Saturation Numbers of Books

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Abstract

A book B_p is a union of p triangles sharing one edge. This idea was extended to a generalized book $B_{b,p}$, which is the union of p copies of a K_{b+1} sharing a common K_b. A graph G is called an H-saturated graph if G does not contain H as a subgraph, but $G \cup \{xy\}$ contains a copy of H, for any two nonadjacent vertices x and y. The *saturation number of* H, denoted by $sat(H,n)$, is the minimum number of edges in G for all H-saturated graphs G of order n. We show that

$$sat(B_p,n) = \frac{1}{2}\Big((p+1)(n-1) - \left\lceil\frac{p}{2}\right\rceil \left\lfloor\frac{p}{2}\right\rfloor + \theta(n,p)\Big),$$

where $\theta(n,p) = \begin{cases} 1 \text{ if } p \equiv n - p/2 \equiv 0 \bmod 2 \\ 0 \text{ otherwise} \end{cases}$, provided $n \geq p^3 + p$.
Moreover, we show that

$$sat(B_{b,p},n) = \frac{1}{2}\Big((p+2b-3)(n-b+1) - \left\lceil\frac{p}{2}\right\rceil \left\lfloor\frac{p}{2}\right\rfloor + \theta(n,p,b) + (b-1)(b-2)\Big),$$

where $\theta(n,p,b) = \begin{cases} 1 \text{ if } p \equiv n - p/2 - b \equiv 0 \bmod 2 \\ 0 \text{ otherwise} \end{cases}$, provided $n \geq 4(p+2b)^b$.

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1 Introduction

In this paper we consider only graphs without loops or multiple edges. For terms not defined here see [1]. We use $A := B$ to define A as B. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We call $n := |G| := |V(G)|$ the order of G and $||G|| := |E(G)|$ the size of G. For any $v \in V(G)$, let $N(v) := \{w \ : \ vw \in E\}$ be the neighborhood of v, $N[v] := N(v) \cup \{v\}$ be the *closed neighborhood* of v, and $d(v) := |N(v)|$ be the degree of v. Furthermore, if $U \subset V(G)$, we will use $\langle U \rangle$ to denote the subgraph of G induced by U. Let $N_U(v) := N(v) \cap U$, and $d_U(v) := |N_U(v)|$. The complement of G is denoted by $\overline{G}$.

Let G and H be graphs. We say that G is H*-saturated* if H is not a subgraph of G, but for any edge uv in $\overline{G}$, H is a subgraph of $G + uv$. For a fixed integer n, the problem of determining the maximum size of an H-saturated graph of order n is equivalent to determining the classical extremal function $ex(H, n)$. In this paper, we are interested in determining the *minimum* size of an H-saturated graph. Erdős, Hajnal and Moon introduced this notion in [3] and studied it for cliques. We let $sat(H, n)$ denote the minimum size of an H-saturated graph on n vertices.

There are very few graphs for which $sat(H, n)$ is known exactly. In addition to cliques, some of the graphs for which $sat(H, n)$ is known include stars, paths and matchings [6], C_4 [7], C_5 [2], certain unions of complete graphs [4] and $K_{2,3}$ in [8]. Some progress has been made for arbitrary cycles and the current best known upper bound on $sat(C_t, n)$ can be found in [5]. The best upper bound on $sat(H, n)$ for an arbitrary graph H appears in [6], and it remains an interesting problem to determine a non-trivial lower bound on $sat(H, n)$

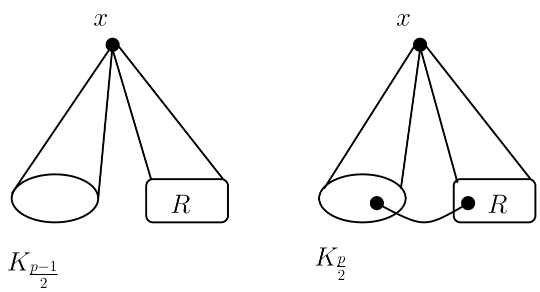
A book B_p is a union of p triangles sharing one edge. This edge is called the *base* of the book. The triangles formed on this edge are called the *pages* of the book. This idea was extended to a generalized book $B_{b,p}$, $b \geq 2$, which is a union of p copies of complete the graph K_{b+1} sharing a base K_b. Again the generalized book has p pages. In particular, $B_p = B_{2,p}$ denotes the standard book and also note that $K_{1,p} = B_{1,p}$.

