Orthogonal polynomials represented by CW-spheres

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Abstract

Given a sequence $\{Q_n(x)\}_{n=0}^{\infty}$ of symmetric orthogonal polynomials, defined by a recurrence formula $Q_n(x) = \nu_n \cdot x \cdot Q_{n-1}(x) - (\nu_n - 1) \cdot Q_{n-2}(x)$ with integer ν_i's satisfying $\nu_i \geq 2$, we construct a sequence of nested Eulerian posets whose ce-index is a non-commutative generalization of these polynomials. Using spherical shellings and direct calculations of the cd-coefficients of the associated Eulerian posets we obtain two new proofs for a bound on the true interval of orthogonality of $\{Q_n(x)\}_{n=0}^{\infty}$. Either argument can replace the use of the theory of chain sequences. Our cd-index calculations allow us to represent the orthogonal polynomials as an explicit positive combination of terms of the form $x^{n-2r}(x^2-1)^r$. Both proofs may be extended to the case when the ν_i's are not integers and the second proof is still valid when only $\nu_i > 1$ is required. The construction provides a new "limited testing ground" for Stanley's non-negativity conjecture for Gorenstein* posets, and suggests the existence of strong links between the theory of orthogonal polynomials and flag-enumeration in Eulerian posets.

Introduction

In a recent paper [13] the present author constructed a sequence of nested Eulerian partially ordered sets whose ce-index generalizes the Tchebyshev polynomials of the first

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kind. The main goal of that paper was to propose a new class of posets to test Stanley's non-negativity conjecture [17, Conjecture 2.1] on the cd-index of Gorenstein* posets.

In this paper we construct a similar sequence $Q_0, Q_1, \ldots$ of nested Eulerian posets for any sequence $\{Q_n(x)\}_{n=0}^{\infty}$ of symmetric orthogonal polynomials satisfying $Q_{-1}(x) = 0$, $Q_0(x) = 1$, $Q_1(x) = x$, and a recursion formula $Q_n(x) = \nu_n \cdot x \cdot Q_{n-1}(x) - (\nu_n - 1) \cdot Q_{n-2}(x)$ for $n \geq 2$ with integers $\nu_n \geq 2$. Since these posets arise as face posets of a sequence of CW-spheres closed under taking (the boundary complexes of) faces, these sequences of posets may help testing Stanley's conjecture the same way as the Tchebyshev posets. (This possibility will be explained in the concluding Section 8.) The study of the structure of these posets, however, opens up also other, potentially even more interesting directions of research.

The fact that the true interval of orthogonality of the orthogonal polynomial systems considered is a subset of $[-1, 1]$ is an easy consequence of the non-negativity of the cd-index of the associated Eulerian posets, which may be shown using spherical shelling. (It is also fairly easy to extract a proof for non-integer ν_i's by inspection of the integral case.) The same result in the classical theory of orthogonal polynomials seems to depend on the theory of chain sequences. Both shellings and chain sequences seem to be a tool to "prove inequalities by induction" in this context. Moreover, the recursion formula for the non-commutative generalization of the orthogonal polynomial systems considered seems to offer a very easy way to find an explicit non-negative representation, which then may be "projected down" to the commutative case. It may be worth finding it out in the future whether the theory of chain sequences is closer to the first or second approach to the cd-coefficients, if it is close to any of them. In this process either a new approach to prove non-negativity results for cd-coefficients or new ways to prove non-negativity results for orthogonal polynomials may be found.

Since it is a goal of the present paper to inspire collaboration between researchers of orthogonal polynomials and Eulerian posets, experts of either field will hopefully find some useful information and sufficient pointers in the preliminary Section 1. Furthermore, this section contains a brief (and somewhat "unorthodox") introduction to spherical coordinates, which will be useful in describing our CW-spheres.

In Section 2 we define complexes of lunes on an $(n-1)$-dimensional sphere. Every partially ordered set in each sequence will be the face poset of a lune complex. We introduce a code system for the faces, and show that each lune complex is a CW-sphere. A fundamental recursion formula for the flag f-vector of lune complexes is shown in Section 3. Instances of the ce-index form of the same recursion clearly generalize of the fundamental recursion formula of the orthogonal polynomials $Q_n(x)$. The fact that the lune complexes are spherically shellable and thus have a non-negative cd-index is shown in Section 4.

The connection between the non-negativity of the cd-index of a lune complex and the statement on the true interval of orthogonality of the polynomials $Q_n(x)$ is explained in Section 5. We also provide the first proof of the non-negativity of the cd-index by the use of spherical shelling. This approach needs the assumption $\nu_n \geq 2$ for $n \geq 2$, while in the traditional approach using chain sequences only $\nu_n > 1$ is needed. This "gap" could probably be filled by using a more general definition for our lune complexes. The study of this option is omitted, since in Section 6 we show how the cd-index recursion may be used to obtain an explicit formula for the cd-coefficients of our face posets, and how these calculations may be "projected down" to obtain an explicit representation of our orthogonal polynomials as a positive combination of terms of the form $x^{n-2r}(x^2-1)^r$. This proof extends also to the weakened condition $\nu_n > 1$, and does not require constructing Eulerian posets. However, it seems more difficult to guess the formula found without the inspiration coming from cd-index calculations.

Section 7 contains the proof of the fact that for the case when $\nu_n = 2$ for $n \geq 2$, the face posets of CW-spheres constructed in this paper are isomorphic to the duals of the Tchebyshev posets constructed in [13]. Finally, we present our suggestions for future research in Section 8.

1 Preliminaries

1.1 Eulerian posets

A partially ordered set P is *graded* if it has a unique minimum element $\widehat{0}$, a unique maximum element $\widehat{1}$, and a rank function ρ. Here $\rho(\widehat{0}) = 0$, and $\rho(\widehat{1})$ is the rank of P. Given a graded partially ordered set P of rank $n+1$ and $S \subseteq \{1, \ldots, n\}$, $f_S(P)$ denotes the number of saturated chains of the S-*rank selected subposet* $P_S = \{x \in P : \rho(x) \in S\} \cup \{\widehat{0}, \widehat{1}\}$. The vector $(f_S(P) : S \subseteq \{1, \ldots, n\})$ is called the *flag f-vector* of P. Equivalent encodings of the flag f-vector include the *flag h-vector* $(h_S(P) : S \subseteq \{1, \ldots, n\})$ (see [17]) and the flag ℓ-vector $(\ell_S(P) : S \subseteq \{1, \ldots, n\})$ (see [6]), given by $h_S(P) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(P)$ and $\ell_S(P) = (-1)^{n-|S|} \sum_{T \supseteq [1,n] \setminus S} (-1)^{|T|} f_T(P)$ respectively. A graded poset is *Eulerian* if every interval $[x, y]$ of positive rank in it satisfies $\sum_{x \leq z \leq y} (-1)^{\rho(z)} = 0$. All linear relations holding for the flag f-vector of an arbitrary Eulerian poset of rank n were determined by Bayer and Billera in [2]. These linear relations were rephrased by J. Fine as follows (see the paper [5] by Bayer and Klapper). For any $S \subseteq \{1, \ldots, n\}$ define the non-commutative monomial $u_S = u_1 \ldots u_n$ by setting

$$u_i = \begin{cases} b & \text{if } i \in S, \\ a & \text{if } i \notin S. \end{cases}$$

Then the polynomial $\Psi_{ab}(P) = \sum_S h_S u_S$ in non-commuting variables a and b, called the *ab-index of P*, is a polynomial of $c = a + b$ and $d = ab + ba$. This form of $\Psi_{ab}(P)$ is called the *cd-index of* P. Further proofs of the existence of the cd-index may be found in [4], in [11], and in [17]. It was noted by Stanley in [17] that the existence of the cd-index is equivalent to saying that the ab-index rewritten as a polynomial of $c = a + b$ and $e = a - b$ involves only even powers of e. It was observed by Bayer and Hetyei in [3] that the coefficients of the resulting *ce-index* may be computed using a formula that is analogous to the definition of the flag ℓ-vector. In fact, given a ce-word $u_1 \cdots u_n$, let S be the set of positions i satisfying $u_i = e$. Then the coefficient $L_S(P)$ of the ce-word is given by

$$L_S(P) = (-1)^{n-|S|} \sum_{T \supseteq [1,n] \setminus S} \left(\frac{-1}{2} \right)^{|T|} f_T(P). \tag{1}$$

The fact that the ce-index is a polynomial of c and e^2 is equivalent to stating that $L_S(P) = 0$ unless S is an *even* set, that is, a union of disjoint intervals of even cardinality.

A poset P is called *near-Eulerian* if it may be obtained from an Eulerian poset $\widetilde{\Sigma} P$, called the *semisuspension* of P, by removing one coatom. The poset $\widetilde{\Sigma} P$ may be uniquely reconstructed from P by adding a coatom x which covers all $y \in P$ for which $[y, \widehat{1}]$ is the three element chain.

1.2 Spherical shellings

The posets we consider in this paper may be represented as face posets of CW-spheres. We call a poset P with $\widehat{0}$ a *CW-poset* when for all $x > \widehat{0}$ in P the geometric realization $|(\widehat{0}, x)|$ of the open interval $(\widehat{0}, x)$ is homeomorphic to a sphere. By [7], P is a CW-poset if and only if it is the face poset $P(\Omega)$ of a regular CW-complex Ω. When Ω is a CW-sphere, the poset $P_1(\Omega)$, obtained from Ω by adding a unique maximum element $\widehat{1}$, is Eulerian. Stanley observed the following; see [17, Lemma 2.1]. Let Ω be an n-dimensional CW-sphere, and σ an (open) facet of Ω. Let Ω' be obtained from Ω by subdividing the closure $\overline{\sigma}$ of σ into a regular CW-complex with two facets σ_1 and σ_2 such that the boundary $\partial\sigma$ remains the same and $\overline{\sigma_1} \cap \overline{\sigma_2}$ is a regular $(n-1)$-dimensional CW-ball Γ. Then we have

$$\Phi(P_1(\Omega')) \;-\; \Phi(P_1(\Omega)) \;=\; \Phi(\widetilde{\Sigma} P_1(\Gamma)) \cdot c \;-\; \Phi(P_1(\partial\Gamma)) \cdot (c^2 - d). \tag{2}$$

In [17] Stanley uses the above observation to prove that the face poset of a *spherically shellable* CW-sphere has non-negative cd-index. A complex Ω or its face poset $P_1(\Omega)$ is called *spherically shellable* (or S–shellable) if either $\Omega = \{\emptyset\}$ (and so $P_1(\Omega)$ is the two-element chain $\{\widehat{0} < \widehat{1}\}$), or else we can linearly order the facets (open n-cells) $F_1, F_2, \ldots, F_m$ of Ω such that for all $1 \leq i \leq m$ the following two conditions hold:

(S-a) $\partial\overline{F_1}$ is S–shellable of dimension $n - 1$.

(S-b) For $2 \leq i \leq m-1$, let $\Gamma_i := \mathrm{cl}[\partial\overline{F_i} - ((\overline{F_1} \cup \cdots \cup \overline{F_{i-1}}) \cap \overline{F_i})]$. (Here both cl and ‾ denote closure.) Then $P_1(\Gamma_i)$ is near-Eulerian of dimension $n - 1$, and the semisuspension $\widetilde{\Sigma}\Gamma_i$ is S–shellable, with the first facet of the shelling being the facet $\tau = \tau_i$ adjoined to Γ_i to obtain $\widetilde{\Sigma}\Gamma_i$.

