the proof of Conjecture 1.6.

Proof of Conjecture 1.6 ('only if' part). Suppose $MG\lambda = GT\lambda$, and that λ has e-weight w. Then we have

$$\begin{aligned} v^{z(T\lambda)} &= d_{(T\lambda)(GT\lambda)}(v) && \text{by Theorem 3.2} \\ &= d_{(T\lambda)(MG\lambda)}(v) && \text{by hypothesis} \\ &= v^w d_{\lambda(G\lambda)}(v^{-1}) && \text{by Theorem 3.1} \\ &= v^w . v^{-z(\lambda)} && \text{by Theorem 3.2} \end{aligned}$$

so that $w = z(\lambda) + z(T\lambda)$. Now Lemma 3.3 gives the result. □

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