Our goal is the saturation number of generalized books. We begin however with B_p. In order for this to be nontrivial, we must have $n \geq |B_p| = p + 2$.

Consider first the graph $G(n, p)$, where p is odd and $n \geq \frac{p+1}{2} + p = \frac{3p+1}{2}$. This graph has a vertex x of degree $n - 1$. On $\frac{p-1}{2}$ of the vertices in $N(x)$ is a complete graph, while on the remaining vertices is a $(p - 1)$-regular graph R (see Figure 1(a)). Then,

$$2||G(n, p)|| = (p + 1)(n - 1) - \frac{p - 1}{2}\frac{p + 1}{2}.$$

Next, suppose p is even. Then a similar graph exists, this time with $K_{p/2}$ in one part of $N(x)$ and again a $(p - 1)$-regular graph R on the rest. (Note that in either situation, the parity of n and p may force the $(p - 1)$-regular graph to be "almost" $(p - 1)$-regular, that is, to have all but one vertex of degree $p - 1$, the other of degree $p - 2$, and this



(a) (b)

Figure 1: Sharpness examples for B_p.

vertex will have one edge to the $K_{p/2}$ (see Figure 1(b)). Here $n \geq \frac{3p+1}{2}$ and $p-1$ and $n-p/2-1$ are odd.

Finally, note that in the small order case when $n < (3p+1)/2$ we have K_s and K_p in $N(x)$ when $n = p+1+s$ and here $2||G(n,p)|| = (p+1)(n-1) - s(p-s)$.

Conjecture 1.1 $sat(B_p, n) \geq ||G(n,p)||$.

We show the above conjecture is true for n much larger than p and a similar result holds for generalized books $B_{b,p}$. However, in order for the reader to follow the main proof ideas without going through too many tedious details and cumbersome notation, we give the proof of the values of $sat(B_p, n)$ first and then prove the generalized case. The following notation and terminology are needed.

Let G be a connected graph. For any two vertices u, $v \in V(G)$, the *distance* $dist(u,v)$ between u and v is the length of a shortest path from u to v. The *diameter*, $diam(G)$, is defined as $\max\{dist(u,v) \ : \ u,v \in V(G)\}$. Clearly, $diam(G) = 1$ if and only if G is a complete graph. For any $v \in V(G)$, let $N_i(v) := \{w \ : \ dist(v,w) = i\}$ for each nonnegative integer i. Clearly, $N_0(v) = \{v\}$ and $N_1(v) = N(v)$. For any two vertex sets $A, B \subseteq V(G)$, let $E(A,B) := \{ab \in E \ : \ a \in A \text{ and } b \in B\}$ and let $e(A,B) := |E(A,B)|$.

2 Basic properties of $B_{b,p}$-saturated graphs

We begin with some useful facts necessary to prove the main results.

Lemma 2.1 *Let $b \geq 2$ be an integer and G be a $B_{b,p}-$saturated graph. Then $diam(G) = 2$.*

Proof: Since G does not contain $B_{b,p}$ as a subgraph, G is not a complete graph; hence $diam(G) \geq 2$ holds. We now show that $diam(G) \leq 2$. Let x and y be two nonadjacent vertices of G. Since $G + xy$ contains a copy of $B_{b,p}$, this book must contain the edge xy. Consequently, $dist(x, y) = 2$. $\square$

Lemma 2.2 *If G is a $B_{b,p}-$saturated graph, then $L := \{v \in V(G) \ : \ d(v) \leq p + b - 3\}$ induces a clique in G.*

Proof: Suppose the result fails to hold. Further, say x, $y \in L$ such that $xy \notin E(G)$. Then, $G + xy$ contains a copy of $B_{b,p}$. Since G does not contain $B_{b,p}$ as a subgraph, at least one of x and y must be in the base of the book $B_{b,p}$. But, every vertex in the base of $B_{b,p}$ has degree at least $p + b - 1$, which leads to a contradiction. $\square$

