A note on $K_{k,k}$-cross free families

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Submitted: Aug 19, 2008; Accepted: Oct 20, 2008; Published: Oct 29, 2008
Mathematics Subject Classifications: 05D05

Abstract

We give a short proof that for any fixed integer k, the maximum number size of a $K_{k,k}$-cross free family is linear in the size of the groundset. We also give tight bounds on the maximum size of a K_k-cross free family in the case when $\mathcal{F}$ is intersecting or an antichain.

Introduction

Let $\mathcal{F} \subset 2^{[n]}$. Two sets $A, B \in \mathcal{F}$ *cross* if

1. $A \cap B \neq \emptyset$.
2. $B \not\subset A$ and $A \not\subset B$.

$\mathcal{F} \subset 2^{[n]}$ is said to be K_k-*cross free* if it does not contain k sets $A_1, ..., A_k$ such that A_i cross A_j for every $i \neq j$. Karzanov and Lomonosov conjectured that for any fixed k, the maximum size of a K_k-cross free family $\mathcal{F} \subset 2^{[n]}$ is $O(n)$ [5], [1]. The conjecture has been proven for $k = 2$ and $k = 3$ [7], [4]. For general k, the best known upperbound is $2(k-1)n \log n$, which can easily be seen by a double counting argument on the number of sets of a fixed size. We say that $\mathcal{F}$ is $K_{k,k}$-*cross free* if it does not contain $2k$ sets $A_1, ..., A_k, B_1, ..., B_k \in \mathcal{F}$ such that A_i crosses B_j for all i, j. In this paper, we prove the following:

Theorem 1: *Let $\mathcal{F} \subset 2^{[n]}$ be a $K_{k,k}$-cross free family. Then $|\mathcal{F}| \leq (2k-1)^2 n$.*

In this section, we give upperbounds on the maximum size of certain classes of K_k-cross free families. By applying Dilworth's Theorem [2], one can obtain a tight bound for *intersecting* k-cross free families. Recall a family $\mathcal{F} \subset 2^{[n]}$ is *intersecting* if for every $A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$.

Theorem 2: *Let $\mathcal{F} \subset 2^{[n]}$ be a family that is k-cross free and intersecting. Then $|\mathcal{F}| \leq (k-1)n$, and this bound is asymptotically tight.*

We also obtain tight bounds for K_k-cross free families that is an *antichain*. Recall $\mathcal{F}$ is an *antichain* if no set in $\mathcal{F}$ is a subset of another.

Theorem 3: *For $k \geq 3$, let $\mathcal{F} \subset 2^{[n]}$ be a family that is k-cross free and an antichain. Then $|\mathcal{F}| \leq (k-1)n/2$, and this bound is asymptotically tight.*

We define $sub(A)$ to be the number of subsets of A in $\mathcal{F}$. Our next Theorem gives a non-trivial upperbound on a K_k-cross free family based on the number of subsets in each set of our family.

Theorem 4: *Let $\mathcal{F} \subset 2^{[n]}$ be a K_k-cross free family and let m be defined as*

$$m = \left\lceil \frac{\sum_{A \in \mathcal{F}} \frac{sub(A)}{|A|}}{|\mathcal{F}|} \right\rceil$$

Then $|\mathcal{F}| \leq 4(k-1)m \cdot n$.

Hence if $sub(A) = c|A|$ for all $A \in \mathcal{F}$ and some constant c, then $|\mathcal{F}| = O(n)$. Now we define the *geometric mean* of $\mathcal{F}$ as

$$\gamma(\mathcal{F}) = \left(\prod_{A \in \mathcal{F}} |A| \right)^{1/|\mathcal{F}|}$$

As an easy corollary to theorem 4, we have

Corollary 5: Let $\mathcal{F} \subset 2^{[n]}$ be a K_k-cross free family. Then

$$|\mathcal{F}| \leq 8(k-1)^2 n \log(\gamma(\mathcal{F})).$$

For simplicity we omit floor and ceiling signs whenever these are not crucial and all logarithms are in the natural base e.

