Circular degree choosability

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Abstract

We extend a characterization of degree-choosable graphs due to Borodin [1], and Erdős, Rubin and Taylor [2], to circular list-colorings.

1 Introduction

Let $G = (V(G), E(G))$ be a graph. A *list assignment* L for G is a mapping which assigns to each vertex v of G a set of non-negative integers $L(v)$. An L*-coloring* of G is a proper coloring c of G such that $c(v) \in L(v)$ for every $v \in V(G)$. A graph G is *degree-choosable* if G admits an L-coloring for every list assignment L, such that $|L(v)| \geq \deg(v)$ for all $v \in V(G)$. Borodin [1] and Erdős, Rubin and Taylor [2] characterized degree-choosable graphs as follows.

Theorem 1. *A graph G is not degree-choosable if and only if each of the blocks of G is a clique or an odd cycle (i. e. G is a* Gallai tree*).*

In this paper we prove an analogue of Theorem 1 for circular colorings. Informally, a circular coloring is a coloring of the vertices of the graph by points of a (possibly discrete) circle, such that the circular distance between the colors assigned to adjacent vertices of the graph is bounded from below. Circular colorings have attracted considerable attention over the last decade (see [9] for a survey of the subject). Circular version of list-colorings has been recently introduced by Mohar [4] and Zhu [8], and has been since studied in [3, 5, 6, 7], among others.

Let us now formally present the relevant definitions and notation. Let p be a positive integer. For an integer a we denote by $[a]_p$ the remainder of a modulo p. Define S_p as $\{0, 1, \ldots, p-1\}$. For $a, b \in S_p$, the *interval* $[a, b]_p$ is defined as

$$[a, b]_p = \{a, a+1, a+2, \cdots, b\},$$

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where the additions are modulo p. The *circular distance* $|a-b|_p$ between a and b is defined as

$$|a-b|_p = \min\{[a-b]_p, [b-a]_p\} = \min\{|[a,b]_p|, |[b,a]_p|\} - 1.$$

Let q be a positive integer, such that $p \geq 2q$, and let $G = (V(G), E(G))$ be a graph. A (p,q)-*coloring* of G is a function $c : V(G) \to S_p$, such that for every edge $uv \in E(G)$ we have $|c(u) - c(v)|_p \geq q$. Given a list assignment L, an L-(p,q)-*coloring* of G is a (p,q)-coloring that is also an L-coloring. We say G is L-(p,q)-*colorable* if there is an L-(p,q)-coloring of G. Let $l : V(G) \to \{0, 1, \ldots, p\}$ be a function. We say that a list-assignment L for G is an l-p-*list assignment* if $L(v) \subseteq S_p$ and $|L(v)| \geq l(v)$ for every $v \in V(G)$. We say that G is l-(p,q)-*choosable* if G is L-(p,q)-colorable for every l-p-list assignment L for G.

We say that a graph G is *circular q-degree-choosable* if G is l_q-(p,q)-choosable for every integer $p \geq (\Delta(G) - 1)(2q - 1) + 1$, where for $v \in V(G)$

$$l_q(v) = (\deg(v) - 1)(2q - 1) + 1. \tag{1}$$

Observe that $l_1(v) = \deg(v)$. So a graph is circular 1-degree-choosable if and only if it is degree-choosable. In the following, we only consider circular q-degree-choosability for $q \geq 2$. We will show that for any integers $q_1, q_2 \geq 2$, a graph G is circular q_1-degree-choosable if and only if G is circular q_2-degree-choosable. We say that a graph is *circular degree-choosable* if it is circular q-degree-choosable for some (and hence for all) integer $q \geq 2$.

Given a graph G, the *heart* of G is the graph obtained from G by repeatedly deleting degree one vertices. In this paper we characterize circular degree-choosable graphs as follows.

Theorem 2. *A connected graph G is not circular degree-choosable if and only if G is a tree, or the heart of G is an odd cycle, or G is isomorphic to K_4.*

In [3] Havet et al. considered notions of circular degree-choosability that are similar to, but distinct from ours. They defined a graph G to be *circularly* $(2d-2)$-*choosable* if G is l'_q-(p,q)-choosable for all integers q and $p \geq (2\Delta(G) - 2)q$, where $l'_q(v) = \max\{(2\deg(v) - 2)q, 1\}$ for $v \in V(G)$. They conjectured a characterization of circularly $(2d-2)$-choosable graphs, namely, that a graph is circularly $(2d-2)$-choosable unless it is a tree or an odd cycle. Note that such a characterization is not implied by Theorem 2, nor would it imply Theorem 2. Our notion of circular degree-choosability is more restrictive, but the class of graphs that are conjectured to be circularly $(2d-2)$-choosable is larger. Our definition of circular degree choosability appears to us to be more natural. In particular, circular q-degree-choosability inherits the following property from degree choosability: For a given integer q the function l_q, defined in (1), is the minimum function of $\deg(v)$, such that G is l_q-(p,q)-choosable for all $p \geq \max_{v \in V(G)} l_q(v)$, if, and only if, some induced subgraph H of G is l_q-(p,q)-choosable.