Lemma 2.3 *Let G be a $B_{b,p}$-saturated graph and let $v \in V(G)$. For any $w \in N_2(v)$, $|N(w) \cap N(v)| \geq b - 1$. Consequently, $|E(N(v), N_2(v))| \geq (b - 1)|N_2(v)|$.*

Proof: Let $B_{b,p}$ be a subgraph of $G + vw$ with base B. Since G does not contain $B_{b,p}$, at least one of v and w must be in the base B. If they are both in B then $|N(v) \cap N(w)| \geq (b - 2) + p \geq b - 1$. If exactly one of them is in B, then $|N(v) \cap N(w)| \geq |B| - 1 = b - 1$. $\square$

3 The saturation numbers for books B_p

Theorem 3.1 *Let n and p be two positive integers such that $n \geq p^3 + p$. Then,*

$$sat(B_p, n) = \frac{1}{2}\left((p + 1)(n - 1) - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor + \theta(n, p)\right),$$

where $\theta(n, p) = \begin{cases} 1 \text{ if } p \equiv n - p/2 \equiv 0 \bmod 2 \\ 0 \text{ otherwise.} \end{cases}$

Proof: It is straight forward to verify that $G(n, p)$ is B_p-saturated in each of the cases. We will show that $sat(B_p, n) \geq ||G(n, p)||$. Suppose the contrary: *There is a B_p-saturated graph G of order $n \geq p^3 + p$ such that $||G|| < ||G(n, p)||$.* If $p = 1$, then $||G(n, 1)|| = n - 1$. Since G is connected, $||G|| \geq n - 1 = ||G(n, 1)||$, so the result is true for $p = 1$. We now assume that $p \geq 2$. Moreover, we notice that $\delta(G) \leq p$ since the average degree of $G(n, p)$ is less than $p + 1$.

The following claim plays the key role in the proof.

Claim 3.2 *There is a unique vertex $u \in V(G)$ such that $d(u) \geq n/2$ and $N(u) \supseteq \{v \in V(G) \ : \ d(v) \leq p\}$.*

To prove this claim, let $v \in V(G)$ such that $d(v) \leq p$. Since $\delta(G) \leq p$ such a vertex v exists. Let $V_i := N_i(v)$ for each nonnegative integer i. By Lemma 2.1, $V(G) = \{v\} \cup V_1 \cup V_2$. Let $n_1 = |V_1|$ and $n_2 = |V_2|$ and let $e_{1,2} = |E(V_1, V_2)|$. Clearly, $n_1 \leq p$. We now obtain $\sum_{w \in V_1} d(w) \geq e(\{v\}, V_1) + e_{1,2} = n_1 + e_{1,2}$. Counting the total degree sum of G, we obtain that

$$2||G|| \geq n_1 + (n_1 + e_{1,2}) + \sum_{w \in V_2} d(w).$$

Using the fact $1 + n_1 + n_2 = n$, we deduce the following inequality from the above.

$$2||G|| \geq (n-1)(p+1) + n_1 - n_1 p + (e_{1,2} - n_2) + \sum_{w \in V_2} (d(w) - p).$$

Since $2||G|| < 2||G(n,p)|| = (n-1)(p+1) - \lceil \frac{p}{2} \rceil \lfloor \frac{p}{2} \rfloor + \theta(n,p)$, the following holds

$$(e_{1,2} - n_2) + \sum_{w \in V_2} (d(w) - p) < n_1 p - n_1 - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor \leq \frac{3}{4} p^2. \tag{3.1}$$

Let $S := \{w \in V_2 \ : \ d(w) = p \ \text{ and } \ d_{V_1}(w) = 1\}$, $T := V_2 - S$, $T_1 := \{w \in T \ : \ d(w) < p\}$, $t_1 := |T_1|$, $T_2 := T - T_1$ and $t_2 := |T_2|$.