1.3 Orthogonal polynomials

For fundamental facts on orthogonal polynomials our main reference is Chihara's book [9]. A *moment functional* $\mathcal{L}$ is a linear map $\mathbb{C}[x] \to \mathbb{C}$. A sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ is an *orthogonal polynomial sequence (OPS)* with respect to $\mathcal{L}$ if $P_n(x)$ has degree n, $\mathcal{L}[P_m(x)P_n(x)] = 0$ for $m \neq n$, and $\mathcal{L}[P_n^2(x)] \neq 0$ for all n. Such a system exists if and only if $\mathcal{L}$ is *quasi-definite* (see [9, Ch. I, Theorem 3.1], the term quasi-definite is introduced in [9, Ch. I, Definition 3.2]). Whenever an OPS exists, each of its elements is determined up to a non-zero constant factor (see [9, Ch. I, Corollary of Theorem 2.2]).

In this paper we consider orthogonal polynomial systems defined recursively. Every monic OPS $\{P_n(x)\}_{n=0}^{\infty}$ may be described by a recurrence formula of the form

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x) \quad n = 1, 2, 3, \ldots \tag{3}$$

where $P_{-1}(x) = 0$, $P_0(x) = 1$, the numbers c_n and λ_n are constants, $\lambda_n \neq 0$ for $n \geq 2$, and λ_1 is arbitrary (see [9, Ch. I, Theorem 4.1]). Conversely, by Favard's theorem [9, Ch. I, Theorem 4.4], for every sequence of monic polynomials defined in the above way there is a unique quasi-definite moment functional $\mathcal{L}$ such that $\mathcal{L}[1] = \lambda_1$ and $\{P_n(x)\}_{n=0}^{\infty}$ is the monic OPS with respect to $\mathcal{L}$.

Due to geometric reasons, the sequences of orthogonal polynomials we consider are *symmetric*, which is equivalent to saying that the coefficients c_n in (3) are all zero, or that $P_n(x) = (-1)^n P_n(-x)$ for all n (see [9, Ch. I, Theorem 4.3]). We will also assume that the coefficients λ_n are *real and positive*. According to the theorems cited above this is equivalent to assuming that $\mathcal{L}$ is *positive definite*, i.e., $\mathcal{L}[\pi(x)] > 0$ for every polynomial $\pi(x)$ that is not identically zero and is non-negative for all real x [9, Ch. I, Definition 3.1]. As a consequence of [9, Ch. I, Theorem 5.2] the zeros of the polynomials $P_n(x)$ are all simple and real.

The smallest closed interval $[\xi_1, \eta_1]$ containing all zeros of an OPS is called the *true interval of orthogonality* of the OPS (see [9, Ch. I, Definition 5.2], and the next sentence). One way to estimate this closed interval is by the use of *chain sequences*. A sequence $\{a_n\}_{n=1}^{\infty}$ is a chain sequence if there is a sequence $\{g_k\}_{k=0}^{\infty}$ satisfying $0 \leq g_0 < 1$ and $0 < g_n < 1$ for $n \geq 1$, such that $a_n = (1 - g_{n-1})g_n$ holds for $n \geq 1$ (see [9, Ch. III, Definition 5.1]). According to [9, Ch. III, Exercise 2.1], for a symmetric OPS given by (3), the true interval of orthogonality is $[-a, a]$ where a is the least positive number for which $\{a^{-2}\lambda_{n+1}\}_{n=1}^{\infty}$ is a chain sequence. (This exercise an easy consequence of [9, Ch. III, Theorem 2.1].)

1.4 Spherical coordinates

Spherical coordinates are often used in mathematical physics and in the theory to of group representations to parameterize the points of an $(n-1)$-dimensional sphere. In this paper we consider the *standard* $(n-1)$*-sphere* $\{(x_1,\ldots,x_n) \ :\ x_1^2+\cdots+x_n^2=1\}$ in an n-dimensional Euclidean space. We parameterize this sphere using the the set of spherical vectors $\{(\theta_1,\ldots,\theta_{n-1}) \ :\ 0\leq\theta_1,\ldots,\theta_{n-2}\leq\pi, 0\leq\theta_{n-1}\leq 2\pi\}$, as given by the system of equations

$$\begin{array}{rcl} x_1 & = & \cos(\theta_1) \\ x_2 & = & \sin(\theta_1)\cos(\theta_2) \\ & \vdots & \\ x_i & = & \sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{i-1})\cos(\theta_i) \\ & \vdots & \\ x_{n-1} & = & \sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-2})\cos(\theta_{n-1}) \\ x_n & = & \sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-1}) \end{array} \tag{4}$$

The classical literature (a sample reference is Vilenkin's [19, Chapter IX, p. 435–437]) seems to be satisfied stating about this (or a similar) parameterization that "for almost all points such a system of parameters is uniquely defined". (To be able to make such a statement, the restrictions on the spherical coordinates need to be strengthened somewhat, for example to $\theta_i<\pi$ for $i<n-1$ and $\theta_{n-1}<2\pi$.)

In this paper we study CW-complexes on the unit sphere, whose combinatorial structure is more transparent if we are allowed to choose some vertices to be points with non-unique spherical coordinates. Thus we need to make our statements little more precise. For completeness sake, we sketch some of the proofs.

Definition 1.1 *We call the spherical vectors* $(\theta_1,\ldots,\theta_{n-1})$ *and* $(\theta_1',\ldots,\theta_{n-1}')$ equivalent *if* $\theta_i-\theta_i'$ *is an integer multiple of* 2π *whenever all* $j<i$ *satisfies* $\theta_j\not\in\{0,\pi\}$.

In other words, we read our spherical vectors from left to right, and stop reading once we find the first 0 or π. No matter what coordinates follow, the spherical vector belongs to the same equivalence class, and we make no other identification. For example, for $n=6$, the spherical vector $(\pi/2,1,\pi,2,3)$ is equivalent to $(\pi/2,1,\pi,1,2\pi)$. We represent the equivalence class of these spherical vectors by $(\pi/2,1,\pi,*,*)$, i.e., we replace the coordinates that "do not matter" with a star. If we are forced to read our vectors till the end, we identify 0 and 2π in the last coordinate. Note also that in this paper we require every n-dimensional spherical vector to belong to $[0,\pi]^{n-1}\times[0,2\pi]$, other sources may use different restrictions.

Definition 1.2 *Assuming $\theta_\ell \in \{0,\pi,2\pi\}$ for some $\ell \leq n-1$, and $\theta_i \notin \{0,\pi,2\pi\}$ for all $i < \ell$, we call the code $(\theta_1,\ldots,\theta_\ell,*,\ldots,*)$ the* simplified code *of the corresponding equivalence class of spherical vectors, and ℓ the* length *of the class, denoted by $\ell(\theta_1,\ldots,\theta_\ell,*,\ldots,*)$. If $\theta_i \notin \{0,\pi,2\pi\}$ for all i, we set $\ell(\theta_1,\ldots,\theta_{n-1}) := n$.*

Proposition 1.3 *The system of equations (4) defines a bijection between equivalence classes of spherical coordinates and points of the unit sphere.*

Proof: The fact that $x_1^2+\cdots+x_n^2 = 1$ for the x_i's given by (4) is well-known and straightforward. Hence we may consider the (4) as the definition of a map

$$\Xi : \{(\theta_1,\ldots,\theta_{n-1}) : 0 \leq \theta_1,\ldots,\theta_{n-2} \leq \pi, 0 \leq \theta_{n-1} \leq 2\pi\} \to \left\{(x_1,\ldots,x_n) : \sum_{i=1}^n x_i^2 = 1\right\}.$$

If θ_i is 0 or π for some i then $x_{i+1} = \cdots = x_n = 0$ no matter what the subsequent spherical coordinates are, so Φ takes equivalent spherical vectors into the same point. The verification of the fact that Φ takes different equivalence classes into different points is straightforward. Surjectivity may be shown by and easy induction on n. $\Diamond$

Introducing $\widetilde{x_k} = \sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_k)$ for $k = 1,2,\ldots,n-1$, it is easy to show that

$$\widetilde{x_k} = \begin{cases} \sqrt{1-x_1^2-\cdots-x_k^2} & \text{if } 1 \leq k \leq n-2, \\ x_n & \text{if } k = n-1. \end{cases}$$

Obviously the length of an equivalence class of spherical vectors determines the length of the "tail of zeros" at the end of the rectangular representation:

Proposition 1.4 *An equivalence class of spherical vectors has length $\ell \leq n-1$ if and only if the rectangular representation of the same point satisfies $x_{\ell+1} = \cdots = x_n = 0$ and $x_\ell \neq 0$. The length is n exactly when $x_n \neq 0$.*

Note next that subjecting θ_{n-1} to the same restrictions as the other coordinates, i.e., restricting θ_{n-1} to $0 \leq \theta_{n-1} \leq \pi$ is equivalent to setting $x_n \geq 0$. In other words:

Proposition 1.5 *The restriction of the parameterization (4) to $\{(\theta_1,\ldots,\theta_{n-1}) : 0 \leq \theta_1,\ldots,\theta_{n-1} \leq \pi\}$ yields the hemisphere $\{(x_1,\ldots,x_n) : x_1^2+\cdots+x_n^2 = 1, x_n \geq 0\}$ as its surjective image.*

Finally, the boundary of this hemisphere is again a sphere:

Proposition 1.6 *The set of points that are representable with spherical coordinates* $(\theta_1, \ldots, \theta_{n-1})$ *satisfying* $\theta_{n-1} \in \{0, \pi\}$ *is the* $(n-2)$*-sphere* $\{(x_1, \ldots, x_n) : x_1^2 + \cdots + x_n^2 = 1, x_n = 0\}$*. The restriction of the projection* $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$ *to this sphere is a homeomorphism with the standard* $(n-2)$*-sphere, which is may be described at the level of spherical coordinates by*

$$\Pi_n : (\theta_1, \ldots, \theta_{n-1}) \mapsto \begin{cases} (\theta_1, \ldots, \theta_{n-3}, \theta_{n-2}) & \text{if } \theta_{n-1} = 0, \\ (\theta_1, \ldots, \theta_{n-3}, 2\pi - \theta_{n-2}) & \text{if } \theta_{n-1} = \pi. \end{cases}$$

In fact, $x_n = 0$ is equivalent to stating that the length of the corresponding spherical vector is at most $n-1$, which is equivalent to allowing $\theta_{n-1} \in \{0, \pi\}$. Comparing the parameterization (4) for the standard $(n-2)$-sphere and its embedding into the hyperplane $x_n = 0$ yields that the first $n-3$ spherical coordinates may be identified, while the only role of choosing $\theta_{n-1} \in \{0, \pi\}$ in the embedded version is to set the sign of x_{n-2} properly: $\theta_{n-1} = 0$ corresponds to $x_{n-2} \geq 0$ while $\theta_{n-1} = \pi$ corresponds to $x_{n-2} \leq 0$. Precisely the same goal may be achieved by replacing $\theta_{n-2} \in [0, \pi]$ with $2\pi - \theta_{n-2} \in [\pi, 2\pi]$ when necessary.

2 The lune complex $L(m_1, \ldots, m_n)$

In this section we construct a spherical CW-complex $L(m_1, \ldots, m_n)$ whose ce-index we use to generalize certain sequences of orthogonal polynomials in Section 3. As a first step, consider the following r-dimensional lunes and hemispheres.