The remainder of the paper is devoted to the proof of Theorem 2. In Section 2 we introduce results from [2, 5, 7] that we will utilize in our proof and prove a couple of simple auxiliary lemmas. The proof itself is presented in Section 3.

2 Prerequisites

Our proof of Theorem 2 follows the general outline of the proof of characterization of degree-choosable graphs in [2]. We prove that a graph G is circular q-degree-choosable if G has an induced subgraph H which is circular q-degree-choosable. Then we show that if the heart of G is non-empty and distinct from an odd cycle then G contains an induced subgraph with a simple, particular structure. Finally, we prove that graphs with such a structure are circular q-degree-choosable. In this section we complete the first two steps of the proof, and introduce the results that we will utilize in the last step. Let us start by introducing some additional notation. For a positive integer p and $x \in S_p$ denote by $B_{p,q}(x)$ the set of elements of S_p at circular distance less than q from x, i.e., $B_{p,q}(x) = [x-q+1, x+q-1]_p$. We write $B(x)$ instead of $B_{p,q}(x)$, when the values of p and q are understood from the context.

Lemma 3. *Let $q \geq 1$ be an integer, let G be a graph and let H be an induced subgraph of G. If H is circular q-degree-choosable, then so is G.*

Proof. It suffices to prove the lemma in the case when H is obtained from G by deleting some vertex $w \in V(G)$. Let $p \geq (\Delta(G)-1)(2q-1)+1$ be an integer. Let L be an l_q-p-list assignment for G, where l_q is defined as in (1). We will prove that G admits an L-(p,q)-coloring. Let $c_0 \in L(w)$ be chosen arbitrarily. Define a list assignment L' for H as follows: Let $L'(v) = L(v) \setminus B_{p,q}(c_0)$ for every $v \in V(H)$, such that $vw \in E(G)$, and let $L'(v) = L(v)$, otherwise. Then $|L'(v)| \geq (\deg_H(v)-1)(2q-1)+1$ for every $v \in V(H)$, and thus there exists an L'-(p,q)-coloring c of H. We extend c to a coloring of G by setting $c(w) = c_0$. Then c is an L-(p,q)-coloring by the choice of L'. □

In proving the "only if" direction of Theorem 2 we will use the following partial converse of Lemma 3.

Lemma 4. *Let $q \geq 1$ be an integer, let G be a graph, and let H be obtained from G by deleting a vertex $u \in V(G)$, such that u has a unique neighbor $w \in V(G)$. Let $l : V(H) \to \mathbb{Z}_+$ be such that H is not l-(p,q)-choosable for all sufficiently large integers p. Let l' be obtained from l by setting $l'(u) = 1$, $l'(w) = l(w) + 2q - 1$, and $l'(v) = l(v)$ for all $v \in V(G) \setminus \{u, w\}$. Then G is not l'-(p,q)-choosable for all sufficiently large integers p.*

Proof. Let $p_0 \geq (2q-1)l(w)+1$ be chosen so that for every integer $p \geq p_0$ the graph H is not l-(p,q)-choosable. Consider arbitrary $p \geq p_0$. By the choice of p, there exists an l-p-list assignment L for H, such that H is not L-(p,q)-colorable. By the choice of p_0 there exists $c \in S_p \setminus \left(\bigcup_{a \in L(w)} B_{p,q}(a)\right)$. We have $B_{p,q}(c) \cap L(w) = \emptyset$. Let $L'(u) = \{c\}$, $L'(w) = L(w) \cup B_{p,q}(c)$, and let $L'(v) = L(v)$ for $v \in V(G) \setminus \{u, w\}$. Then L' is an l'-p-list assignment for G, and, clearly, there exists no L'-(p,q)-coloring of G. □

In [2], typical induced subgraphs of graphs that contain a block distinct from a clique or an odd cycle are described as follows.