By Lemma 2.2, T_1 is a clique and every vertex in T_1 has degree at least $|T_1|$, and so

$$\sum_{w \in T_1} (d(w) - p) \geq |T_1|^2 - |T_1| p \geq - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor,$$

which, combining with (3.1), gives

$$e_{1,2} - n_2 + \sum_{w \in T_2} (d(w) - p) < n_1 p - n_1 \leq p^2 - p. \tag{3.2}$$

Since, for each $w \in T_2$, either $d(w) \geq p+1$ or $d_{V_1}(w) \geq 2$, $t_2 \leq e_{1,2} - n_2 + \sum_{w \in T_2} (d(w) - p)$. So,

$$t_2 \leq p^2 - p. \tag{3.3}$$

Since $e_{1,2} - n_2 \geq 0$, inequalities (3.1) and (3.3) give the following

$$\sum_{w \in T_2} d(w) < p^2 - p + p t_2 \leq p^3 - p. \tag{3.4}$$

and

$$\sum_{w \in T_2} (d(w) - 1) < p^2 - p + (p-1) t_2 \leq p^3 - p^2. \tag{3.5}$$

The remainder of the proof of this claim is divided into a few sub-claims.

(A). Let s_1 and $s_2 \in S$ and let x_1 and x_2 be the corresponding neighbors in V_1 of s_1 and s_2, respectively. If $x_1 \neq x_2$ and $s_1s_2 \notin E(G)$ then $N(s_1) \cap N(s_2) \cap T_2 \neq \emptyset$.

To prove **(A)**, let B_p be obtained from $G + s_1s_2$ and B be the base. Since $d(s_1) = d(s_2) = p$ and $N(s_1) \neq N(s_2)$, the edge $s_1s_2 \neq B$. Let $w \in B$ such that $w \notin \{s_1, s_2\}$. Since w is one vertex in the base of B_p, $d(w) \geq p + 1$. Consequently, $w \notin S \cup T_1$. Since $d_{V_1}(s_1) = d_{V_1}(s_2) = 1$ and $x_1 \neq x_2$, $w \notin V_1$, this leaves $w \in T_2$ as the only possibility. Thus, $N(s_1) \cap N(s_2) \cap T_2 \neq \emptyset$.

Let $x \in V_1$ such that $d_S(x)$ is maximum among all vertices $w \in V_1$ and let $Y = N_S(x)$ and $Z = S - Y$.

(B). $|S| \geq n - p^2 - p$ and $|Y| \geq |S|/n_1 \geq |S|/p \geq p$.

We note that $n_1 \leq p$ and $t_1 \leq p - 1$ since T_1 is a clique and connected to the rest of the graph. Now the first inequality follows since $|S| = n - 1 - n_1 - t_1 - t_2 \geq n-1-p-(p-1)-(p^2-p) = n-p^2-p$. Since $d_{V_1}(s) = 1$ for each $s \in S$, $\{N_S(u) \,:\, u \in V_1\}$ gives a partition of S, so that

$$|Y| \geq |S|/n_1 \geq |S|/p \geq (n - p^2 - p)/p \geq p^2 - p \geq p.$$

(C). $|Z| \leq p - 1$. Consequently, $d(x) \geq |Y| = |S| - |Z| \geq n/2$.

Assume $|Z| \geq p$. For each $y \in Y \subseteq S$, since $d(y) = p$, $Z - N(y) \neq \emptyset$; since $|Y| \geq p$, for each $z \in Z$, $Y - N(z) \neq \emptyset$. So for any $s \in S$ there exists $s_1 \in S$ such that $ss_1 \notin E(G)$. Thus by **(A)**, $S \subseteq N(T_2)$. Since every vertex $w \in T_2$ has a neighbor in V_1, $\sum_{w \in T_2}(d(w) - 1) \geq |S|$. Using (3.5) and (B) we obtain

$$n - p^2 - p \leq |S| \leq \sum_{w \in T_2}(d(w) - 1) < p^3 - p^2,$$

so $n < p^3 + p$, a contradiction.

(D). For each $y \neq x$, $d(y) < n/2$.

Suppose to the contrary that there is a $y \neq x$ such that $d(y) \geq n/2$. Then a contradiction is reached by the followings facts. (1) $y \neq v$ since $d(v) \leq p < n/2$;
(2) $y \notin V_1 - \{x\}$ since $N(Y) \cap V_1 = \{x\}$ and $|Y| \geq n/2$;
(3) $y \notin S \cup T_1$ since $d(w) \leq p$ for every vertex $w \in S \cup T_1$, and
(4) $y \notin T_2$ since, by (3.2) and $e_{1,2} - n_2 \geq 0$ and $d(w) \geq p$ for each $w \in T_2$, we have, for each $u \in T_2$,

$$d(u) - p \leq e_{1,2} - n_2 + \sum_{w \in T_2}(d(w) - p) \leq p^2 - p,$$

which gives $d(u) \leq p^2 < n/2$.