Proposition 2.1 *Assume* $0 \leq r \leq n-2$ *is an integer. Given* $\sigma_i \in [0, \pi]$ *for* $r+1 \leq i \leq n-2$ *and* $\sigma_{n-1} \in [0, 2\pi]$*, the set of spherical vectors*

$$(*, *, \ldots, *, \sigma_{r+1}, \ldots, \sigma_{n-1}) := \{(\theta_1, \ldots, \theta_{n-1}) : 0 \leq \theta_1, \ldots, \theta_r \leq \pi, \quad \theta_i = \sigma_i \quad \text{for } i \geq r\}$$

is an r*-dimensional hemisphere, i.e., the intersection of an* r*-dimensional sphere centered at the origin with a half-space whose boundary contains the origin.*

Proof: We proceed by induction on $n-1-r$. If $r = n-2$ then, by (4), there is no restriction on $x_1, \ldots, x_{n-2}$, while the last two rectangular coordinates are constrained by $x_{n-1} = \cos(\sigma_{n-1}) \cdot \widetilde{x}_{n-2}$ and $x_n = \sin(\sigma_{n-1}) \cdot \widetilde{x}_{n-2}$. It is easy to show that these restrictions are equivalent to setting

$$\begin{aligned} x_1^2 + \cdots + x_n^2 &= 1, \\ -\sin(\sigma_{n-1}) \cdot x_{n-1} + \cos(\sigma_{n-1}) \cdot x_n &= 0, \quad \text{and} \\ \cos(\sigma_{n-1}) \cdot x_{n-1} + \sin(\sigma_{n-1}) \cdot x_n &\geq 0. \end{aligned} \tag{5}$$

In fact, the second equation is equivalent to guaranteeing that the vector (x_{n-1}, x_n) is a multiple of $(\cos(\sigma_{n-1}), \sin(\sigma_{n-1}))$, the first restricts its value to $\pm(\cos(\sigma_{n-1}) \cdot \widetilde{x}_{n-2}, \sin(\sigma_{n-1}) \cdot \widetilde{x}_{n-2})$, while the last one picks the correct sign.

The resulting hemisphere may be parameterized by $(\theta_1, \ldots, \theta_{n-2})$ in spherical coordinates and $(x_1, \ldots, x_{n-2}, \widetilde{x}_{n-2})$ in rectangular coordinates. This parameterization represents the hemisphere as the set of vectors with non-negative last coordinate. The above argument does not change if we start with the hemisphere given by $x_n \geq 0$ instead of the entire sphere. Hence we may repeat it to prove our claim for $r = n-3$, and so on. $\Diamond$

Proposition 2.2 *Assume* $1 \leq r \leq n-2$ *is an integer, and* $0 \leq \alpha < \beta \leq \pi$ *are such that* $[\alpha, \beta] \neq [0, \pi]$. *Given* $\sigma_i \in [0, \pi]$ *for* $r+1 \leq i \leq n-2$, *and* $\sigma_{n-1} \in [0, 2\pi]$, *the set of spherical vectors* $(*, \ldots, *, [\alpha, \beta], \sigma_{r+1}, \ldots, \sigma_{n-1})$, *satisfying* $\alpha \leq \theta_r \leq \beta$ *and* $\theta_i = \sigma_i$ *for* $i \geq r+1$, *is an* r*-dimensional closed region. The boundary of this region is the union of the* $(r-1)$*-dimensional hemispheres* $(*, \ldots, *, \alpha, \sigma_{r+1}, \ldots, \sigma_{n-1})$ *and* $(*, \ldots, *, \beta, \sigma_{r+1}, \ldots, \sigma_{n-1})$. *Similarly, for* $r = n-1$, *given* $0 \leq \alpha < \beta \leq 2\pi$, *where* $[\alpha, \beta] \neq [0, 2\pi]$, *the set of spherical vectors* $(*, \ldots, *, [\alpha, \beta])$ *defined by* $\alpha \leq \theta_{n-1} \leq \beta$ *is an* $(n-1)$*-dimensional closed region with boundary* $(*, \ldots, *, \alpha) \cup (*, \ldots, *, \beta)$.

Proof: In analogy to the proof of Proposition 2.1 we may proceed by induction on $n-1-r$ and the only interesting case is the induction basis $r = n-1$, since the lower dimensional cases may be obtained by reparameterizing the hemispheres obtained along the way. Again, there is no essential restriction on $x_1, \ldots, x_{n-2}$. Let us fix these coordinates. Then $\theta_{n-1} \in [\alpha, \beta]$ is equivalent to stating that the vector (x_{n-1}, x_n) is on an arc of radius $\widetilde{x}_{n-2}$ with endpoints corresponding $\widetilde{x}_{n-2} \cdot (\cos(\alpha), \sin(\alpha))$ and $\widetilde{x}_{n-2} \cdot (\cos(\beta), \sin(\beta))$. Equivalently, (x_{n-1}, x_n) is either $(0,0)$, or it is on the same side of the line connecting $(0,0)$ with $(\cos(\alpha), \sin(\alpha))$ as $(\cos(\beta), \sin(\beta))$, and vice versa. In analogy to (5) we may obtain the following equivalent description of $(*, \ldots, *, [\alpha, \beta])$:

$$\begin{array}{rcl} x_1^2 + \cdots + x_n^2 & = & 1, \\ \sin(\beta - \alpha) \cdot (-\sin(\alpha) \cdot x_{n-1} + \cos(\alpha) \cdot x_n) & \geq & 0, \quad \text{and} \\ \sin(\alpha - \beta) \cdot (-\sin(\beta) \cdot x_{n-1} + \cos(\beta) \cdot x_n) & \geq & 0. \end{array} \tag{6}$$

Hence $(*, \ldots, *, [\alpha, \beta])$ is the intersection of two half-spaces, containing the origin on their boundary, and of the unit $(n-1)$-sphere. The boundary of the resulting region is the intersection of the $(n-1)$-sphere with either of the hyperplanes defining the two half-spaces. $\Diamond$

Generalizing the 3-dimensional terminology, we call a region

$$(*, \ldots, *, [\alpha, \beta], \sigma_{r+1}, \ldots, \sigma_{n-1})$$

an r*-dimensional lune*. Obviously, each equivalence class of spherical vectors is either completely contained in a lune $(*,\ldots,*,[\alpha,\beta],\sigma_{r+1},\ldots,\sigma_{n-1})$ or it is disjoint from it. Hence we may extend our equivalence relation to the code of the lunes considered in the obvious way.

Corollary 2.3 *The* r*-dimensional lunes*

$$(*,\ldots,*,[\alpha,\beta],\sigma_{r+1},\ldots,\sigma_{n-1}) \text{ and } (*,\ldots,*,[\alpha',\beta'],\sigma'_{r+1},\ldots,\sigma'_{n-1})$$

are equal if and only if $\alpha=\alpha'$, $\beta=\beta'$, *and* $\sigma_i=\sigma'_i$ *whenever* $\sigma_j\notin\{0,\pi\}$ *holds for* $r+1\le j<i$.

Thus we may extend our simplified notation for equivalence classes of spherical vectors to lunes. For example, for $n=6$, the lune $(*,[1,2],2,\pi,3)$ is equal to $(*,[1,2],2,\pi,\sqrt{2})$. Both codes of this same 2-dimensional lune may be simplified to $(*,[1,2],2,\pi,*)$. Using this simplified notation, every lune considered has a unique code of the form

$$(*,\ldots,*,[\alpha,\beta],\sigma_{r+1},\ldots,\sigma_\ell,*,\ldots,*)$$

where $\sigma_i\notin\{0,\pi,2\pi\}$ for $r+1\le i\le\min(\ell-1,n-1)$, and $\sigma_\ell\in\{0,\pi\}$ if $\ell\le n-2$.

Definition 2.4 *Extending Definition 1.2, we call* $(*,\ldots,*,[\alpha,\beta],\sigma_{r+1},\ldots,\sigma_\ell,*,\ldots,*)$ *above the* simplified code *of the lune, and* ℓ *its* length *if* $\sigma_\ell\in\{0,\pi,2\pi\}$. *We set the length to be* n *if* $r=n-1$ *or the simplified code is* $(*,\ldots,*,[\alpha,\beta],\sigma_{r+1},\ldots,\sigma_{n-1})$ *where* $\sigma_{n-1}\notin\{0,\pi,2\pi\}$.

It is easy to verify the length of a lune is greater or equal to the length of any lune or spherical vector contained in it.

Remark 2.5 A hemisphere $(*,*,\ldots,*,\sigma_{r+1},\ldots,\sigma_{n-1})$ may be considered as a generalized lune
$(*,\ldots,*,[\alpha,\beta],\sigma_{r+1},\ldots,\sigma_{n-1})$, satisfying $\alpha=0$ and $\beta=\pi$. (This was exactly the excluded choice of α and β above.) In the study of lune complex $L(m_1,\ldots,m_n)$ we will use the obvious homeomorphism

$$\phi^r_{\alpha,\beta}\big|_{(*,\ldots,*,[\alpha,\beta],\sigma_{r+1},\ldots,\sigma_{n-1})} : (*,\ldots,*,[\alpha,\beta],\sigma_{r+1},\ldots,\sigma_{n-1})\to(*,\ldots,*,\sigma_{r+1},\ldots,\sigma_{n-1})$$

given by

$$\phi^r_{\alpha,\beta}\left((\theta_1,\ldots,\theta_{n-1})\right):=\left(\theta_1,\ldots,\theta_{r-1},(\theta_r-\alpha)\cdot\frac{\pi}{\beta-\alpha},\theta_{r+1},\ldots,\theta_{n-1}\right).$$

Definition 2.6 *Given a vector of positive integers* $(m_1, \ldots, m_{n-1})$ *satisfying* $m_i \geq 2$, *we define the* lune complex $L(m_1, \ldots, m_{n-1})$ *as the following CW-complex on the* $(n-1)$-*sphere:*

(i) Its vertices are all points with spherical coordinates $(t_1 \cdot \frac{\pi}{m_1}, \ldots, t_{n-2} \cdot \frac{\pi}{m_{n-2}}, t_{n-1} \cdot \frac{2\pi}{m_{n-1}})$, *where each* $t_j \in [0, m_j]$ *is an integer.*

(ii) For $1 \leq r \leq n-2$, *its* r*-dimensional faces are lunes*

$$\left(*, \ldots, *, \left[s_r \cdot \frac{\pi}{m_r}, (s_r+1) \cdot \frac{\pi}{m_r}\right], s_{r+1} \cdot \frac{\pi}{m_{r+1}}, \ldots, s_{n-2} \cdot \frac{\pi}{m_{n-2}}, s_{n-1} \cdot \frac{2\pi}{m_{n-1}}\right),$$

where each s_j *is an integer,* $s_r \in [0, m_r - 1]$, *and* $s_j \in [0, m_j]$ *for* $j > r$.