Lemma 5. [2] *Let G be a graph that contains a block distinct from a clique or an odd cycle. Then G contains an induced subgraph H, such that H is an even cycle with at most one chord.*

The next description of typical induced subgraphs of graphs with the heart non-empty and distinct from an odd cycle follows immediately from Lemma 5.

Corollary 6. *Let G be a connected graph such that G is not a tree and the heart of G is not an odd cycle. Then G contains an induced subgraph H, such that*

- *H is a clique on 4 vertices, or*
- *H is an even cycle with at most one chord, or*
- *H consists of two odd cycles C_1 and C_2 joined by a path P, such that P is internally disjoint from C_1 and C_2, and C_1 and C_2 are vertex disjoint, unless P has zero length, in which case C_1 and C_2 share a single vertex.*

Proof. If the heart of G contains a block that is not an odd cycle, then the lemma follows from Lemma 5. If not, then G contains two blocks, each of which is an odd cycle. By joining these blocks by an induced path in G (possibly of length zero), we obtain the required subgraph H. □

By Lemma 3 and Corollary 6, it suffices to prove Theorem 2 for graphs with simple structure. We will use the following results on circular colorings of trees and cycles in further analysis of colorings of these graphs.

Lemma 7. [7] *Let T be a tree, let $p \geq 2q$ be positive integers and let $l : V(T) \to \{0, 1, \ldots, p\}$. Then T is l-(p, q)-choosable if and only if for each subtree T' of T we have*

$$\sum_{v \in V(T')} l(v) \geq 2q(|V(T')| - 1) + 1.$$

Lemma 8. [5] *Let $p \geq 2q$ be positive integers. Let G be an even cycle, and let $l(v) = 2q$ for every $v \in V(G)$. Then G is l-(p, q)-choosable.*

3 Proof of Theorem 2

In this section we prove Theorem 2. We prove the "if" direction of the theorem by considering graphs that serve as outcomes of Corollary 6. We start by disposing of the first outcome.

Lemma 9. *Suppose $q \geq 2$, $p \geq 4q$ and L is a list assignment of K_4 with $|L(x)| \geq 4q - 1$ for each vertex x. Then G has an L-(p, q)-coloring.*

Proof. Without loss of generality, we may assume that $0 \in L(v_1)$ and $1 \notin L(v_2)$. Color v_1 with 0. Let c_1 be the least color in $\cup_{i=2}^{4} L(v_i) \setminus [0, q-1]_p$. If $c_1 \in L(v_i)$ for some $i \neq 2$, then color v_i (say v_3) with color c_1. Otherwise, color v_2 with color c_1. At this moment, there are two uncolored vertices, that are either v_2, v_4 or v_3, v_4.

In the former case, $c_1 \notin L(v_3) \cup L(v_4)$. Hence for $i = 3, 4$, $L(v_i) \setminus [0, c_1 + q - 1]_p$ contains at least $2q$ colors. Let c_2 be the least color in $(L(v_3) \cup L(v_4)) \setminus [0, c_1 + q - 1]_p$. Assume $c_2 \in L(v_3)$, then color v_3 with color c_2. Then $L(v_4) \setminus [0, c_2 + q - 1]_p$ contains at least q colors. Color v_4 with the least color in $L(v_4) \setminus [0, c_2 + q - 1]_p$, we obtain an L-(p, q)-coloring of K_4.

In the later case, let c_2 be the least color in $(L(v_2) \cup L(v_4)) \setminus [0, c_1 + q - 1]_p$. If $c_2 \in L(v_4)$, then color v_4 with color c_2. Since $1 \notin L(v_2)$, we conclude that $L(v_2) \setminus [0, c_2 + q - 1]_p$ contains at least q colors. Color v_2 with the least color in $L(v_2) \setminus [0, c_2 + q - 1]_p$, we obtain an L-(p, q)-coloring of K_4. If $c_2 \notin L(v_4)$, then color v_2 with color c_2. Now $L(v_4) \setminus [0, c_2 + q - 1]_p$ contains at least q colors. Color v_4 with the least color in $L(v_4) \setminus [0, c_2 + q - 1]_p$, we obtain an L-(p, q)-coloring of K_4. □

Corollary 10. *Let G be a connected graph with $V(G) \geq 5$ and suppose that G contains a subgraph isomorphic to K_4. Then G is circular q-degree choosable for every integer $q \geq 2$.*