Thus, x is the unique vertex of G such that $d(x) \geq n/2$. Since v is an arbitrary vertex such that $d(v) \leq p$, we conclude that x is adjacent to all vertices of degree at most p. This completes the proof of Claim 3.2.

We are now in the position to finish the proof of Theorem 3.1.

Let $L := \{v \in V(G) \ : \ d(v) < p\}$, $M := \{v \in V(G) \ : \ d(v) = p\}$, and $Q := \{v \in V(G) - \{x\} \ : \ d(v) \geq p+1\}$. Let $\ell = |L|$, $m = |M|$, and $q = |Q|$. By Lemma 2.2, we have L induces a clique and each vertex in L has degree at least ℓ. By counting degrees in $\{x\}$, L, M, and Q, we obtain the following set of inequalities.

$$\begin{aligned} 2||G|| &\geq (\ell + m) + \ell^2 + mp + q(p+1) \\ &= (p+1)(\ell + m + q) - \ell p + \ell^2 \\ &\geq (p+1)(n-1) - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor . \end{aligned}$$

Thus, Theorem 3.1 holds with only one exception, when $p \equiv n - p/2 \equiv 0 \bmod 2$. But this is also true if one of the inequalities above is strict. So we may assume that all equalities hold in the set of inequalities above, which gives us the following statements:

- $\ell = p/2$;
- each vertex in L is only adjacent to x and all other vertices in L;
- $N(x) \cap Q = \emptyset$.

If $Q \neq \emptyset$, we have $dist(v, w) \geq 3$ for any $v \in L$ and $w \in Q$, which contradicts $diam(G) = 2$. Therefore, $Q = \emptyset$. In this case $m = n - p/2 - 1 \equiv 1 \bmod 2$ and the subgraph $\langle M \rangle$ induced by M is a $p-1$ regular graph, which is impossible since both m and $p-1$ are odd. This contradiction completes the proof of Theorem 3.1 $\square$

4 Generalized books $B_{b,p}$

We first generalize the graph $G(n,p)$ to $G(n,b,p)$. Suppose p is odd and $n \geq \frac{p+1}{2} + p + b - 2 = \frac{3p+1}{2} + b - 2$. The graph $G(n,b,p)$ contains a set X of $b-1$ vertices of degrees $n-1$, a clique L of $\frac{p-1}{2}$ vertices, a subgraph T of $n - (p-1)/2 - b + 1$ vertices inducing a $(p-1)$-regular graph where $E(L,T) = \emptyset$. Then,

$$2||G(n,b,p)|| = (p + 2b - 3)(n - b + 1) - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor + (b-1)(b-2).$$

Suppose p is even and $n - p/2 - b + 1$ is even. Then a similar graph exists, that is, the graph has a set X of $b-1$ vertices, each of degree $n-1$, a clique L of $\frac{p}{2}$ vertices, a set T of $n-p/2-b+1$ vertices inducing a $(p-1)$-regular graph, and $E(L,T)=\emptyset$. Then again ,

$$2||G(n,b,p)|| = (p+2b-3)(n-b+1) - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor + (b-1)(b-2).$$

Suppose p is even and $n-p/2-b+1$ is odd. Then again a similar graph exists with some modification due to parities (see Figure 2). This time, the graph has a set X of $b-1$ vertices, each of degree $n-1$, a clique L of $\frac{p}{2}$ vertices, a set T of $n-p/2-b+1$ vertices inducing an almost $(p-1)$-regular graph which contains a vertex y of degree $p-2$, and $E(L,T)=\{xy\}$, where x is a vertex in L. Then,

$$2||G(n,b,p)|| = (p+2b-3)(n-b+1) - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor + 1 + (b-1)(b-2). \tag{4.1}$$

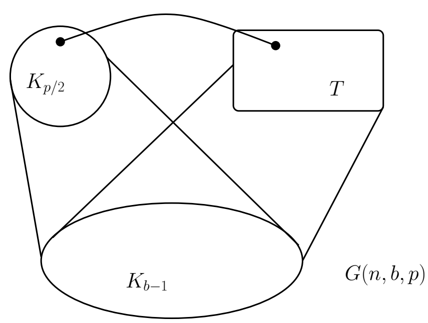


Figure 2: A sharpness example for $B_{b,p}$.