(iii) Its $(n-1)$*-faces (facets) are lunes* $\left(*, \ldots, *, \left[s_{n-1} \cdot \frac{2\pi}{m_{n-1}}, (s_{n-1}+1) \cdot \frac{2\pi}{m_{n-1}}\right]\right)$, *where* $0 \leq s_{n-1} \leq m_{n-1} - 1$ *is an integer.*

Theorem 2.7 *Assuming* $m_1, \ldots, m_{n-1} \geq 2$, *the lune complex* $L(m_1, \ldots, m_{n-1})$ *is a CW-complex, homeomorphic to an* $(n-1)$*-sphere.*

Proof: Observe first that the union of the facets indeed covers the $(n-1)$-sphere, and that the intersection of any two facets is either empty or a hemisphere of the form $(*, \ldots, *, t_{n-1} \cdot \frac{2\pi}{m_{n-1}})$, where t_{n-1} is any integer from $[0, m_{n-1}]$. This set is a union of $(n-2)$-faces:

$$\left(*, \ldots, *, t_{n-1} \cdot \frac{2\pi}{m_{n-1}}\right) = \bigcup_{s_{n-1}=0}^{m_{n-1}-1} \left(*, \ldots, *, \left[s_{n-2} \cdot \frac{\pi}{m_{n-1}}, (s_{n-2}+1) \cdot \frac{\pi}{m_{n-1}}\right], t_{n-1} \cdot \frac{2\pi}{m_{n-1}}\right).$$

Rather than repeating a similar argument in lower dimensions, let us observe that the faces contained in any facet replicate the face structure of $L(m_1, \ldots, m_{n-2}, 2 \cdot m_{n-1})$. For that purpose, consider a facet $F := \left(*, \ldots, *, \left[s_{n-1} \cdot \frac{2\pi}{m_{n-1}}, (s_{n-1}+1) \cdot \frac{2\pi}{m_{n-1}}\right]\right)$. As noted in Remark 2.5, the homeomorphism

$$\phi^{n-1}_{s_{n-1} \cdot \frac{2\pi}{m_{n-1}}, (s_{n-1}+1) \cdot \frac{2\pi}{m_{n-1}}} : (\theta_1, \ldots, \theta_{n-1}) \mapsto \left(\theta_1, \ldots, \theta_{n-2}, \left(\theta_{n-1} - s_{n-1} \cdot \frac{2\pi}{m_{n-1}}\right) \cdot \frac{m_{n-1}}{2}\right)$$

sends the facet F into $(*, \ldots, *, [0, \pi])$, and its boundary into $(*, \ldots, *, 0) \cup (*, \ldots, *, \pi)$. The boundary $(*, \ldots, *, 0) \cup (*, \ldots, *, \pi)$ of the hemisphere $(*, \ldots, *, [0, \pi])$ is an $(n-2)$-dimensional sphere, let us use the homeomorphism Π_n defined in Proposition 1.5 to send this into the standard $(n-2)$-sphere. It is easy to verify that $\Pi_n \circ \phi^{n-1}_{s_{n-1} \cdot \frac{2\pi}{m_{n-1}}, (s_{n-1}+1) \cdot \frac{2\pi}{m_{n-1}}}$ establishes a bijection between the faces contained in F and the faces of $L(m_1, \ldots, m_{n-3}, 2 \cdot m_{n-2})$. By induction we may thus state that the faces properly contained in F form a CW-complex covering the boundary of F. $\Diamond$

As a consequence of our proof we see that the lune complexes $L(m_1, \ldots, m_{n-1})$ have the following recursive property:

Corollary 2.8 *The poset of all faces contained in an arbitrary facet of $L(m_1, \ldots, m_{n-1})$ is isomorphic to the face poset of $L(m_1, \ldots, m_{n-3}, 2 \cdot m_{n-2})$.*

Example 2.9 Figure 1 represents the lune complex $L(3, 3)$. It has 8 vertices (of which 6 are visible on the picture, the invisible ones are marked with an empty circle), 9 edges (the 3 invisible ones are marked with dashed lines), and 3 facets (of which only $(*, [2\pi/3, 4\pi/3])$ is entirely visible.) The boundary of each facet is a hexagon, isomorphic to $L(6)$.

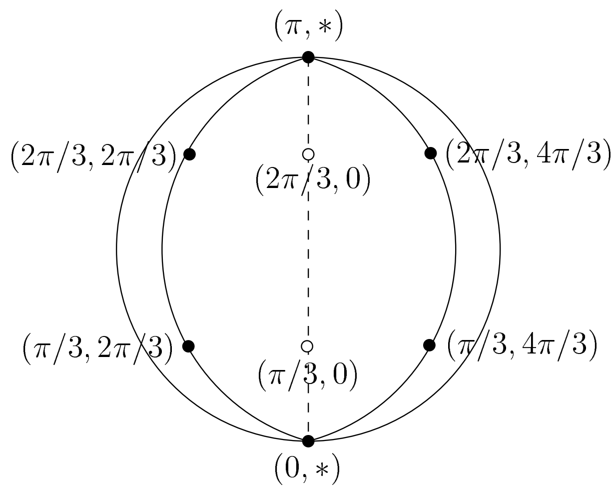


Figure 1: The complex $L(3, 3)$

We conclude this section with another embedding result that sometimes complements the role of Corollary 2.8.

Proposition 2.10 *The partially ordered set of faces of length at most $n - 2$ of $L(m_1, \ldots, m_{n-1})$ is isomorphic to the face poset of $L(m_1, \ldots, m_{n-4}, 2 \cdot m_{n-3})$.*

Proof: If a spherical vector has length at most $n - 2$ then the $(n - 2)$-nd coordinate in its simplified code is 0, π, or $*$, and the last coordinate is always $*$. Removing the last coordinate from all such spherical vectors establishes a bijection with the hemisphere $\{(x_1, \ldots, x_{n-1}) : x_1^2 + \cdots + x_{n-1}^2 = 1, x_{n-1} = 0\}$. Consider the projection Π_{n-1} described in Proposition 1.6, taking this set into the standard $(n - 3)$-sphere. This projection, combined with removing the last star, takes a face with code

$$\left(*, \ldots, *, \left[s_r \cdot \frac{\pi}{m_r}, (s_r + 1) \cdot \frac{\pi}{m_r}\right], s_{r+1} \cdot \frac{\pi}{m_{r+1}}, \ldots, s_{n-3} \cdot \frac{\pi}{m_{n-3}}, 0, *\right)$$

into

$$\left(*,\ldots,*,\left[s_r\cdot\frac{\pi}{m_r},(s_r+1)\cdot\frac{\pi}{m_r}\right],s_{r+1}\cdot\frac{\pi}{m_{r+1}},\ldots,s_{n-3}\cdot\frac{\pi}{m_{n-3}}\right),$$

and a face with code

$$\left(*,\ldots,*,\left[s_r\cdot\frac{\pi}{m_r},(s_r+1)\cdot\frac{\pi}{m_r}\right],s_{r+1}\cdot\frac{\pi}{m_{r+1}},\ldots,s_{n-3}\cdot\frac{\pi}{m_{n-3}},\pi,*\right)$$

into

$$\left(*,\ldots,*,\left[s_r\cdot\frac{\pi}{m_r},(s_r+1)\cdot\frac{\pi}{m_r}\right],s_{r+1}\cdot\frac{\pi}{m_{r+1}},\ldots,(2m_{n-3}-s_{n-3})\cdot\frac{\pi}{m_{n-3}}\right).$$

Considering also the other possible face codes, the statement becomes a trivial verification of the definitions. $\diamondsuit$

3 The flag f-vector of the lune complex

In this section we present a fundamental recursion formula for the flag f-vectors of Eulerian posets of the form $P_1\left(L(m_1,\ldots,m_{n-1})\right)$. (Since the lune complexes are CW-spheres, the partially ordered sets considered are in fact Eulerian.) To simplify our notation, for every CW-sphere Ω we will use $f_S(\Omega)$ as a shorthand for $f_S(P_1(\Omega))$. This can not lead to confusion, since every saturated chain enumerated in $f_S(P_1(\Omega))$ contains $\widehat{1}$, so this element may be removed from all chains at once and we are left with an equivalent enumeration question.

Proposition 3.1 *For $n\geq 4$ and $S\neq\emptyset$ the flag number $f_S(L(m_1,\ldots,m_{n-1}))$ is equal to*

$$\frac{m_{n-1}}{2}\cdot f_S(L(m_1,\ldots,m_{n-3},2m_{n-2}))-\frac{m_{n-1}-2}{2}\cdot f_S(L(m_1,\ldots,m_{n-4},2m_{n-3}))$$

if $S\cap\{n,n-1\}=\emptyset$, and

$$2^{|S\cap\{n\}|}\cdot\frac{m_{n-1}}{2}\cdot f_{S\setminus\{n\}}(L(m_1,\ldots,m_{n-3},2m_{n-2}))$$

if $S\cap\{n,n-1\}\neq\emptyset$.

Proof: Consider the case $S\cap\{n,n-1\}\neq\emptyset$ first. Every sub-coatom in an Eulerian poset is covered by exactly two atoms. Hence, a set S not containing n (but containing $n-1$) satisfies

$$f_{S\cup\{n\}}(L(m_1,\ldots,m_{n-1}))=2\cdot f_S(L(m_1,\ldots,m_{n-1})).$$

By this observation, if our formula is correct when $n \in S$ then it is also correct in the case when $n \notin S$ but $n-1 \in S$. Therefore, w.l.o.g. we may assume $n \in S$. First we choose the facet F in the S-chain and then the rest below it. There are m_{n-1} ways to choose F. By Corollary 2.8, the interval $[\widehat{0}, F]$ is isomorphic to the face poset of $L(m_1, \ldots, m_{n-3}, 2m_{n-2})$. Hence there are $f_{S \setminus \{n\}}(L(m_1, \ldots, m_{n-3}, 2m_{n-2})$ options to choose the rest of the S-chain.

Assume from now on $S \cap \{n, n-1\} = \emptyset$. We distinguish two sub-cases depending on whether the top element of the S-chain has length at least $n-1$ or less. If the length of the top element is at least $n-1$ then its simplified code has last coordinate $t_{n-1} \cdot \frac{2\pi}{m_{n-1}}$ for some integer $t_{n-1} \in [0, m_{n-1}]$, and it is contained in precisely two facets of $L(m_1, \ldots, m_{n-1})$:

$$\begin{array}{rl} \text{in} & \left(*, \ldots, *, \left[t_{n-1} \cdot \frac{2\pi}{m_{n-1}}, (t_{n-1}+1) \cdot \frac{2\pi}{m_{n-1}}\right]\right), \\ \text{and in} & \left(*, \ldots, *, \left[(t_{n-1}-1) \cdot \frac{2\pi}{m_{n-1}}, t_{n-1} \cdot \frac{2\pi}{m_{n-1}}\right]\right). \end{array}$$

(If t_{n-1} is 0 resp. m_{n-1} then $t_{n-1}-1$ resp. $t_{n-1}+1$ must be understood "modulo m_{n-1}".) If we count each such chain below each facet, then we count each such chain exactly twice. By Corollary 2.8, below each facet we have a copy of $L(m_1, \ldots, m_{n-3}, 2m_{n-2})$, hence each such chain is counted in $\frac{m_{n-1}}{2} \cdot f_S(L(m_1, \ldots, m_{n-3}, 2m_{n-2}))$ exactly once. If the length of the top element is less than $n-1$, then it (and the rest of the chain) is contained in all facets of $L(m_1, \ldots, m_{n-1})$. Such chains are thus overcounted in $\frac{m_{n-1}}{2} \cdot f_S(L(m_1, \ldots, m_{n-3}, 2m_{n-2}))$ precisely $\frac{m_{n-1}}{2} - 1$ times. By Proposition 2.10 the effect of this overcounting may be offset by subtracting $(m_{n-1}-2)/2 \cdot f_S(L(m_1, \ldots, m_{n-4}, 2m_{n-3}))$. $\diamondsuit$

We may transform Proposition 3.1 into the following recursion formula for the ce-index:

Proposition 3.2 *The ce-index of $L(m_1, \ldots, m_{n-1})$ satisfies*

$$\begin{aligned} \Phi_{ce}(L(m_1, \ldots, m_{n-1})) &= \frac{m_{n-1}}{2} \cdot \Phi_{ce}(L(m_1, \ldots, m_{n-3}, 2m_{n-2})) \cdot c \\ &\quad - \frac{m_{n-1}-2}{2} \cdot \Phi_{ce}(L(m_1, \ldots, m_{n-4}, 2m_{n-3})) \cdot e^2 \end{aligned}$$

for $n \geq 4$.