Proof. We need to prove that G is l_q-(p, q)-choosable for all integers $p \geq (\Delta(G) - 1)(2q - 1) + 1$, and the function l_q as in (1). By repeating the argument in the proof of Lemma 3, we can see that it suffices to prove that some subgraph H is l_q^H-(p, q)-choosable, where the superscript H indicates that the degrees in the formula (1) for l_q are taken in H. Choose H isomorphic to K_4. Note that $\Delta(G) \geq 4$, and, thus, $p \geq 4q$. The corollary now follows from Lemma 9. □

When considering two of the remaining outcomes of Corollary 6, we produce the required coloring by precoloring some vertices and applying a variant of Lemma 7 to color the remaining ones. Unfortunately, for our purposes Lemma 7 is not always sufficient. Our next goal is to prove a slightly more technical result.

Lemma 11. *Let T be a tree, and let p and q be positive integers, such that $p \geq 4q - 1$. Then T is L-(p, q)-choosable for every list assignment L with the following properties*

(a) $L(v) \subseteq S_p$ *for every* $v \in V(T)$,

(b) $\sum_{v \in V(T')} |L(v)| \geq 2q(|V(T')| - 1) + 1$ *for each proper subtree T' of T,*

(c) $\sum_{v \in V(T)} |L(v)| \geq 2q(|V(T)| - 1)$, *and if the equality holds, then $L(t)$ is not an interval for some leaf t of T.*

Proof. If the inequality in the condition (c) does not hold with equality then the lemma follows from Lemma 7. Thus, we assume that it does hold with equality, and that $L(t)$ is not an interval for some leaf t of T. Let u be the unique neighbor of t in T. Define a list assignment L' for $T - t$ as follows: Let $L'(u) = \{x \in L(u) \mid L(t) \not\subseteq B_{p,q}(x)\}$, and let $L'(v) = L(v)$ for all $v \in V(T) \setminus \{w, u\}$. Note that L' satisfies conditions of Lemma 7, as long as $|L'(u)| \geq |L(u)| + |L(t)| - 2q + 1$. Moreover, any L'-(p,q)-coloring of $T - w$ extends to an L-(p,q)-coloring of T by the choice of L'.

Thus, it remains to show that $|L'(u)| \geq |L(u)| + |L(t)| - 2q + 1$. By (b) and our assumptions, we have $|L(t)| \leq 2q - 1$. Therefore, we assume that there exists $z \in L(u) \setminus L'(u)$. Then $L(t) \subseteq B(z)$. Let x and y be the elements of $L(t)$ closest to $z - q + 1$ and $z + q - 1$, respectively. We have $L(t) \subsetneq [x, y]_p$ and therefore $|y - x|_p \geq |L(t)|$. Finally, we have $L(u) \setminus L'(u) = \bigcap_{s \in L(t)} B(s) \subseteq B(x) \cap B(y)$, and $|B(x) \cap B(y)| = 2q - 1 - |y - x|_p \leq 2q - 1 - |L(t)|$. The desired inequality follows. $\square$

Next we prove another technical result. It will later allow us to precolor vertices of the graph in such a way that sufficiently many colors remain available for the remaining vertices, and Lemma 11 is applicable.

Lemma 12. *Let p and q be positive integers, such that $p \geq 4q - 1$. Let $w : S_p \to \{0, 1, 2\}$ be a weight function, such that $w(S_p) = 4q$, where $w(S_p) = \sum_{i \in S_p} w(i)$. Let*

$$A = \{a \in S_p \mid w(B_{p,q}(a)) \geq 2q\}.$$

Then $|A| \leq 3q - 1$. Moreover, if $|A| = 3q - 1$ then $w(x) \in \{0, 2\}$ for every $x \in S_p$, and either $q = 1$, or there exists $a \in A$ such that $\operatorname{supp}(w) \setminus B_{p,q}(a)$ is not an interval, where $\operatorname{supp}(w) = \{i \mid w(i) \neq 0\}$.

Proof. We prove the lemma for fixed q by induction on p. We start by considering the base case $p = 4q - 1$. Let $X = \{(a, b) \mid a \in A, |a - b|_p = q\}$. Then $|X| = 2|A|$. On the other hand, $|\{a \in S_p \mid (a, b_0) \in X\}| \leq 2$ for every $b_0 \in S_p$. The inequality is strict for every $b_0 \in \operatorname{supp}(w)$, as $B(b_0 - q), B(b_0 + q)$ and $\{b_0\}$ are pairwise disjoint subsets of S_p. Thus

$$2|A| \leq |X| \leq 2(4q - 1) - |\operatorname{supp}(w)|.$$

It follows that $|A| \leq 3q - 1$.