Let $f(n,b,p) = (p+2b-3)(n-b+1) - \lceil \frac{p}{2} \rceil \lfloor \frac{p}{2} \rfloor + (b-1)(b-2) + \theta(n,b,p)$, where $\theta(n,b,p)=1$ if $p \equiv n-p/2-b \equiv 0 \bmod 2$ and 0 otherwise.

Theorem 4.1 *Let n, $b \geq 3$ and p be three positive integers such that $n \geq 4(p+2b)^b$. Then, $sat(B_{b,p},n) = \frac{1}{2}f(n,b,p)$.*

Proof: It is readily seen that graphs $G(n,b,p)$ defined above are $B_{b,p}$-saturated graphs of size $\frac{1}{2}f(n,b,p)$, so that $sat(B_{b,p},n) \leq \frac{1}{2}f(n,b,p)$. We will show that $sat(B_{b,p},n) \geq \frac{1}{2}f(n,b,p)$. Suppose the contrary: *There is a $B_{b,p}$-saturated graph G with n vertices such that $2||G|| < f(n,b,p)$.*

The main part of the proof is dedicated to establishing the following claim which plays a key role in calculating the total degree sum of G.

Claim 4.2 *There exists a clique X in G of order $b-1$ such that $|\cap_{x\in X} N(x)| \geq n/2$ and $\cap_{x\in X} N(x) \supseteq \{v \ : \ d(v) < p+2b-3\}$.*

To prove Claim 4.2, let v be an arbitrary vertex of $V(G)$ such that $d(v) \leq p+2b-4$. Since $2||G|| < f(n,b,p) < (p+2b-3)n$ such a vertex v exists. Let $V_i := N_i(v)$ for each nonnegative integer i. By Lemma 2.1, $V(G) = \{v\} \cup V_1 \cup V_2$. Let $n_1 = |V_1|$, $n_2 = |V_2|$, and $e_{1,2} := |E(V_1, V_2)|$. Clearly, $n_1 = d(v) \leq p+2b-4$. By Lemma 2.3, $d_{V_1}(w) \geq b-1$ for each $w \in V_2$. Clearly,

$$\sum_{u \in V_1} d_{V_2}(u) = \sum_{w \in V_2} d_{V_1}(w) = e_{1,2} \geq (b-1)n_2. \tag{4.2}$$

Counting the total degree sum of G, we obtain the following inequalities:

$$\begin{aligned}
2||G|| &= d(v) + \sum_{u \in V_1} d(u) + \sum_{w \in V_2} d(w) \\
&\geq n_1 + (n_1 + e_{1,2}) + \sum_{w \in V_2} d(w) \\
&= n_1 + (n_1 + (b-1)n_2) + \sum_{w \in V_2} (d_{V_1}(w) - (b-1)) + \\
&\quad\; n_2(p+b-2) + \sum_{w \in V_2} (d(w) - p - b + 2) \\
&= (p+2b-3)n_2 + 2n_1 + \sum_{w \in V_2} ((d_{V_1}(w) - b + 1) + (d(w) - p - b + 2)) \\
&= (p+2b-3)(n-b+1) - (p+2b-3)(n_1+2-b) + 2n_1 + \\
&\quad\; \sum_{w \in V_2} ((d_{V_1}(w) - b + 1) + (d(w) - p - b + 2)).
\end{aligned}$$

Using (4.1), we obtain that

$$\begin{aligned}
&\sum_{w \in V_2} ((d_{V_1}(w) - b + 1) + (d(w) - p - b + 2)) \\
\leq\;& (b-1)(b-2) - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor - 2n_1 + (p+2b-3)(n_1+2-b).
\end{aligned} \tag{4.3}$$