Proof: Considering (1) it is sufficient to prove the appropriate formula for each entry L_S in the flag L-vector of the posets involved. We may restrict our attention to even sets S (since the ce-index is a polynomial of c and e^2).

Assume first $n \notin S$. Then every set T containing $[1, n] \setminus S$ contains $\{n\}$, and so applying Proposition 3.1 to the right hand side of

$$L_S(L(m_1, \ldots, m_{n-1})) = (-1)^{n-|S|} \sum_{T \supseteq [1,n] \setminus S} \left(\frac{-1}{2}\right)^{|T|} f_T(L(m_1, \ldots, m_{n-1})) \qquad (7)$$

yields only terms of the form $2\cdot\frac{m_{n-1}}{2}\cdot f_{T\setminus\{n\}}(L(m_1,\ldots,m_{n-3},2m_{n-2}))$. Replacing T with $T\setminus\{n\}$ in each summand on the right hand side yields

$$(-1)^{n-|S|}\sum_{[1,n-1]\setminus S\subseteq T\subseteq[1,n-1]}\left(\frac{-1}{2}\right)^{|T|}2\cdot\frac{m_{n-1}}{2}\cdot f_T(L(m_1,\ldots,\ldots,m_{n-3},2m_{n-2}))$$

which is exactly $\frac{m_{n-1}}{2}\cdot L_S(L(m_1,\ldots,m_{n-3},2m_{n-2}))$. Therefore our recursion formula is true for the coefficients of ce words ending with c.

We are left with the case $n\in S$, i.e., the proof of our statement for the coefficients of ce-words ending with e . Since S must be even, in this case S must also contain $n-1$, and $[1,n]\setminus S$ is a subset of $[1,n-2]$. The right hand side of (7) may be rewritten as

$$(-1)^{n-|S|}\sum_{[1,n\setminus S\subseteq T\subseteq[1,n-2]}\left(\frac{-1}{2}\right)^{|T|}\left(f_T(L(m_1,\ldots,m_{n-1}))-\frac{1}{2}f_{T\cup\{n-1\}}(L(m_1,\ldots,m_{n-1}))\right.$$
$$\left.-\frac{1}{2}f_{T\cup\{n\}}(L(m_1,\ldots,m_{n-1}))-\frac{1}{4}f_{T\cup\{n-1,n\}}(L(m_1,\ldots,m_{n-1}))\right)$$

Applying Proposition 3.1 to each term in this sum, the multiples of the terms $f_T(L(m_1,\ldots,m_{n-3},2m_{n-2}))$ cancel, and the multiples of the terms $f_T(L(m_1,\ldots,m_{n-3},2m_{n-2}))$ add up to

$$L_S(L(m_1,\ldots,m_{n-1}))=-\frac{m_{n-1}-2}{2}L_{S\setminus\{n-1,n\}}(L(m_1,\ldots,m_{n-3},2m_{n-2})).$$

$\diamondsuit$

Introducing the symbol $L()$ for the zero-dimensional sphere (with two vertices), the recursion formulas of Propositions 3.1 and 3.2 can be easily shown to extend to the case $n=3$. The ce-index of the (face poset of) $L(m_1,\ldots,m_{n-1})$ for $n\le 4$ is shown in Table 1. (Note that $L(m_1)$ is a cycle with m_1 vertices and m_1 edges.)

4 Spherical shellability of the lune complex

The main result of this section is the following.

Theorem 4.1 *For $n\ge 2$ and $m_1,\ldots,m_{n-2}\ge 0$, the lune complex $L(m_1,\ldots,m_{n-1})$ is spherically shellable.*

Proof: We proceed by induction on n and the m_i's. For $n=2$ the lune complex $L(m_1)$ is a cycle with m_1 vertices and m_1 edges, easily shown to be spherically shellable. Consider

$$
\begin{aligned}
\Psi_{ce}(\emptyset) &= 1 \\
\Psi_{ce}(L()) &= c \\
\Psi_{ce}(L(m_1)) &= \frac{m_1}{2}c^2 - \left(\frac{m_1}{2} - 1\right) e^2 \\
\Psi_{ce}(L(m_1, m_2)) &= m_1 \frac{m_2}{2} c^3 - (m_1 - 1)\frac{m_2}{2} e^2 c - \left(\frac{m_2}{2} - 1\right) ce^2 \\
\Psi_{ce}(L(m_1, m_2, m_3)) &= m_1 m_2 \frac{m_3}{2} c^4 - (m_1 - 1) m_2 \frac{m_3}{2} e^2 c^2 - (m_2 - 1)\frac{m_3}{2} ce^2 c \\
&\quad -m_1 \left(\frac{m_3}{2} - 1\right) c^2 e^2 + (m_1 - 1)\left(\frac{m_3}{2} - 1\right) e^4.
\end{aligned}
$$

Table 1: The ce-index of the face poset of $L(m_1, \ldots, m_{n-1})$ for $n \leq 4$.

next $n \geq 3$ and $m_{n-1} = 2$. The lune complex $L(m_1, \ldots, m_{n-2}, 2)$ has two (closed) facets: the hemisphere $\overline{F_1} = (*, \ldots, [0, \pi])$ and the hemisphere $\overline{F_2}(*, \ldots, [\pi, 2\pi])$. The boundary of both facets is the same, and it is isomorphic to the CW-complex $L(m_1, \ldots, m_{n-3}, 2m_{n-2})$, as noted in the proof of Theorem 2.7. Hence axiom (S-a) is satisfied by the induction hypothesis, while (S-b) is never applicable when we have only two facets.

Assume finally $n \geq 3$ and $m_{n-1} \geq 3$. By our induction hypothesis we may assume that the complex $L(m_1, \ldots, m_{n-1} - 1)$ has an S-shelling $F_1, \ldots, F_{m_{n-1}}$. Due to the "rotational symmetry" (the map $(\theta_1, \ldots, \theta_{n-1}) \mapsto (\theta_1, \ldots, \theta_{n-2}, \theta_{n-1} + 2\pi/(m_{n-1} - 1))$ induces an automorphism of $L(m_1, \ldots, m_{n-1} - 1))$ we may assume that the closure of the last facet is

$$\overline{F_{m_{n-1}-1}} = \left(*, \ldots, *, \left[\frac{m_{n-1} - 2}{m_{n-1} - 1} \cdot 2\pi, 2\pi\right]\right).$$

A homeomorphic copy of $L(m_1, \ldots, m_{n-1})$ may be obtained from $L(m_1, \ldots, m_{n-1} - 1)$ by subdividing $\overline{F_{m_{n-1}-1}}$ into two (closed) facets

$$\overline{F'_{m_{n-1}-1}} = \left(*, \ldots, *, \left[\frac{m_{n-1} - 2}{m_{n-1} - 1} \cdot 2\pi, \frac{m_{n-1} - 1.5}{m_{n-1} - 1} \cdot 2\pi\right]\right) \quad \text{and}$$

$$\overline{F''_{m_{n-1}-1}} = \left(*, \ldots, *, \left[\frac{m_{n-1} - 1.5}{m_{n-1} - 1} \cdot 2\pi, 2\pi\right]\right)$$

and replicating the appropriate face structure at the intersection of the subdividing closed facets. The homeomorphism from the subdivided complex to $L(m_1, \ldots, m_{n-1})$ may be given by

$$(\theta_1, \ldots, \theta_{n-1}) \mapsto (\theta_1, \ldots, \theta_{n-2}, \theta'_{n-1})$$

where

$$\theta'_{n-1} = \begin{cases} \frac{m_{n-1}-1}{m_{n-1}} \cdot \theta_{n-1} & \text{when } 0 \leq \theta_{n-1} \leq \frac{m_{n-1}-2}{m_{n-1}-1} \cdot 2\pi, \\ 2 \cdot \frac{m_{n-1}-1}{m_{n-1}} \cdot \theta_{n-1} + \frac{2-m_{n-1}}{m_{n-1}} \cdot 2\pi & \text{when } \frac{m_{n-1}-2}{m_{n-1}-1} \cdot 2\pi \leq \theta_{n-1} \leq 2\pi. \end{cases}$$

All we need to show then is that $F_1, \ldots, F_{m_{n-2}}, F'_{m_{n-1}}, F''_{m_{n-1}}$ is an S-shelling of the subdivided complex. In other words, we only need to verify that (S-b) is satisfied by the facet $F'_{m_{n-1}}$. The intersection of the boundary of $F'_{m_{n-1}}$ with the closure of the previously added facets is isomorphic to the positive hemisphere $0 \leq \theta_{n-2} \leq \pi$ in $L(m_1, \ldots, m_{n-3}, 2m_{n-2})$. Its semisuspension may be geometrically realized by adding the lune $(*, \ldots, *, [\pi, 2\pi])$ to those facets of $L(m_1, \ldots, m_{n-3}, 2m_{n-2})$ which are contained in the positive hemisphere. Thus the semisuspension is isomorphic to $L(m_1, \ldots, m_{n-3}, m_{n-2}+1)$. This isomorphism is induced by the homeomorphism $(\theta_1, \ldots, \theta_{n-2}) \mapsto (\theta_1, \ldots, \theta_{n-3}, \theta''_{n-2})$ where

$$\theta''_{n-2} = \begin{cases} \frac{2m_{n-2}}{m_{n-2}+1} \cdot \theta_{n-2} & \text{when } 0 \leq \theta_{n-2} \leq \pi, \\ \frac{2\theta_{n-2}}{m_{n-2}+1} + \frac{2(m_{n-2}-1)\pi}{m_{n-2}+1} & \text{when } \pi \leq \theta_{n-2} \leq 2\pi. \end{cases}$$

Again, by our induction hypothesis, $L(m_1, \ldots, m_{n-3}, m_{n-2}+1)$ has an S-shelling and, because of the "rotational symmetry" mentioned above, we may choose any of its facets to be the first one. $\Diamond$

The proof of Theorem 4.1, together with (2) provides the following recursion formula for the cd-index of $L(m_1, \ldots, m_{n-1})$:

$$\begin{aligned} \Phi_{cd}(L(m_1, \ldots, m_{n-1})) \;=\; & \Phi_{cd}(L(m_1, \ldots, m_{n-3}, 2 \cdot m_{n-2}))c \\ & +(m_{n-1}-2)\Phi_{cd}(L(m_1, \ldots, m_{n-3}, m_{n-2}+1))c \\ & -(m_{n-1}-2)\Phi_{cd}(L(m_1, \ldots, m_{n-4}, 2m_{n-3}))(c^2-d). \end{aligned} \tag{8}$$

In fact, every time we increase m_{n-1} by 1, we subdivide a closed facet $\overline{\sigma}$ into two facets $\overline{\sigma_1}$ and $\overline{\sigma_2}$ in such a way that the semisuspension of $\Gamma = \overline{\sigma_1} \cap \overline{\sigma_2}$ is isomorphic to $L(m_1, \ldots, m_{n-3}, m_{n-2}+1)$, while the boundary of Γ is clearly isomorphic to $L(m_1, \ldots, m_{n-4}, 2m_{n-3})$. To obtain $L(m_1, \ldots, m_{n-1})$ from $L(m_1, \ldots, m_{n-2}, 2)$ we need to perform such a subdivision $(m_{n-1}-2)$ times. Finally,

$$\Phi_{cd}(L(m_1, \ldots, m_{n-2}, 2)) = \Phi_{cd}(L(m_1, \ldots, m_{n-3}, 2m_{n-2}))c$$

is obvious.

