A Combinatorial Proof of the Log-concavity of a famous sequence counting permutations

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Submitted: Nov 24, 2004; Accepted: Jan 11, 2005; Published: Jan 24, 2005
Mathematics Subject Classifications: 05A05, 05A15

To Richard Stanley, who introduced me to the area of log-concave sequences.

Abstract

We provide a combinatorial proof for the fact that for any fixed n, the sequence $\{i(n,k)\}_{0\leq k\leq \binom{n}{2}}$ of the numbers of permutations of length n having k inversions is log-concave.

1 Introduction

Let $p = p_1p_2\cdots p_n$ be a permutation of length n, or, in what follows, an n-permutation. An *inversion* of p is a pair (i,j) of indices so that $i < j$, but $p_i > p_j$. The enumeration of n-permutations according to their number $i(p)$ of inversions, and the study of numbers $i(n,k)$ of n-permutations having k inversions, is a classic area of combinatorics. The best-known result is the following [4].

Theorem 1.1 *Let $n \geq 2$. Then we have*

$$\sum_{p\in S_n} x^{i(p)} = \sum_{k=0}^{\binom{n}{2}} i(n,k)x^k = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1}).$$

Another classic result [3] is that the numbers $i(n,k)$ also count n-permutations having *major index* k. Details about this result, and other related results can be found in [1].

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A somewhat less explored property of the numbers $i(n,k)$ is log-concavity. The sequence $(a_k)_{0\leq k\leq m}$ is called *log-concave* if $a_k a_{k+2} \leq a_{k+1}^2$ for all k. See [5] for a classic survey of log-concave sequences, and see [2] for an update on that survey. A *polynomial* is called log-concave if its coefficients form a log-concave sequence. It is a classic result (see for instance [1] for a proof) that the product of log-concave polynomials is log-concave. Therefore, Theorem 1.1 immediately implies that the polynomial $\sum_{k=0}^{\binom{n}{2}} i(n,k)x^k$ is log-concave, that is, the sequence $i(n,0), i(n,1), \cdots, i(n,\binom{n}{2})$ is log-concave. We could not find any previous proof of this fact that does not use generating functions. In this paper, we will provide such a proof. It is also the first non-generating function proof we know of in which a sequence whose length is quadratic in terms of the length of the input objects is shown to be log-concave.

2 The proof of our claim

2.1 The outline of the proof

It is easy to see that the sequence $(a_k)_{0\leq k\leq m}$ is log-concave if and only if $a_k a_l \leq a_{k+1}a_{l-1}$ for all $k \leq l-2$. One implication is trivial, and the other becomes obvious if we note that log-concavity is equivalent to the sequence a_{k+1}/a_k being weakly decreasing.

Let p be an n-permutation, and set $p = p_1p_2\cdots p_n$. Define $I_{n,k}$ to be the set of all n-permutations with exactly k inversions. When there is no danger of confusion about what n is, we will just write I_k instead of $I_{n,k}$.

The structure of our proof will be as follows. We want to prove the following theorem.

Theorem 2.1 *For all integers n, k and l satisfying $0 \leq k \leq l-2 \leq \binom{n}{2} - 2$, there exists an injection $f_{n,k,l} : I_k \times I_l \to I_{k+1} \times I_{l+1}$.*

Theorem 2.1 is clearly equivalent to what we want to prove. We will prove our claim by induction on n. That is, first, we will construct the injections $f_{n,k,l}$ for the smallest meaningful value of n, which is $n = 3$. Then, in the induction step, we will use the assumption that the maps $f_{n-1,k,l}$ exist for *all* allowed values of k and l to create the maps $f_{n,k,k+2}$. We will not create the maps $f_{n,k,l}$ for $k < l-2$, but we do not have to, since the existence of the maps $f_{n,k,k+2}$ in itself implies the log-concavity of the sequence $\{i(n,k)\}_{0\leq k\leq\binom{n}{2}}$, and therefore, it implies the existence of the maps $f_{n,k,l}$ for $k < l-2$. That will complete the induction step of our proof.

2.2 The details of the proof

It is time that we carried out the strategy to prove Theorem 2.1 that we discussed in the previous subsection.

The smallest value of n for which the domains of the maps $f_{n,k,l}$ are not all empty is $n = 3$. In this case, $f_{n,k,l}$ is defined for the (k, l)-pairs (0,2), (0,3) and (1,3). In those cases, we define $f_{3,0,2}(123, 231) = (213, 132)$, $f_{3,0,2}(123, 312) = (213, 213)$, and $f_{3,0,3}(123, 321) = (213, 231)$, as well as $f_{3,1,3}(132, 321) = (231, 231)$, and $f_{3,1,3}(213, 321) = (312, 231)$. It will soon become obvious why we define f this way.