Let

$$\begin{aligned}
S &:= \{w \in V_2 \;:\; d(w) = p+b-2 \;\text{ and }\; d_{V_1}(w) = b-1\}, \\
T &:= V_2 - S, \\
T_1 &:= \{w \in T \;:\; d(w) < p+b-2\}, \\
T_2 &:= T - T_1 = \{w \in V_2 \;: d(w) > p+b-2 \text{ or } (d(w) = p+b-2 \text{ and } d_{V_1}(w) \geq b)\},
\end{aligned}$$

and

$$s := |S|, \quad t_1 := |T_1|, \quad t_2 := |T_2|.$$

By the definition, we have $s + t_1 + t_2 = n_2$ and

$$\sum_{w \in S} ((d_{V_1}(w) - b + 1) + (d(w) - p - b + 2)) = 0. \tag{4.4}$$

By Lemma 2.2, T_1 is a clique, and so, for each $w \in T_1$, $d(w) = d_{V_1}(w) + d_{V_2}(w) \geq b - 1 + t_1 - 1 = t_1 + b - 2$. Hence,

$$\sum_{w \in T_1} ((d_{V_1}(w) - b + 1) + (d(w) - p - b + 2)) \geq t_1(t_1 - p) \geq - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor . \tag{4.5}$$

Combining (4.3), (4.4), and (4.5), we obtain

$$\begin{aligned} & \sum_{w \in T_2} ((d_{V_1}(w) - b + 1) + (d(w) - p - b + 2)) \\ \leq \; & (b-1)(b-2) - 2n_1 + (p + 2b - 3)(n_1 + 2 - b) \leq (p + 2b)^2. \end{aligned} \tag{4.6}$$

Since, for each $w \in T_2$, either $d_{V_1}(w) > b - 1$ or $d(w) > p + b - 2$,

$$t_2 \leq \sum_{w \in T_2} ((d_{V_1}(w) - b + 1) + (d(w) - p - b + 2)) \leq (p + 2b)^2. \tag{4.7}$$

Using (4.6), (4.7), and that $d_{V_1}(w) \geq b - 1$ for each $w \in T_2 \subseteq V_2$, we obtain

$$\sum_{w \in T_2} d(w) \leq (p + 2b)^2 + (p + b - 2)t_2 \leq (p + 2b)^3. \tag{4.8}$$

The remainder of the proof consists of a few sub-claims.

(A). For any s_1 and $s_2 \in S$ and $x_i \in N(s_i) \cap V_1$ for each $i = 1, 2$. If $x_1 \neq x_2$ and $s_1 s_2 \notin E(G)$ then $N(s_1) \cap N(s_2) \cap T_2 \neq \emptyset$.

Let $B_{b,p}$ be obtained from $G + s_1 s_2$ and B be the base. Since $d(s_1) = d(s_2) = p + b - 2$ and $N(s_1) \neq N(s_2)$, $\{s_1, s_2\} \not\subseteq B$. Thus, $B - (V_1 \cup \{s_1, s_2\}) \neq \emptyset$ thanks to $d_{V_1}(s_1) = d_{V_1}(s_2) = b - 1$. So there exists a $w \in T_2 \cap B$. Since $w \in B$, $w \in N(s_1) \cap N(s_2)$, which completes the proof of **(A)**.

Let $X \subseteq V_1$ such that $|X| = b - 1$ and $|(\cap_{x \in X} N(x)) \cap S|$ is maximum among all subsets $X^* \subseteq V_1$ with $|X^*| = b - 1$. Let $Y = (\cap_{x \in X} N(x)) \cap S$ and $Z = S - Y$. Using inequalities $n_1 \leq p + 2b - 4$, $t_2 \leq (p + 2b)^2$, and $n \geq 4(p + 2b)^b$, we obtain $S \neq \emptyset$, which in turn shows that such an X exists. Considering the $B_{b,p}$ obtained by adding the edge vw for a $w \in Y$, we conclude X is a clique.

(B). $|S| \geq n/2 + p + b - 2 > 2 \sum_{w \in T_2} d(w)$ and $|Y| \geq |S| / \binom{n_1}{b-1} \geq |S|/(p + 2b - 4)^{b-1} \geq p + 2b$.