5 A sequence of orthogonal polynomials represented by lune complexes

In the following we assume that $\nu_1, \nu_2, \ldots$ is an infinite sequence of positive numbers satisfying $\nu_1 = 1$ and $\nu_n \geq 2$ for $n \geq 2$. We define the polynomials $Q_{-1}(x), Q_0(x), Q_1(x), \ldots$ by setting $Q_{-1}(x) = 0$, $Q_0(x) = 1$, and the recursion formula

$$Q_n(x) = \nu_n \cdot x \cdot Q_{n-1}(x) - (\nu_n - 1) \cdot Q_{n-2}(x) \quad \text{for } n \geq 1. \tag{9}$$

(As a consequence, $Q_1(x) = x$.) This sequence is not monic, the leading coefficient of $Q_n(x)$ is $\nu_1 \cdots \nu_n$. Introducing $P_{-1}(x) = 0$ and $P_n(x) = Q_n(x)/(\nu_1 \cdots \nu_n)$ for $n \geq 1$ we obtain the monic and symmetric OPS $\{P_n(x)\}_{n=0}^{\infty}$ satisfying the recurrence

$$P_n(x) = x \cdot P_{n-1}(x) - \frac{\nu_n - 1}{\nu_{n-1}\nu_n} P_{n-2}(x) \quad \text{for } n \geq 1.$$

Since, for each n, $Q_n(x)$ differs from $P_n(x)$ only by a nonzero constant factor, the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ form a (non-monic) symmetric OPS. As an illustration of the power of spherical shellings we provide a new proof of the following theorem.

Theorem 5.1 *Each polynomial $Q_n(x)$ may be written as a non-negative combination of products of powers of x and $(x^2 - 1)$.*

Before explaining how spherical shellability of the lune complexes may be used to prove this theorem, let us review how Theorem 5.1 follows from the classical theory. Introducing $\lambda_1 = 1$ and $\lambda_n = \frac{\nu_n - 1}{\nu_{n-1}\nu_n}$ for $n \geq 2$, the sequence $\{\lambda_{n+1}\}_{n=1}^{\infty}$ is a chain sequence with parameter sequence $g_n = (1 - 1/\nu_n)$. Hence, as noted at the end of the preliminary Section 1.3, the true interval of orthogonality of the OPS $\{P_n(x)\}_{n=0}^{\infty}$ is a subset of $[-1, 1]$. In other words, every zero of each $P_n(x)$ is contained in $[-1, 1]$, and the same holds for the zeros of the $Q_n(x)$, since for every n, the polynomials $Q_n(x)$ and $P_n(x)$ differ at most by a nonzero constant factor. A "classical" proof of Theorem 5.1 may be then concluded using the following observation, due to Ismail and Stanton [14].

Lemma 5.2 *Assume that all zeros of the polynomial $q(x)$ are simple and real, and that $q(x)$ is a linear combination of only even or only odd powers of x. Then the polynomial $q(x)$ is a non-negative linear combination of polynomials of the form $(x^2-1)^n$ or $x(x^2-1)^n$ if and only if all zeros of $q(x)$ lie in the interval $[-1, 1]$.*

Proof: Assume first $q(x)$ is a non-negative linear combination of polynomials of the form $(x^2 - 1)^n$ or $x(x^2 - 1)^n$. Assume also that $q(x)$ is a combination of even powers of x (the odd case is similar). Then only terms of the form $(x^2 - 1)^n$ occur in $q(x)$ and

$q(x) > 0$ when $|x| > 1$. Thus all zeros lie in $[-1, 1]$. Assume, conversely, that all zeros of $q(x)$ lie in $[-1, 1]$. We restrict our attention again to the case of even powers, the odd case is similar. Then $q(x) = (x^2 - r_1^2) \cdots (x^2 - r_n^2)$ for some $r_1, .., r_n \in [0, 1]$. Since $x^2 - r_i^2 = (x^2 - 1) + (1 - r_i^2)$, each factor has a non-negative expansion so the product will too. $\diamondsuit$

For sequences of integer ν_i's, Theorem 5.1 is a relatively easy consequence of Theorem 4.1. To see this, let us introduce the auxiliary polynomials $R_n(y_1, y_2, \ldots, y_{n-1}; x)$ given by $R_0 = 1$, $R_1(x) = x$, $R_2(y_1; x) = x + \dfrac{y_1 - 2}{2}(x^2 - 1)$, and the recursion formula

$$\begin{aligned} R_n(y_1, \ldots, y_{n-1}; x) \quad = \quad & R_{n-1}(y_1, \ldots, y_{n-3}, 2 \cdot y_{n-2}; x)x \\ & +(y_{n-1} - 2)R_{n-1}(y_1, \ldots, y_{n-3}, y_{n-2} + 1; x)x \\ & -(y_{n-1} - 2)R_{n-2}(y_1, \ldots, y_{n-4}, 2y_{n-3}; x)\left(x^2 - \tfrac{x^2-1}{2}\right). \end{aligned} \tag{10}$$

As an immediate consequence of equation (8) we may observe that for any sequence of integers $m_1, m_2, \ldots, m_n, \ldots$, satisfying $m_i \geq 2$ for all i, the value of $R(y_1, \ldots, y_{n-1}; x)$ is the image of
$\Phi_{ce}(L(m_1, \ldots, m_{n-1}))$ under the homomorphism induced by $c \mapsto x$ and $e \mapsto 1$. (Under this homomorphism, $d = (c^2 - e^2)/2$ goes into $(x^2 - 1)/2$.) Applying the same homomorphism to Proposition 3.2 we obtain the recursion formula

$$\begin{aligned} R(m_1, \ldots, m_{n-1}; x) \quad = \quad & \tfrac{m_{n-1}}{2} \cdot R(m_1, \ldots, m_{n-3}, 2m_{n-2}; x) \cdot x \\ & -\tfrac{m_{n-1}-2}{2} \cdot R(m_1, \ldots, m_{n-4}, 2m_{n-3}; x) \end{aligned} \tag{11}$$

Comparing this recursion formula to (9) it follows by trivial induction on n that

$$Q_n(x) = R_n(\nu_2, \nu_3, \ldots, \nu_{n-1}, 2 \cdot \nu_n; x) \tag{12}$$

for any $n \geq 1$, if all the ν_i's are integers and at least 2. As an immediate consequence of Theorem 4.1, the polynomials $R(m_1, \ldots, m_{n-1}; x)$ are non-negative combinations of terms of the form $x^i(x^2 - 1)^j$. (By Stanley's result [17, Theorem 2.2], the cd-index associated to a spherically shellable CW-sphere has nonnegative coefficients.) This concludes the proof for sequences of integer ν_i's.

However, only two small observations are necessary to extend the validity of our new argument to sequences of arbitrary real ν_i's. First, we may observe that equation (12) is valid for any sequence of real numbers $\nu_2, \nu_3, \ldots$. In fact, keeping the ν_i's as variables it is obvious from (9) that $Q_n(x)$ is a polynomial expression of $\nu_2, \ldots, \nu_n$ and x, and so is $R_n(\nu_2, \nu_3, \ldots, \nu_{n-1}, 2 \cdot \nu_n; x)$ in light of (10). Both polynomials agree for infinitely many independent (integer) values of $\nu_2, \ldots, \nu_n$ and x, so they are equal as polynomial expressions. But then they are also equal when we substitute non-integer values as ν_i's. Hence it is sufficient to show that the polynomials $R(r_1, \ldots, r_{n-1}; x)$ are non-negative combinations of terms of the form $x^i(x^2 - 1)^j$ whenever all r_i's are at least 2, even if they are not integers. A slightly stronger statement is easily proven by induction:

Proposition 5.3 *Given $n \geq 1$, and any sequence of real numbers $r_1, \ldots, r_{n-1}$ satisfying $r_i \geq 2$ for all i, the polynomial $R_n(r_1, \ldots, r_{n-1}; x)$ is a non-negative combinations of terms of the form $x^i(x^2-1)^j$. Moreover, increasing r_n while leaving all other r_i's unchanged cannot decrease any coefficient in such a combination.*

Proof: As an immediate consequence of (10) we may write

$$R_n(y_1, \ldots, y_{n-2}, 2; x) = R_{n-1}(y_1, \ldots, y_{n-3}, 2 \cdot y_{n-2}; x)x. \tag{13}$$

Using this observation, we may rewrite (10) as

$$\begin{array}{rcl} R_n(y_1, \ldots, y_{n-1}; x) & = & R_{n-1}(y_1, \ldots, y_{n-3}, 2 \cdot y_{n-2}; x)x \\ & & +(y_{n-1}-2)\left(R_{n-1}(y_1, \ldots, y_{n-3}, y_{n-2}+1; x)\right. \\ & & \quad \left.-R_{n-1}(y_1, \ldots, y_{n-4}, y_{n-3}, 2; x)\right) x \\ & & +(y_{n-1}-2) R_{n-2}(y_1, \ldots, y_{n-4}, 2y_{n-3}; x)\frac{x^2-1}{2}. \end{array} \tag{14}$$

By our induction hypothesis, $R_{n-1}(r_1, \ldots, r_{n-3}, 2r_{n-2}; x)$ and $R_{n-2}(r_1, \ldots, r_{n-4}, 2r_{n-3}; x)$ are non-negative combinations of terms of the form $x^i(x^2-1)^j$ whenever the r_i's are at least 2. The same also holds for the difference

$$R_{n-1}(r_1, \ldots, r_{n-3}, r_{n-2}+1; x) - R_{n-1}(r_1, \ldots, r_{n-4}, r_{n-3}, 2; x)$$

since, by our induction hypothesis, the coefficients cannot decrease when we increase y_{n-2} from 2 to $r_{n-2}+1$. Finally the last variable y_{n-1} occurs only as factor $(y_{n-2}-2)$ in two of the terms, so the coefficients of the terms of the form $x^i(x^2-1)^j$ in $R_n(r_1, \ldots, r_{n-1}; x)$ can not decrease when we increase r_{n-1}. ◊

Remark 5.4 Although the chain-sequence approach, as well as the spherical shelling argument, prove "essentially" the same result, the actual representation that could be obtained following either argument is different. Using Lemma 5.2 we obtain a positive combination where each power of (x^2-1) is multiplied by at most the first power of x, while the spherical shelling approach yields a positive combination of terms of the form $x^{n-2r}(x^2-1)^r$.

Remark 5.5 The careful reader will notice that in the "classical" approach the condition $\nu_i \geq 2$ may be relaxed to $\nu_i > 1$, since this condition already guarantees $0 < 1 - 1/\nu_i < 1$, and so λ_n is a chain sequence. The same is probably also true about the spherical shelling approach, since it is possible to construct lune complexes $L(m_1, \ldots, m_{n-1})$ even when some m_i's satisfying $i < n-1$ are equal to 1. For example, removing all vertices except $(0, *)$ and $(\pi, *)$ from Figure 1 (and merging the edges meeting at the removed vertices) yields the lune complex $L(1,3)$, with 2 vertices: $(0, *)$ and $(\pi, *)$; 3 edges: $([0,\pi], 0)$, $([0,\pi], 2\pi/3)$, and $([0,\pi], 2\pi/3)$; and 3 faces (having the same codes as in $L(3,3)$. As indicated by the notational ambivalence $([0,\pi],0)=(*,0)$, allowing $m_i=1$ would induce some confusion that would need extra consideration at every step. We prefer to avoid this complication in this first presentation of lune complexes. Theorem 5.1 easily follows from the classical theory anyway, and Theorem 6.5, inspired by a more direct approach to the cd-index calculation, gives a much more explicit statement under the more general conditions. This section is only an illustration of the possible usefulness of spherical shellings in the theory of orthogonal polynomials.