Now let $n \geq 4$, and assume we have defined $f_{n-1,k,l}$ for all allowed values of k and l.

Let $(p, q) \in I_k \times I_{k+2}$, with $p = p_1p_2 \cdots p_n$ and $q = q_1q_2 \cdots q_n$. Proceed as follows.

(Rule 1) If $p_1 < n$ and $q_1 > 1$, increase p_1 by one, and decrease the entry of p that was one larger than p_1 by one. Let the obtained permutation be p'. Similarly, decrease q_1 by 1, and increase the entry of q that was one larger than q_1 by 1. Let the obtained permutation be q'. Set $f_{n,k,k+2}(p, q) = (p', q')$.

Note that p' starts with an entry larger than 1, and q' starts with an entry less than n.

Example 2.2 *If $p = 2134$ and $q = 3142$, then we have $f_{4,1,3}(p, q) = (3124, 2143)$.*

(Rule 2) If $p_1 = n$, or $q_1 = 1$, then remove these entries, to get the permutations $p*$ and $q*$. (After natural relabeling, these are both permutations of length $n - 1$.) Because of the extreme values of at least one of the omitted elements, we have $i(q*) - i(p*) \geq i(q) - i(p) = 2$. Therefore, there exist positive integers r and s, with $r \leq s - 2$, so that $(p*, q*)$ is in the domain of $f_{n-1,r,s}$.

Take $f_{n-1,r,s}(p*, q*) = (\bar{p}, \bar{q}) \in I(n-1, r+1) \times I(n-1, s-1)$. Now prepend $\bar{p}$ by p_1, and prepend $\bar{q}$ by q_1. In both cases, entries larger than or equal to the prepended entry have to be increased by 1. Call this new pair of n-permutations $(p_1\bar{p}, q_1\bar{q})$. Finally, set $f_{n,k,k+2}(p, q) = (q_1\bar{q}, p_1\bar{p})$. We point out that we *swapped* p and q.

Note that either $q_1\bar{q}$ starts in 1 or $p_1\bar{p}$ starts in n.

Example 2.3 *If $p = 1324$ and $q = 1432$, then we have $(p*, q*) = (213, 321)$, therefore, recalling that we have already defined $f_{3,1,3}$ for 3-permutations, $f_{3,1,3}(p*, q*) = (\bar{p}, \bar{q}) = (312, 231)$. Reinserting the removed first entries, we get $(p_1\bar{p}, q_1\bar{q}) = (1423, 1342)$. Finally, after swapping the two permutations of the last pair, we get $f_{4,1,3}(p, q) = (1342, 1423)$.*

Lemma 2.4 *The map $f_{n,k,k+2} : I_k \times I_{k+2} \to I_{k+1} \times I_{k+1}$ is an injection.*

Proof: First, it is clear that $f_{n,k,k+2}$ maps into $I_{k+1} \times I_{k+1}$ since both rules increase the number of inversions of the first permutation by one, and decrease the number of inversions of the second permutation by one.

Now we prove that $f_{n,k,k+2}$ is one-to-one. We achieve this by induction on n, the initial case of $n = 3$ being obvious. Assume now that the statement is true for $n - 1$.

Let $(t,u) \in I_{k+1} \times I_{k+1}$, with $t = t_1 t_2 \cdots t_n$, and $u = u_1 u_2 \cdots u_n$. We show that (t,u) can have at most one preimage under $f_{n,k,k+2}$. There are two cases.

1. If $t_1 > 1$ and $u_n < n$, then (t,u) could only be obtained as a result of applying $f_{n,k,k+2}$ if Rule 1 was used. In that case, we have $f^{-1}_{n,k,k+2}(t,u) = ((t_1 - 1)t_2 \cdots t_n, (u_1 + 1)u_2 \cdots u_n)$.

2. If $t_1 = 1$, or $u_1 = n$, then (t,u) could only be obtained as a result of applying $f_{n,k,k+2}$ if Rule 2 was used. In that case, to get the preimage of (t,u), we need to remove the first entry of t and the first entry of u, swap the permutations, and find the preimage of the resulting pair $(\bar{u}, \bar{t})$ under the appropriate map $f_{n-1,r,s}$.

   However, the preimage of $(\bar{u}, \bar{t})$ under $f_{n-1,r,s}$ is unique by the induction hypothesis, therefore so is $f^{-1}_{n,k,k+2}(t,u)$.

This completes our proof. $\diamond$

Consequently, the sequence $\{i(n,k)\}_{0 \leq k \leq \binom{n}{2}}$ is log-concave, and the injections $f_{n,k,l}$ exist for all values k and l satisfying $0 \leq k \leq l - 2 \leq \binom{n}{2} - 2$.

Acknowledgement

I am grateful to the anonymous referee for a careful reading of the manuscript.

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