Since $n \geq 4(p + 2b)^b$ and $b \geq 3$, $|S| = n - 1 - n_1 - t_2 - t_1 \geq n - 1 - (p + 2b - 4) - (p + 2b)^2 - (p + b - 3) \geq n/2 + p + b - 2$. Using (4.8) and $n \geq 4(p + 2b)^b$, we obtain that $n/2 + p + b - 2 > 2 \sum_{w \in T_2} d(w)$. Since $d_{V_1}(w) = b - 1$ for each $w \in S$, $\{\cap_{x \in X} N_S(x) \ : \ X \subseteq V_1 \text{ and } |X| = b - 1\}$ gives a partition of S. Hence, $|Y| \geq |S| / \binom{n_1}{b-1}$. The last two inequalities follow from $|S| \geq n/2 + p + b - 2 \geq (p + 2b)^b$ and the choice of v satisfying $n_1 \leq p + 2b - 4$. $\square$

(C). $|Z| < p + b - 2$. Consequently, $|Y| \geq n/2$.

Otherwise, assume $|Z| \geq p + b - 2$. For every $y \in Y$ there exists a $z \in Z$ such that $yz \notin E(G)$. On the other hand, since $|Y| \geq p + 2b$, for every $z \in Z$ there exists a $y \in Y$ such that $yz \notin E(G)$. Using **(A)**, we obtain $S \subseteq N(T_2)$, so that $\sum_{w \in T_2} d(w) \geq |S|$, which contradicts **(B)**. $\square$

(D). For each clique W with $|W| = b - 1$ and $W \neq X$, we have $|\cap_{w \in W} N(w)| < n/2$.

Suppose the contrary: *There is a clique* $W \neq X$ *such that* $|W| = b - 1$ *and* $|\cap_{w \in W} N(w)| \geq n/2$. Then a contradiction is reached by the following listed facts:
(1). $W \cap (\{v\} \cup S \cup T_1) = \emptyset$ since vertices in $\{v\} \cup S \cup T_1$ have degree less than $p+2b-3 < n/2$;
(2). $W \not\subseteq V_1$ since $N(Y) \cap V_1 = X \neq W$ and $|Y| = |S| - |Z| \geq n/2$; and
(3). $W \cap T_2 = \emptyset$ since $\sum_{w \in T_2} d(w) \leq (p + 2b)^3 < n/2$, a contradiction. $\square$

From **D**, we obtain that X is the unique clique of G such that $|X| = b - 1$ and $|\cap_{x \in X} N(x)| \geq n/2$. Since v is an arbitrary vertex such that $d(v) \leq p + 2b - 4$, we conclude that $\cap_{x \in X} N(x)$ contains all vertices of degree at most $p + 2b - 4$. So we have completed the proof of Claim 4.2. $\square$

Let $L := \{v \in V(G) \ : \ d(v) < p + b - 2\}$, $M := \{v \in V(G) \ : \ p + b - 2 \leq d(v) \leq p + 2b - 4\}$, and $Q := \{v \in V(G) - X \ : \ d(v) \geq p + 2b - 3\}$. Let $\ell = |L|$, $m = |M|$, and $q = |Q|$. By Lemma 2.2, we have L is a clique and each vertex in L has degree at least $\ell + b - 2$. We then have the following inequalities on the total degree by counting degrees in X, L, M, and Q.

$$\begin{aligned}
2||G|| &\geq (b-1)(\ell + m + b - 2) + \ell(\ell + b - 2) + m(p + b - 2) + q(p + 2b - 3) \\
&= (p + 2b - 3)(\ell + m + q) - \ell(p - \ell) + (b - 1)(b - 2) \\
&\geq (p + 2b - 3)(n - b + 1) - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor + (b - 1)(b - 2).
\end{aligned}$$

So Theorem 4.1 holds with one exception, that $p \equiv n - p/2 - b \equiv 0 \bmod 2$. Furthermore, Theorem 4.1 holds if one of the inequalities is strict. Thus we may assume that p is even and $n - p/2 - b + 1$ is odd, and all equalities hold in the set of inequalities above, which gives us the following statements:

- $\ell = p/2$;
- each vertex in L is only adjacent to vertices in $L \cup X$;
- $(\cup_{x \in X} N(x)) \cap Q = \emptyset$.

If $Q \neq \emptyset$, we have $dist(v, w) \geq 3$ for any $v \in L$ and $w \in Q$, which contradicts $diam(G) = 2$. Therefore, $Q = \emptyset$. In this case $m = n - p/2 - b + 1 \equiv 1 \bmod 2$ and the subgraph $\langle M \rangle$

induced by M is a $p-1$ regular graph, which is impossible since both m and $p-1$ are odd. □

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