6 Explicit cd-index formula

The proof of Theorem 5.1 motivates to introduce the following sequence of partially ordered sets.

Definition 6.1 *Given a sequence of positive integers $\nu_1, \nu_2, \ldots$ satisfying $\nu_1 = 1$ and $\nu_n \geq 2$ for $n \geq 2$ let $Q_n = Q_n(\nu_2, \ldots, \nu_n)$ be the face poset of the lune complex $L(\nu_2, \ldots, \nu_{n-1}, 2\nu_n)$.*

As we have seen in the previous section, the sequence $\Phi_{ce}(Q_n)$ is a non-commutative generalization of the OPS $\{Q_n(x)\}_{n=0}^{\infty}$ defined by (9). As noted in the proof of Theorem 2.7, the faces contained in any facet of $L(\nu_2, \ldots, \nu_{n-1}, 2\nu_n)$ replicate the face structure of $L(\nu_2, \ldots, \nu_{n-2}, 2\nu_{n-1})$.

Corollary 6.2 *For every coatom c of Q_n, the interval $[\widehat{0}, c] \subset Q_n$ is isomorphic to Q_{n-1}.*

As an immediate consequence of Proposition 3.2 we have

$$\Phi_{ce}(Q_n) = \nu_n \Phi_{ce}(Q_{n-1})c + (1-\nu_n)\Phi_{ce}(Q_{n-2})e^2. \tag{15}$$

Similarly, (8) implies

$$\Phi_{cd}(Q_n) = \nu_n \Phi_{cd}(Q_{n-1})c + (1-\nu_n)\Phi_{cd}(Q_{n-2})(c^2-2d). \tag{16}$$

Introducing the sequence of polynomials $\imath_0, \imath_1(t_1), \ldots, \imath_n(t_1, \ldots, t_n), \ldots$ given by

$$\begin{aligned} \imath_0 &= 1 \quad \text{and} \\ \imath_n(t_1, \ldots, t_n) &= 1 + (t_1-1) + (t_1-1)(t_2-1) + \cdots + (t_1-1)\cdots(t_n-1) \quad \text{for } n \geq 1, \end{aligned} \tag{17}$$

the coefficient of a cd-word in $\Phi_{cd}(Q_n)$ may be described as follows:

Proposition 6.3 *The coefficient of $c^{k_1}dc^{k_2}d\cdots dc^{k_{r-1}}dc^{k_r}$ in $\Phi_{cd}(Q_n)$ is*

$$2^r\prod_{i=1}^{r-1}(\nu_{k_1+\cdots+k_i+2i}-1)\cdot\prod_{j=1}^{r}\imath_{k_j}\left(\nu_{k_1+\cdots+k_{j-1}+2j-1},\ldots,\nu_{k_1+\cdots+k_{j-1}+k_j+2j-2}\right).$$

Proof: Recalling the fact that c has degree 1 and d has degree 2, the i-th d from the left arises from the letters at position $k_1+\cdots+k_i+2i-1$ and $k_1+\cdots+k_i+2i$ in the ab-index. By inspection of (16) it is clear that a letter d is introduced only when the contribution of the second term is considered at the appropriate place, and so the i-th d contributes precisely a factor of $2\,(\nu_{k_1+\cdots+k_i+2i}-1)$ to the coefficient of the cd-word.

The contribution of the term c^{k_j} is obtained by substituting the ν_i's corresponding to the positions covered into a function $I_{k_j}(t_1,\cdots,t_{k_j})$, described recursively by

$$I_k(t_1,\ldots,t_k)=t_kI_{k-1}(t_1,\ldots,t_{k-1})+(1-t_k)\cdot I_{k-2}(t_1,\ldots,t_{k-2})$$

where $I_0=1$ and $I_1(t_1)=t_1=t_1-1+1$ are the appropriate initial conditions. Straightforward induction shows $I_n(t_1,\ldots,t_n)=\imath_n(t_1,\ldots,t_n)$. $\Diamond$

Corollary 6.4 *Every cd-word in $\Phi_{cd}(Q_n)$ has a strictly positive coefficient.*

Let us observe now, that in the proof Proposition 6.3 we used no other properties of of the posets Q_n than the recursion (16) for their cd-indices. Given any sequence $\{\widetilde{Q}_n\}_{n=0}^{\infty}$ of cd-polynomials satisfying $\widetilde{Q}_0=1$, $\widetilde{Q}_1=c$, and a recursion formula $\widetilde{Q}_n=\nu_n\widetilde{Q}_{n-1}c+(1-\nu_n)\widetilde{Q}_{n-2}(c^2-2d)$ for all $n\geq 2$, the coefficients of the cd-words in $\widetilde{Q}_n$ are given by Proposition 6.3. As a consequence, all cd-words have non-negative coefficients if all ν_i's satisfy $\nu_i\geq 1$. Consider now the linear transformation from cd-polynomials to polynomials in one variable induced by sending c into x and d into $(x^2-1)/2$. (Equivalently, we send e^2 into 1.) Then the sequence of polynomials $\{\widetilde{Q}_n\}_{n=0}^{\infty}$ goes into a sequence $\{Q_n(x)\}_{n=0}^{\infty}$ that satisfies $Q_0(x)=1$, $Q_1(x)=x$ and the recursion formula $Q_n(x)=\nu_nxQ_{n-1}(x)+(1-\nu_n)Q_{n-2}(x)$. If $\nu_i>1$ for $i\geq 2$ then we obtain an OPS, and now it is an immediate consequence of Proposition 6.3 that every $Q_n(x)$ in this sequence is a positive combination of terms of the form $x^i(x^2-1)^j$. Moreover, we obtain the following explicit formula for the coefficients.

Theorem 6.5 *Assume that the OPS $\{Q_n(x)\}_{n=0}^{\infty}$ is given by $Q_0(x)=1$, $Q_1(x)=x$, and*

$$Q_n(x)=\nu_nxQ_{n-1}(x)+(1-\nu_n)Q_{n-2}(x)\quad\text{for } n\geq 2,$$

where $\nu_n > 1$ for $n \geq 2$. Then each $Q_n(x)$ may be written as a positive combination of terms of the form $x^{n-2r}(x^2-1)^r$. The coefficient of $x^{n-2r}(x^2-1)^r$ is

$$\sum_{k_1+\cdots+k_r=n-2r} \prod_{i=1}^{r-1} (\nu_{k_1+\cdots+k_i+2i} - 1) \cdot \prod_{j=1}^{r} \imath_{k_j} \left(\nu_{k_1+\cdots+k_{j-1}+2j-1}, \ldots, \nu_{k_1+\cdots+k_{j-1}+k_j+2j-2}\right).$$

Here the functions $\imath_k(t_1,\ldots,t_k)$ are the ones given in (17).

In fact, the image of $c^{k_1}dc^{k_2}d\cdots dc^{k_{r-1}}dc^{k_r}$ is $x^{k_1+\cdots+k_r}\left(\frac{x^2-1}{2}\right)^r$ (here $k_1+\cdots+k_r = n-2r$), so Theorem 6.5 follows from the proof of Proposition 6.3 and the observation that the factors 2^r and $\left(\frac{1}{2}\right)^r$ cancel.

Admittedly, the proof of Theorem 6.5 may be presented without any reference to the face posets of lune complexes, but it seems to be more difficult to come up with the idea of the "underlying" non-commutative polynomials without the the inspiration from the theory of *cd*-indices of Eulerian posets.

7 Connection to the Tchebyshev posets

The self-similarity property of Corollary 6.2 also holds for the duals of the *Tchebyshev posets* introduced by the present author in [13]. Moreover, the setting $\nu_n = 2$ for $n \geq 2$ in (9) yields precisely the Tchebyshev polynomials of the first kind. Hence it is worth observing the following connection:

Theorem 7.1 *The Eulerian poset $Q_n(2,\ldots,2)$ is isomorphic to the dual of the Tchebyshev poset T_n introduced in [13].*

Before outlining the proof, let us recall the (momentarily) most convenient definition of T_n. The Tchebyshev poset T_n has a unique minimum element $\widehat{0} = (-1,1)$ and a unique maximum element $\widehat{1} = (-(n+1), -(n+2))$. All other elements of T_n are pairs of nonzero integers (x,y) such that $|x| < |y|$ and $|x|, |y| \in \{1,2,\ldots,n+1\}$, with the restriction that for $|y| = n+1$ we must have $y = -(n+1)$. For $(x_1,y_1),(x_2,y_2) \in T_n$ the partial order is defined by

$$(x_1,y_1) < (x_2,y_2) \Leftrightarrow ((|y_1| < |y_2|) \wedge ((|y_1| < |x_2|) \vee (y_1 = x_2) \vee (x_1 = x_2))). \tag{18}$$

This definition is easily dualized as follows. Let us introduce

$$x' = \operatorname{sign}(y)(n+2-|y|) \quad \text{and} \quad y' = \operatorname{sign}(x)(n+2-|x|) \quad \text{for every } (x,y) \in T_n \setminus \{\widehat{0},\widehat{1}\}.$$

Then $|x| < |y|$ is equivalent to $|x'| < |y'|$ and $|x|, |y| \in \{1, 2, \ldots, n+1\}$ is equivalent to $|x'|, |y'| \in \{1, 2, \ldots, n+1\}$. The restriction on the sign of y when $|y| = n+1$ is equivalent to setting $x = -1$ whenever $|x| = 1$. Given (x_1, y_1) and (x_2, y_2) from $T_n \setminus \{\widehat{0}, \widehat{1}\}$, let us describe the condition for $(x_1, y_1) <^* (x_2, y_2)$ in the dual order, in terms of (x_1', y_1') and (x_2', y_2'). In other words, we are describing the condition for $(x_1, y_1) > (x_2, y_2)$ in the original order. Easy substitution into (18) yields:

$$(x_1', y_1') <^* (x_2', y_2') \Leftrightarrow ((|x_1'| < |x_2'|) \wedge ((|x_2'| > |y_1'|) \vee (x_2' = y_1') \vee (y_2' = y_1'))) . \tag{19}$$

It is easy to verify that $|x'|$ is the rank of $(x', y') \in T_n^* \setminus \{\widehat{0}, \widehat{1}\}$. We will use this labeling of the elements of $T_n^* \setminus \{\widehat{0}, \widehat{1}\}$ to encode the elements of $Q_n(2, \ldots, 2) \setminus \{\widehat{0}, \widehat{1}\}$, i.e., the proper faces of the $(n-1)$-dimensional lune complex $L(2, \ldots, 2, 4)$. Using Corollary 2.3, consider the simplified code of each face in this complex, that is, let us replace every coordinate after the first 0 or π with $*$. Thus the code of every vertex will be of one of the following forms:

$(0, *, \ldots, *)$, $(\pi, *, \ldots, *)$,
$(\pi/2, \ldots, \pi/2, 0, *, \ldots, *)$, $(\pi/2, \ldots, \pi/2, \pi, *, \ldots, *)$,
$(\pi/2, \ldots, \pi/2, 0)$, $(\pi/2, \ldots, \pi/2)$, $(\pi/2, \ldots, \pi/2, \pi)$, or $(\pi/2, \ldots, \pi/2, 3\pi/2)$.