$K_{k,k}$-cross free family

Proof of Theorem 1: Induction on n. BASE CASE: $n = 1$ is trivial. INDUCTIVE STEP: For $x \in [n]$, let

$$\mathcal{F}_1 = \{A \in \mathcal{F} : x \in A \quad \text{and} \quad A \setminus x \in \mathcal{F}\}$$

and

$$\mathcal{F}_2 = \{A \setminus x : A \in \mathcal{F}\}.$$

Now notice that there does not exists $2k$ sets $A_1, ..., A_{2k} \in \mathcal{F}_1$ such that $A_1 \subset A_2 \subset \cdots \subset A_{2k}$, since otherwise in $\mathcal{F}$, A_i crosses $A_j \setminus x$ for each $i \leq k$ and $j \geq k+1$. Hence the longest chain in $\mathcal{F}_1$ is $2k-1$ and since $\mathcal{F}_1$ is intersecting, the largest antichain in $\mathcal{F}_1$ is $2k-1$. By Dilworth's Theorem [2], this implies

$$|\mathcal{F}_1| \leq (2k-1)^2.$$

Since $\mathcal{F}_2 \subset 2^{[n-1]}$ is a $K_{k,k}$-cross free family, by the induction hypothesis, we have

$$|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| \leq (2k-1)^2(n-1) + (2k-1)^2 \leq (2k-1)^2 n.$$

□

For the lower bound of a $K_{k,k}$-cross free family, One can consider the edges of a $(k-1)/2$ regular graph on n vertices plus the singletons. Here we have a family with $(k+1)n/2$ sets, and each set crosses at most $k-1$ other sets. Hence this family is $K_{k,k}$-cross free with $(k-1)/2$ sets.

On the maximum size of certain K_k-cross free families

In this section, we will prove Theorems 2,3,4, and Corollary 5.

Proof of Theorem 2: Notice that the largest anitchain must be of size at most $k-1$. Hence by Dilworth's Theorem [2], we can decompose $(\mathcal{F}, \subset)$ into $(k-1)$ chains. Since each chain has length at most n, this implies $|\mathcal{F}| \leq (k-1)n$. Notice that this bound is asymptotically tight. For $i \leq j$, let $[i,j] \in 2^{[n]}$ denote the set $[j] \setminus [i-1]$, and let C_l be a chain of $n-1$ sets defined as

$$C_l = \left(\bigcup_{j=l+1}^{n} [l+1, j] \cup \{1\}\right) \bigcup \left(\bigcup_{j=1}^{l-1} [l+1, n] \cup [1, j]\right) \bigcup \{1\}$$

for $l \geq 2$. Then the family $\mathcal{F} = \left(\bigcup_{l=2}^{k} C_l\right) \bigcup [n]$ is K_k-cross free intersecting family with $(k-1)(n-2)+2$ sets and is intersecting.

□

Proof of Theorem 3: Induction on n. BASE CASE: $n=1$ is trivial. INDUCTIVE STEP: **(case 1)** suppose there is a singleton set $\{x\} \in \mathcal{F}$. Then define $\mathcal{F}' = \{A : x \notin A\}$. Then notice

$$|\mathcal{F}| = 1 + |\mathcal{F}'|.$$

Since $\mathcal{F}'$ is a $K_{k,k}$-cross free family and an antichain, by the induction hypothesis we have

$$|\mathcal{F}| = 1 + |\mathcal{F}'| \leq 1 + (k-1)(n-1)/2 = 1 + (k-1)n/2 - (k-1)/2 \leq (k-1)n/2.$$

Since $k \geq 3$. **(case 2)** Now we can assume all sets in $\mathcal{F}$ has size at least 2. Recall that the *fractional chromatic number* $\chi_f(G)$ of a graph G is defined as the minimum of the fractions a/b such that $V(G)$ can be covered by a indepdendent sets in such a way that every vertex is covered at least b times [6]. Let $G = (V, E)$ be the non-crossing graph of $\mathcal{F}$. I.e. $V(G) = F$ and $(A, B) \in E(G)$ if A and B do not cross. Then for each set $A \in V$, we will assign any two number $(a, b) \subset A$ to A. This is possible since all sets in $\mathcal{F}$ have size at least 2. Since $\mathcal{F}$ is an antichain, this implies that $\chi_f(G) \leq n/2$. Hence by using the inequality $\frac{|G|}{\alpha(G)} \leq \chi_f(G)$, we have

$$\frac{|\mathcal{F}|}{(k-1)} \leq \frac{n}{2} \quad \Rightarrow \quad |\mathcal{F}| \leq \frac{(k-1)n}{2}.$$

Notice that this bound is tight since we can consider the edges of a $(k-1)$ regular bipartite graph. Clearly this family has $(k-1)n/2$ sets and is an antichain since every set is of size 2. By Hall's Theorem [8], the edges of this graph decomposes into $k-1$ perfect matchings, which implies this family is K_k-cross free.