Suppose now that $|A| = 3q - 1$. Then $|\operatorname{supp}(w)| = 2q$, and therefore $w(x) \in \{0, 2\}$ for every $x \in S_p$. If $\operatorname{supp}(w)$ is an interval, then $A = \operatorname{supp}(w)$, so $|A| = 2q$ and, thus, $q = 1$.

Assume $\operatorname{supp}(w)$ is not an interval. Then there exists $x_0 \in S_p$, such that $\operatorname{supp}(w) \setminus B(x_0)$ is not an interval. If $x_0 \in A$, or $q = 1$, then the lemma holds, and so we assume $x_0 \notin A$ and $q > 1$.

Since $|X| = 2(4q - 1) - |\operatorname{supp}(w)|$, then, using the argument from the first paragraph of the proof, we derive that for $a \in \operatorname{supp}(w)$, exactly one of $a - q$ and $a + q$ is in A, and for $a \notin \operatorname{supp}(w)$, we have $a - q, a + q \in A$. To simplify the notation we assume for the rest of the paragraph that $x_0 = 0$. Since $0 \notin A$, we conclude that q, $3q - 1 \in \operatorname{supp}(w)$ and $2q - 1$, $2q \in A$. Note that $\operatorname{supp}(w) \setminus B(2q)$ and $\operatorname{supp}(w) \setminus B(2q - 1)$ can not be contained in a common interval of $\operatorname{supp}(w)$, as such an interval would have to contain q and $3q - 1$ and would have to have length at least $2q$, in contradiction with the assumption that $\operatorname{supp}(w)$ is not an interval (as q is bigger than 1). Thus either one of $\operatorname{supp}(w) \setminus B(2q)$ and $\operatorname{supp}(w) \setminus B(2q - 1)$ is not an interval, in which case the lemma holds, or $\operatorname{supp}(w) \setminus B(2q) = \{q\}$ and $\operatorname{supp}(w) \setminus B(2q - 1) = \{3q - 1\}$. In the second case, we have $w(q - 1) = 0$, and, thus $4q-2 \in A$. It remains to note that $B(4q-2) \cap B(2q-1) = \emptyset$, and, therefore, $2q \le w(B(4q-2)) = w(2q-1) \le 2$. This contradicts our assumption that $q > 1$, and so the proof in the base case is finished.

It remains to consider the induction step. If $\mathrm{supp}(w) = S_p$ then $p = 4q$, $w(x) = 1$ for every $x \in S_p$, and the lemma trivially holds as $|A| = \emptyset$. Thus we assume $S_p \setminus \mathrm{supp}(w) \neq \emptyset$. We construct an auxiliary graph H with $V(H) = S_p$, joining $x, y \in S_p$ by an edge in H if, and only if, $|x-y|_p \ge 2q-1$. The graph H is connected, and so there exist $x \in \mathrm{supp}(w), y \notin \mathrm{supp}(w)$ such that $xy \in E(H)$. We have $w(B(y-q+1) \cap B(y+q-1)) = 0$, and $w(B(y-q+1) \cup B(y+q-1)) \le 4q - w(x)$. Therefore, not both of $y-q+1$ and $y+q-1$ belong to A. We assume, by symmetry, that $y = p$ and that $p-q+1 \notin A$. Consider the restriction w' of w to S_{p-1}, and let $A' = \{a \in S_{p-1} \,|\, w(B_{p-1,q}(a)) \ge 2q\}$. Let $\phi : S_p \to S_{p-1}$ be defined as $\phi(z) = z$ if $z \le p-q$, and $\phi(z) = z-1$ if $z \ge p-q+1$. Then $\phi(A) \subseteq A'$, as $B_{p,q}(z) \cap S_{p-1} \subseteq B_{p-1,q}(\phi(z))$ for every $z \in S_p$. Moreover, we have $p-q+1 \notin A$, and, thus, ϕ is injective when restricted to A. It follows that $|A| \le |A'|$, and $A' \le 3q-1$, by the induction hypothesis. The remaining conclusions of the lemma also follow straightforwardly from the induction hypothesis applied to w' and A'. □

Finally, we are ready to prove Theorem 2.