All the vertices have rank 1 in the face poset, hence the first coordinate of their code (x', y') should satisfy $x' = 1$. Let us set $|y'| = \ell + 1$ where ℓ is the length of code the vertex. Choose y' to be negative if $\theta_\ell = 0$ and let y' have positive sign if $\theta_\ell = \pi$. This rule uniquely determines the code (x', y') of a vertex, except for the vertices $(\pi/2, \ldots, \pi/2)$ and $(\pi/2, \ldots, 3\pi/2)$ which are the only vertices of length n. Let us associate $(-1, -(n+1))$ to $(\pi/2, \ldots, \pi/2)$ and $(-1, (n+1))$ to $(\pi/2, \ldots, 3\pi/2)$. Obviously we defined a bijection between the vertices of the lune complex and the rank 1 elements of T_n^*.

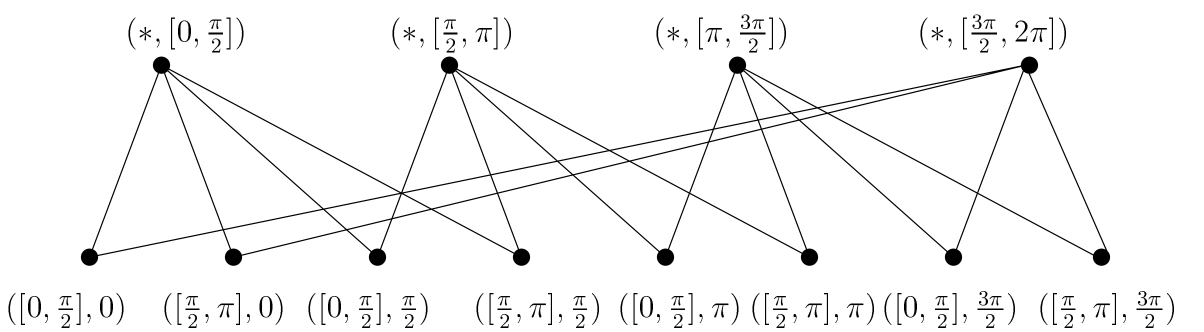
A lune of dimension more than zero but less than $(n-1)$ will have a code of the form

$$(*, \ldots, *, [\alpha, \beta], \pi/2, \ldots, \pi/2, \theta, *, \ldots, *)$$

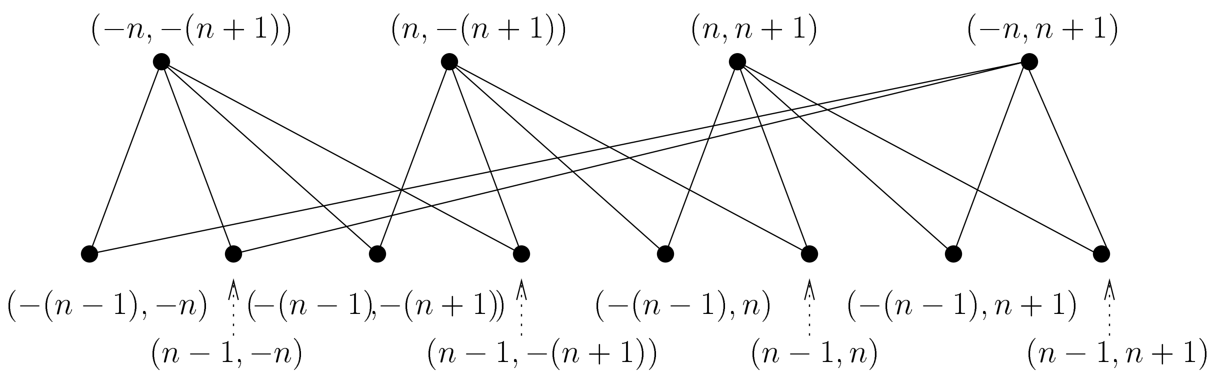
where $[\alpha, \beta]$ is either $[0, \pi/2]$ or $[\pi/2, \pi]$ and $\theta \in \{0, \pi\}$ unless θ is the last coordinate. (The number of entries between $[\alpha, \beta]$ and θ may be zero.) Again we must choose $|x'|$ to be the rank of our lune, which is the dimension of the lune plus one. Consistently with the sign choice for the vertices let us set x' to be negative if $[\alpha, \beta]$ contains 0 (i.e., $[\alpha, \beta] = [0, \pi/2]$), and let us set x' to be positive of $[\alpha, \beta]$ contains π (i.e., $[\alpha, \beta] = [\pi/2, \pi]$). Set again $|y'| = \ell + 1$, where ℓ is the length of the lune. Consistently with the sign choice for the vertices, set the sign of y' to be negative if $\theta \in \{0, \pi/2\}$ and positive if $\theta \in \{\pi, 3\pi/2\}$. (The possibilities $\theta = \pi/2$ or $\theta = 3\pi/2$ occur only if the lune has length n.)

It is easy to verify that this correspondence between the faces of the lune complex and elements of T_n^* up to rank at most $(n-2)$ is order preserving. In fact, if the lune (or vertex) λ_1 associated (x_1', y_1') to is contained in the lune λ_2 associated to (x_2', y_2'), then the λ_2 must have larger dimension, and so larger rank (this is condition $|x_1'| < |x_2'|$). If this is satisfied, then $\lambda_1 \subset \lambda_2$ holds when either the first non-star coordinate of λ_2 has higher index than the length λ_1 (equivalent to $|y_1| < |x_2|$) or the interval $[\alpha, \beta]$ in the code λ_2 exactly at index $\ell(\lambda_1)$ (thus containment at this coordinate is the only issue, and it is equivalent to $x_2' = y_1'$), or the interval $[\alpha, \beta]$ in the code λ_2 occurs at a coordinate where the coordinate of λ_1 is $\pi/2$. In this last case the subsequent coordinates of λ_1 and λ_2 must agree, which is equivalent to $x_2' = y_1'$.

Finally, we may extend our order preserving bijection to the elements of rank n by associating $(-n, -(n+1))$ to $(*, \ldots, *, [0, \pi/2])$, $(n, -(n+1))$ to $(*, \ldots, *, [\pi/2, \pi])$, $(n, (n+1))$ to
$(*, \ldots, *, [\pi, 3\pi/2])$, and $(-n, (n+1))$ to $(*, \ldots, *, [3\pi/2, 2\pi])$. Since we are constructing a bijection between graded partially ordered sets, it is sufficient to verify that the corresponding coatoms cover the corresponding sub-coatoms. This is most easily shown "pictorially":



corresponds to



Note that on the first picture the labels correspond to $n = 3$, but for larger n the only difference is that each label needs to be prepended with the appropriate number of stars.

8 Concluding remarks

As mentioned in the Introduction, an interesting continuation of the research presented in this paper could be exploring the potential connections between the theory of chain sequences and flag-enumeration in Eulerian posets, deciding along the way whether there is a closer relation to spherical shellings or to the approach taken in Section 6. If the theory of chain sequences turns out to be unrelated to either of these methods, then its non-commutative generalization (if it exists) could provide a new approach to proving non-negativity results for Eulerian posets.

It is also worth exploring what the study of lune complexes may tell us about systems of symmetric orthogonal polynomials. The first question is whether the polynomials $R(y_1, \ldots, y_n; x)$ given by (10) (introduced in connection with spherical shellings) have any further significance in the theory of orthogonal polynomials. Second, the issue of "translating" invariants of Eulerian posets could be raised. If we do not insist on finding non-negativity results, "almost every" symmetric OPS is equivalent (up to replacing each polynomial with a nonzero constant multiple) to an OPS defined by a recursion formula of the form $Q_n(x) = \nu_n x Q_{n-1}(x) + (1 - \nu_n) Q_{n-2}(x)$. Whenever the ν_i's are positive integers such that an associated poset $Q_n(\nu_2, \ldots, \nu_n)$ exists, invariants like the "toric h-vector" (for the definition see [16, Section 3.14]) are polynomial expressions of the ν_i's. Hence the definition of such invariants may be extended to almost all symmetric OPS (and perhaps even to the "singular ones", if "taking limits" is possible). It is very natural to ask, what is the meaning of such invariants in the theory of orthogonal polynomials. Finally, a new connection between certain orthogonal polynomials and statistics on words may be established by passing through the lune complex representation. In fact, the non-negativity of the cd-index of the Tchebyshev posets was shown in [13] using a shelling of the order complex, and not spherical shelling. That approach not only provided explicit formulas for the coefficients of the cd-index, but also established a connection to a certain statistics on words. It is worth trying to generalize that method to the order complexes of the face posets of lune complexes with the same aim.

To conclude, let us remind the reader how having a sequence of face posets of CW-spheres closed under taking boundary complexes of facets may bring us closer to proving Stanley's conjecture [17, Conjecture 2.1]. Given a sequence $\Omega_1, \Omega_2, \ldots$ of CW-spheres (one for each dimension), such that each face of each Ω_i has the same face structure as some Ω_j, we may restrict our attention to those Eulerian posets P for which every interval $[\widehat{0}, x]$ satisfying $x \neq \widehat{1}$ is isomorphic to some $P_1(\Omega_i)$. In the case when each Ω_i is the boundary complex of a simplex, one obtains the class of Gorenstein* simplicial posets, that was treated by Stanley in [17]. A first step towards handling the cubical case (where each Ω_i is the boundary complex of a cube) was proposed by Ehrenborg and Hetyei in [10]. Consideration of the special case when each Ω_i (for $i \geq 2$) is of the form $L(2, 2, \ldots, 4)$ was proposed in [13] and, before considering $\Omega_i = L(\nu_2, \ldots, \nu_i, 2\nu_i)$ with other ν_i's, it is still advisable to investigate that special case first. As noted in [13], it is easy to derive the analogues of the Dehn-Sommerville equations for "spheres of dual Tchebyshev cells", but it is unknown whether a sufficient number of linearly independent examples exists (in analogy to the simplicial and the cubical case). It is probably harder to construct such examples (because of the lack of polytopality), but if they exist, one obtains a special case of Stanley's conjecture that is very different from the simplicial and cubical cases yet "not more difficult" (as far as the number of unknowns per rank is concerned). It is to be expected that, for any such restricted case, the study of cd-indices (or, equivalently, flag f-vectors) may be reduced to the study of (non-flag) f-vectors of certain CW-spheres. In particular, Stanley reduces to the nonnegativity of the cd-index of a simplicial Gorenstein* poset to the nonnegativity of its simplicial h-vector in [17], and Ehrenborg and Hetyei show an analogous reduction for cubical posets in [10]. The nonnegativity of the simplicial h-vector of simplicial Cohen-Macaulay posets was shown in Stanley's earlier paper [18]. In the cubical case, the nonnegativity of the analogous h-vector (originally defined by Adin [1]) is still a conjecture, i.e., the cubical analogue of [18] is still missing. It is not implausible to suspect that, if finding the analogous results for for $\Omega_i = L(2, 2, \ldots, 2, 4)$ is feasible, then the corresponding argument for $\Omega_i = L(\nu_2, \ldots, \nu_i, 2\nu_i)$ with arbitrary ν_i's would involve finding a generalized h-vector that is a linear expression of the face numbers, with coefficients from $\mathbb{Q}[\nu_2, \nu_3, \ldots]$. Once a sufficient number of such simplified questions is answered, the proof of Stanley's general conjecture may perhaps be found more easily.

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