$\square$

Proof of Theorem 4: We will start by blowing up each vertex by a factor of $2m$, i.e. each vertex $x \in [n]$ is replaced by $2m$ vertices $\{x_1, x_2, \ldots, x_{2m}\}$ such that for every $A \in \mathcal{F}$ such that $x \in A$, all $x_1, \ldots, x_{2m} \in A$. Now let G be the non-crossing graph of $\mathcal{F}$. Then we will assign a random color to A by picking a vertex $x \in A$. Then for any $B \in \mathcal{F}$ such that $B \subset A$,

$$\mathbb{P}[B \text{ and } A \text{ are the same color}] = \frac{1}{2m|A|}\frac{1}{2m|B|}2m|B| = \frac{1}{2m|A|}$$

Let X denote the number of monochromatic edges in G. Then

$$\mathbb{E}[X] = \sum_{A \in \mathcal{F}} \sum_{B \subset A} \frac{1}{2m|A|}$$

by definition of m, we have

$$\mathbb{E}[X] = \sum_{A \in \mathcal{F}} \sum_{B \subset A} \frac{1}{2m|A|} \leq \sum_{A \in \mathcal{F}} \frac{1}{2} = |\mathcal{F}|/2.$$

Now we delete one set from each monochromatic edge to obtain a K_k-cross free family $\mathcal{F}'$ with at least $|\mathcal{F}|/2$ sets and is properly colored. Hence by the inequality $|G|/\alpha(G) \leq \chi(G)$, we have

$$\frac{|\mathcal{F}|/2}{k-1} \leq 2mn.$$

Hence $|\mathcal{F}| \leq 4(k-1)mn$.

$\square$

Proof of Corollary 5: Since $sub(A) \leq 2(k-1)|A|\log(|A|)$, this implies

$$\frac{\sum_{A\in\mathcal{F}} \frac{sub(A)}{|A|}}{|\mathcal{F}|} \leq \frac{\sum_{A\in\mathcal{F}} 2(k-1)\log(|A|)}{|\mathcal{F}|} = 2(k-1)\log(\gamma(\mathcal{F})).$$

By Theorem 4, we have

$$|\mathcal{F}| \leq 8(k-1)^2 n \log(\gamma(\mathcal{F})).$$

$\square$

Cross versus strongly-cross

In other places, two sets *cross* are defined a bit differently. To avoid confusion, we say that two sets $A, B \in 2^{[n]}$ *strongly-cross* if $A \cap B \neq \emptyset$, $A \not\subset B$, $B \not\subset A$, and $A \cup B \neq [n]$ (This is how *cross* is defined in [4]). However one can obtain asymptotically similar results for *strongly-crossing* by the next Theorem. Let G be a graph on k vertices $v_1, ..., v_k$. Then $\mathcal{F}$ is a *G-strongly-cross* free family if there does not exist k sets $A_1, ..., A_k \in \mathcal{F}$ such that A_i *strongly crosses* A_j if and only if v_i is adjacent to v_j in G. Likewise $\mathcal{F}$ is a *G-cross* free family if there does not exist k sets $A_1, ..., A_k \in \mathcal{F}$ such that A_i *crosses* A_j if and only if v_i is adjacent to v_j in G.

Theorem 6: *Let $\mathcal{F} \subset 2^{[n]}$ be a maximum G-strongly-cross free family and $\mathcal{H} \subset 2^{[n]}$ be a maximum G-cross free family. Then*

$$|\mathcal{H}| \leq |\mathcal{F}| \leq 2|\mathcal{H}|$$

Proof: Clearly $|\mathcal{H}| \leq |\mathcal{F}|$. Now let $F_1 = \{A \in \mathcal{F} : |A| \leq \lfloor n/2 \rfloor\}$ and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$. Then notice that if $A, B \in \mathcal{F}_1$ intersect, then $A \cup B \neq [n]$. Hence $\mathcal{F}_1$ is a *G-cross* free family, which implies $|\mathcal{F}_1| \leq |\mathcal{H}|$. Now define $\mathcal{F}_2^c = \{A^c : A \in \mathcal{F}_2\}$, where $A^c = [n] \setminus A$. Then notice that $A, B \in \mathcal{F}_2$ strongly-cross if and only if $A^c, B^c \in \mathcal{F}_2^c$ strongly-cross. Also notice for $A^c, B^c \in \mathcal{F}_2^c$ such that $A^c \cap B^c \neq \emptyset$, then $A^c \cup B^c \neq [n]$. Hence $\mathcal{F}_2^c$ is a G-cross free family, which implies $|\mathcal{F}_2| = |\mathcal{F}_2^c| \leq |H|$. Therefore $|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| \leq 2|\mathcal{H}|$.

$\square$

Acknowledgment

I would like to thank Janos Pach for introducing me to the Karzanov-Lomonosov Conjecture.

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