Proof of Theorem 2. We start by proving the "if" direction of the theorem. If G contains a subgraph isomorphic to K_4 then the result follows from Corollary 10. Therefore, by Lemma 3 and Corollary 6, it suffices to prove that G is q-degree-choosable if G is isomorphic to one of the other outcomes of Corollary 6. Let $p \ge (\Delta(G)-1)(2q-1)+1$ be a positive integer, and let L be an l_q-p-list assignment for G, where l_q is defined as in (1). It suffices to prove that G is L-(p,q)-colorable for all such p and L. If G is an even cycle then the result follows from Lemma 8.

Suppose next that G is an even cycle with a single chord uu'. Let s and t be the neighbors of u in the cycle. For $x \in S_p$ let $w(x) = \chi_{L(s)}(x) + \chi_{L(t)}(x)$. (Here $\chi_X : S_p \to \{0,1\}$ denotes the characteristic function of a subset X of S_p.) By Lemma 12, applied to w, we conclude that there exists $c_0 \in L(u)$ such that $|B(c_0) \cap L(s)| + |B(c_0) \cap L(t)| \le 2q-1$. Let L' be a list assignment for $G \setminus u$, such that $L'(v) = L(v) \setminus B(c_0)$ for $v \in \{u', s, t\}$, and let $L'(v) = L(v)$, otherwise. Then $|L'(u')| \ge |L(u')| - 2q + 1 = 2q$, and $|L'(s)| + |L'(t)| \ge |L(s)| + |L(t)| - 2q + 1 = 2q + 1$. It follows that L' satisfies the conditions of Lemma 7, and, thus, there exists an L'-(p,q)-coloring of $G \setminus u$, which can be extended to an L-(p,q)-coloring of G by coloring u in color c_0.

It remains to prove the claim in the case when G consists of two odd cycles C_1 and C_2 joined by a path P, such that C_1, C_2 and P satisfy the conditions in the statement of Corollary 6. Let u_1 and u_2 be the vertices that P shares with C_1 and C_2, respectively. Let s_i and t_i be the neighbors of u_i in C_i for $i \in \{1,2\}$. Let R_i be the set of colors c in $L(u_i)$, such that either $|B(c) \cap L(s_i)| + |B(c) \cap L(t_i)| > 2q$; or $|B(c) \cap L(s_i)| + |B(c) \cap L(t_i)| = 2q$, and both $|B(c) \cap L(s_i)|$ and $|B(c) \cap L(t_i)|$ are intervals in S_p. Note that if $c \in L(u_i) \setminus R_i$ then every L-(p,q)-coloring of $G \setminus (C_i \setminus u_i)$, where u_i is colored in color c, can be extended to an L-(p,q)-coloring of G by Lemma 11. For $x \in S_p$ let $w_i(x) = \chi_{L(s_i)}(x) + \chi_{L(t_i)}(x)$. By Lemma 12 applied to w_i, we conclude that $|R_i| \le 3q-2$.

We are now ready to construct an L-(p,q)-coloring of G. Let L' be the following list assignment for P. Let $L'(u_1) = L(u_1) \setminus R_1$, $L'(u_2) = L(u_2) \setminus R_2$. If $u_1 = u_2$ then let $L'(u_1)$=$L(u_1) \setminus R_1 \setminus R_2$, instead. Let $L'(v) = L(v)$ for $v \in V(P) \setminus \{u_1, u_2\}$. We have $|L'(u_1)| + |L'(u_2)| \geq 2((4q-1) - (3q-2)) = 2q+2$ if $u_1 \neq u_2$, and we have $|L'(u_1)| \geq 6q - 2 - 2(3q-2) = 2$ if $u_1 = u_2$. Thus, L' satisfies the requirements of Lemma 7, and so there exists an L'-(p,q)-coloring of P. As noted above, such an L'-(p,q)-coloring can be extended to an L-(p,q)-coloring of G.

It remains to prove that if a graph G is a tree, or G has an odd cycle as its heart, or G is isomorphic to K_4, then for all positive integers q the graph G is not circular q-degree-choosable.

If G is isomorphic to K_4 then there exists no $(4q-1,q)$-coloring of G. If G is a tree then the required result follows from Lemma 4 and the fact that K_1 is not l_q-(p,q)-choosable for all p. If G has an odd cycle as its heart then we claim that G is not l_q-(p,q)-choosable for all sufficiently big p. By Lemma 4 it suffices to consider the case when G is an odd cycle. Let $L(v) = \{0, 1, \ldots, 2q-1\}$ for all $v \in V(G)$. It is easy to see that G is not L-(p,q)-colorable. Therefore, G is not l_q-(p,q)-choosable for all $p \geq 2q$, as claimed. $\square$

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